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POSNER'S SECOND THEOREM, MULTILINEAR POLYNOMIALS AND VANISHING DERIVATIONS

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Abstract

Let K be a commutative ring with unity, R a prime K-algebra of characteristic different from 2, d and δ non-zero derivations of R, $f(x_1, \ldots, x_n)$ a multilinear polynomial over K. If

$$\delta([d(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)]) = 0 \quad \text{for all } r_1, \ldots, r_n \in R,$$

then $f(x_1, \ldots, x_n)$ is central-valued on R.

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A well-known Posner's result states that if R is a prime ring and d is a non-zero derivation of R such that $[d(r), r] \in Z(R)$, the center of R, for all $r \in R$, then R is commutative [17]. This result is included in a line of investigation concerning the relationship between the structure of R and the behaviour of some derivation defined on R. It is possible to formulate many results obtained in the literature in this context by considering appropriate conditions on the subset $P(d, k, S) = \{[d(s), s]_k : s \in S\}$, where S is a suitable subset of R, k is a positive integer and the k-commutator $[d(x), x]_k$, for $k > 1$, is defined by $[d(x), x]_k = [[d(x), x]_{k-1}, x]$. For instance, we can read the result of Lansky [11] as follows: If L is a noncentral Lie ideal of R and $P(d, k, L) = 0$ then R satisfies the standard polynomial identity $S_4(x_1, \ldots, x_4)$ and it is of characteristic 2. More generally, in the case when $f(x_1, \ldots, x_n)$ is a multilinear polynomial, I is a non-zero twosided ideal of R, Lee and Lee [12] proved that if $P(d, k, f(I)) = 0$ then either $f(x_1, \ldots, x_n)$ is central valued on R or $\text{char}(R) = 2$ and R satisfies the standard identity $S_4(x_1, \ldots, x_4)$. On the other hand, if $P(d, 1, R) \neq 0$ then it is a large subset of R, and as showed by Brešar and Vukman in [4], it generates a subring which contains a non-zero right and a non-zero left ideal of

R. More recently, in [6] and [7], we considered the case when R is a prime algebra over a commutative ring K, $f(x_1,\ldots,x_n)$ is a multilinear polynomial with coefficients in K and $P(d,1,f(R)) = \{[d(f(r_1,\ldots,r_n)), f(r_1,\ldots,r_n)] : r_1,\ldots,r_n \in R\}$ is not zero. More precisely, if $\mathrm{char}(R) \neq 2$, we proved that the left annihilator of $P(d,1,f(R))$ in R must be zero [7]. Moreover, if the non-zero elements of $P(d,1,f(R))$ are invertible then R is a division ring [6, Corollary 1].

The previous results also say that the subset $P(d,1,f(R))$ is rather large in R.

It would seem natural to ask what happens if there exists a non-zero derivation δ of R, such that $\delta(a) = 0$ for all $a \in P(d,1,f(R))$. In this paper we will give an answer and prove the following:

THEOREM 1. *Let K be a commutative ring with unity, R a prime K-algebra of characteristic different from 2, d and δ non-zero derivations of R, $f(x_1,\ldots,x_n)$ a multilinear polynomial over K. If $\delta([d(f(r_1,\ldots,r_n)), f(r_1,\ldots,r_n)]) = 0$ for all $r_1,\ldots,r_n \in R$, then $f(x_1,\ldots,x_n)$ is central-valued on R.*

We begin with the case when R is a ring of matrices over a field and d and δ are inner derivations. As above, for any elements s,t in a ring, we shall denote $[s,t]_2$ the triple commutator $[[s,t],t]$, and we shall use this notation through the rest of the paper. We have:

LEMMA 1. *Let $R = M_k(F)$ be the ring of $k \times k$ matrices over the field F, with $k > 1$, a,b non-central elements of R such that $[a,[b,f(r_1,\ldots,r_n)]_2] = 0$ for all $r_1,\ldots,r_n \in R$. Then $f(x_1,\ldots,x_n)$ is central-valued on R.*

PROOF. We suppose that $f(x_1,\ldots,x_n)$ is not central-valued on R and prove that in this case either a or b fall in $Z(R)$. The first aim is to prove that, if b is not a diagonal matrix, then a must be a central matrix. We will divide the proof in two cases: $k=2$ and $k \geq 3$.

Case 1: $k = 2$. Say $a = \sum_{ij} a_{ij}e_{ij}$, $b = \sum_{ij} b_{ij}e_{ij}$, where $a_{ij}, b_{ij} \in F$, and e_{ij} are the usual unit matrices. Suppose that b is not a diagonal matrix, for example let $b_{21} \neq 0$.

Since $f(x_1,\ldots,x_n)$ is not central on R, there exists an odd sequence of matrices $r_1,\ldots,r_n \in R$ such that $f(r_1,\ldots,r_n) = \gamma e_{ij}$, with $0 \neq \gamma \in F$ and $i \neq j$ [14, Lemma]. In particular, we may assume that $f(r_1,\ldots,r_n) = \gamma e_{12}$, because the set $f(R) = \{f(s_1,\ldots,s_n) : s_1,\ldots,s_n \in R\}$ is invariant under the action of all inner automorphisms of R. Thus

$$0 = [a,[b,f(r_1,\ldots,r_n)]_2] = -2\gamma^2(ae_{12}be_{12} - e_{12}be_{12}a)$$

and multiplying on the right by e_{12} we have:

$$e_{12}be_{12}ae_{12} = 0, \quad \text{that is,} \quad b_{21}a_{21} = 0.$$

Since $b_{21} \neq 0$, we have $a_{21} = 0$. Moreover by [15, Lemmas 2 and 9] there exists an even sequence of matrices $s_1, \ldots, s_n \in R$ such that $f(s_1, \ldots, s_n) = \alpha e_{11} + \beta e_{22}$, with $\alpha \neq \beta$. Then

$$[b, f(s_1, \ldots, s_n)]_2 = \begin{bmatrix} 0 & (\beta-\alpha)^2 b_{12} \\ (\alpha-\beta)^2 b_{21} & 0 \end{bmatrix}$$

and

$$0 = [a, [b, f(s_1, \ldots, s_n)]_2] = \begin{bmatrix} a_{12}b_{21}(\alpha-\beta)^2 & (a_{11}-a_{22})b_{12}(\beta-\alpha)^2 \\ (a_{22}-a_{11})b_{21}(\alpha-\beta)^2 & -a_{12}b_{21}(\alpha-\beta)^2 \end{bmatrix}.$$

Since $b_{21} \neq 0$, then $a_{12} = 0$ and $a_{11} = a_{22}$, which means that a is central in R, a contradiction.

Analogously we have the same contradiction if we suppose $b_{12} \neq 0$ and $a_{12} = 0$. Hence b must be a diagonal matrix in $R = M_2(F)$.

Case 2: $k \geq 3$. As above, since $f(x_1, \ldots, x_n)$ is not central on R, and $f(R)$ is invariant under the action of all F-automorphisms of R, for all $i \neq j$, there exist $r_1, \ldots, r_n \in R$ such that $f(r_1, \ldots, r_n) = \alpha e_{ij} \neq 0$. Thus

$$0 = [a, [b, f(r_1, \ldots, r_n)]_2] = -2\alpha^2(ae_{ij}be_{ij} - e_{ij}be_{ij}a)$$

and multiplying on the right by e_{ll}, with $l \neq j$ we have:

(1) $$e_{ij}be_{ij}ae_{ll} = 0, \quad \text{that is,} \quad b_{ji}a_{jl} = 0, \quad \forall j \neq i, l.$$

Analogously, left multiplying by e_{pp}, with $p \neq i$,

(1′) $$e_{pp}ae_{ij}be_{ij} = 0, \quad \text{that is,} \quad a_{pi}b_{ji} = 0 \quad \forall i \neq j, p.$$

Suppose b is not a diagonal matrix. Let $i \neq j$ such that $b_{ji} \neq 0$. Hence

(2) $$a_{pi} = 0, \quad \forall p \neq i, \quad \text{and} \quad a_{jl} = 0, \quad \forall l \neq j.$$

Moreover, we know that

$$(1 + e_{qi})(\alpha e_{ij})(1 - e_{qi}) = \alpha(e_{ij} + e_{qj}) \quad \forall q \neq i, j$$

is also a valuation of $f(x_1, \ldots, x_n)$ in R.

So, $[a, [b, \alpha(e_{ij} + e_{qj})]_2] = 0$, and left multiplying the last equation by e_{hh}, with $h \neq i, q$, we have

(3) $$e_{hh}ae_{ij}be_{ij} + e_{hh}ae_{ij}be_{qj} + e_{hh}ae_{qj}be_{ij} + e_{hh}ae_{qj}be_{qj} = 0.$$

By (3) using (1′), and (2) we obtain

$$a_{hq}b_{ji} = 0, \quad \text{that is} \quad a_{hq} = 0 \quad \forall h \neq i, q \quad \forall q \neq i, j.$$

This fact and (2) means that

(A) ‘If $b_{ji} \neq 0$ then the non-zero entries of the matrix a are just in the i-th row, in j-th column or in the main diagonal.’

As above, we assume $b_{ji} \neq 0$ and let $m \neq i, j$. Denote by σ_m and τ_m the following automorphisms of R:

$$\sigma_m(x) = (1 + e_{jm})x(1 - e_{jm}) = x + e_{jm}x - xe_{jm} - e_{jm}xe_{jm},$$
$$\tau_m(x) = (1 - e_{jm})x(1 + e_{jm}) = x - e_{jm}x + xe_{jm} - e_{jm}xe_{jm}$$

and say $\sigma_m(b) = \sum \sigma_{rs}e_{rs}$, $\tau_m(b) = \sum \tau_{rs}e_{rs}$ where $\sigma_{rs}, \tau_{rs} \in F$. We have

$$\sigma_{ji} = b_{ji} + b_{mi} \quad \text{and} \quad \tau_{ji} = b_{ji} - b_{mi}.$$

If there exists m such that $\sigma_{ji} = b_{ji} + b_{mi} = 0$ or $\tau_{ji} = b_{ji} - b_{mi} = 0$ then $b_{mi} = -b_{ji} \neq 0$ or $b_{mi} = b_{ji} \neq 0$. Therefore $b_{ji} \neq 0$ and $b_{mi} \neq 0$, and so, using (A), the non-zero entries of the matrix a are just in the i-row or on the main diagonal, since $m \neq j$. Hence

(4) $$a = \sum_{r,r\neq i} a_{rr}e_{rr} + \sum_{s} a_{is}e_{is}, \quad \text{with } a_{rs} \in F.$$

Now assume that $\sigma_{ji} \neq 0$ and $\tau_{ji} \neq 0$, for all $m \neq i, j$, and recall that, for any F-automorphism φ of R, the following holds

$$[\varphi(a), [\varphi(b), f(r_1, \dots, r_n)]_2] = 0, \quad \text{for all} \quad r_1, \dots, r_n \in R.$$

Thus in this case by (A), for any $m \neq i, j$, the non-zero entries of the matrices $\sigma_m(a)$ and $\tau_m(a)$ are just in the i-th row, in j-th column or on the main diagonal. In particular, since

$$\sigma_m(a) = a + e_{jm}a - ae_{jm} - e_{jm}ae_{jm},$$
$$\tau_m(a) = a - e_{jm}a + ae_{jm} - e_{jm}ae_{jm}$$

then both of the above matrices have zero in the (j, m) entry, that is,

$$a_{jm} + a_{mm} - a_{jj} - a_{mj} = 0, \quad a_{jm} - a_{mm} + a_{jj} - a_{mj} = 0, \quad \forall m \neq i, j.$$

Moreover, by (A), $a_{jm} = 0$, because $m \neq i, j$ and so $a_{mm} - a_{jj} = a_{mj} = a_{jj} - a_{mm}$, which implies $a_{mj} = 0$, for all $m \neq i, j$. At this point we can write again the matrix a as follows:

(4′) $$a = \sum_{r,r\neq i} a_{rr}e_{rr} + \sum_{s} a_{is}e_{is}.$$

In other words, by (4) and (4′), we have:

(B) 'If $b_{ji} \neq 0$ then the non-zero entries of the matrix a are just in the i-th row or on the main diagonal.'

Let again $b_{ji} \neq 0$ and $m \neq i, j$. Denote

$$\lambda_m(x) = (1 + e_{mi})x(1 - e_{mi}) = x + e_{mi}x - xe_{mi} - e_{mi}xe_{mi},$$
$$\mu_m(x) = (1 - e_{mi})x(1 + e_{mi}) = x - e_{mi}x + xe_{mi} - e_{mi}xe_{mi}$$

and say $\lambda_m(b) = \sum \lambda_{rs}e_{rs}$, $\mu(b) = \sum \mu_{rs}e_{rs}$ with $\lambda_{rs}, \mu_{rs} \in F$. We have that

$$\lambda_{ji} = b_{ji} - b_{jm} \quad \text{and} \quad \mu_{ji} = b_{ji} + b_{jm}.$$

If there exists $m \neq i, j$ such that $\lambda_{ji} = b_{ji} - b_{jm} = 0$ or $\mu_{ji} = b_{ji} + b_{jm} = 0$ then $b_{jm} = b_{ji} \neq 0$ or $b_{jm} = -b_{ji} \neq 0$. Thus, by (B), a is just a diagonal matrix because $b_{ji} \neq 0$, $b_{jm} \neq 0$ and $m \neq i, j$.

On the other hand, if $\lambda_{ji} \neq 0$ and $\mu_{ji} \neq 0$, for all $m \neq i, j$, then the non-zero entries of the matrices $\lambda_m(a)$ and $\mu_m(a)$ are just in the i-th row and on the main diagonal. In particular, since

$$\lambda_m(a) = a + e_{mi}a - ae_{mi} - e_{mi}ae_{mi},$$
$$\mu_m(a) = a - e_{mi}a + ae_{mi} - e_{mi}ae_{mi}$$

then both the matrices have zero in the (m, i) entry, that is,

$$a_{mi} + a_{ii} - a_{mm} - a_{im} = 0, \quad a_{mi} - a_{ii} + a_{mm} - a_{im} = 0, \quad \forall m \neq i, j.$$

Moreover, by (B), $a_{mi} = 0$, because $m \neq i, j$, and so $a_{mm} - a_{ii} = a_{im} = a_{ii} - a_{mm}$, which implies $a_{im} = 0$, for all $m \neq i, j$. Finally in any case, if $b_{ji} \neq 0$, we can write the matrix a as follows:

(5) $$a = \sum_r a_{rr}e_{rr} + a_{ij}e_{ij}.$$

Since $f(x_1, \ldots, x_n)$ is not central valued on R, by [15, Lemmas 2 and 9] there exists an even sequence of matrices $s_1, \ldots, s_n \in R$, such that $f(s_1, \ldots, s_n) = \sum_l \alpha_l e_{ll}$, with $\alpha_p \neq \alpha_q$, for some $p \neq q$. Moreover, since $f(R)$ is invariant under the action of all F−automorphisms of R, we may assume $p = i$ and $q = j$. By the above argument, $a = \sum_r a_{rr}e_{rr} + a_{ij}e_{ij}$, moreover $\left[b, \sum_l \alpha_l e_{ll}\right]_2 = \sum_{rs} b_{rs}(\alpha_s - \alpha_r)^2 e_{rs}$ and

(6) $$0 = \left[\sum_l a_{ll}e_{ll} + a_{ij}e_{ij}, \sum_{rs} b_{rs}(\alpha_s - \alpha_r)^2 e_{rs}\right].$$

In particular, the (i, i) entry of the matrix (6) is zero, that is, $b_{ji} a_{ij} (\alpha_i - \alpha_j)^2 = 0$. Since $b_{ji} \neq 0$ and $\alpha_i \neq \alpha_j$, we get $a_{ij} = 0$, which means that a is a diagonal matrix.

Let now, for all $m \neq i, j$, $\chi_m \in \mathrm{Aut}_F(R)$ with $\chi_m(x) = (1 + e_{im}) x (1 - e_{im})$. Since $[\chi_m(a), [\chi_m(b), f(s_1, \dots, s_n)]_2] = 0$, for all $s_1, \dots, s_n \in R$ and the (j, i)-entry of the matrix $\chi_m(b)$ is not zero, then $\chi_m(a) = a - a e_{im} + e_{im} a - e_{im} a e_{im}$ is diagonal, which implies

$$a_{mm} = a_{ii}, \quad \forall m \neq j. \tag{7}$$

Analogously, for all $t \neq i, j$, let $\psi_t(x) = (1 + e_{tj}) x (1 - e_{tj})$. Also in this case the (j, i)-entry of $\psi_t(b)$ is not zero, then $\psi_t(a) = a - a e_{tj} + e_{tj} a - e_{tj} a e_{tj}$ is diagonal, which implies

$$a_{tt} = a_{jj}, \quad \forall t \neq i. \tag{7'}$$

Thus by (7) and (7′) we conclude that if b is not diagonal then a must be central, which is a contradiction.

Therefore, we can assume that b is a diagonal matrix in $M_k(F)$ also in the case $k \geq 3$.

Finally, for any $\varphi \in \mathrm{Aut}_F(R)$, we have $[\varphi(a), [\varphi(b), \varphi(f(r_1, \dots, r_n))]_2] = 0$ for all $r_1, \dots, r_n \in R$, and so, by the previous cases, $\varphi(b)$ must be a diagonal matrix in $M_k(F)$ for any $k \geq 2$.

In particular, for any $r \neq s$, if $\varphi(x) = (1 + e_{rs}) x (1 - e_{rs})$, then

$$\varphi(b) = b + e_{rs} b - b e_{rs} - e_{rs} b e_{rs} = b + (b_{ss} - b_{rr}) e_{rs}.$$

This means $b_{rr} = b_{ss}$, for all $r \neq s$, that is b must be central, a contradiction again.

The previous argument says that $f(x_1, \dots, x_n)$ must be central-valued on R. ☐

Before beginnig the proof of the main theorem, for the sake of completeness we recall some basic notations, definitions and some easy consequences of the result of Kharchenko [10] about the differential identities on a prime ring R. We refer to [2, Chapter 7] for a complete and detailed description of the theory of generalized polynomial identities involving derivations.

We denote by Q the Martindale quotients ring of R and let $C = Z(Q)$ be the extended centroid of R [2, Chapter 2]. It is well known that any derivation of a prime ring R can be uniquely extended to a derivation of its Martindale quotients ring Q, and so any derivation of R can be defined on the whole Q [2, page 87]. Moreover, if R is a K-algebra we can assume that K is a subring of C.

Now, we denote by $\mathrm{Der}(Q)$ the set of all derivations on Q. By a derivation word we mean an additive map Δ of the form $\Delta = d_1 d_2 \cdots d_m$, with each $d_i \in \mathrm{Der}(Q)$. Then a differential polynomial is a generalized polynomial, with coefficients in Q, of the form $\Phi({}^{\Delta_j}x_i)$ involving noncommutative indeterminates x_i on which the derivations words Δ_j act as unary operations. The differential polynomial $\Phi({}^{\Delta_j}x_i)$ is said to be *a differential identity on a subset T of Q* if it vanishes for any assignment of values from T to its indeterminates x_i.

Let D_{int} be the C-subspace of $\mathrm{Der}(Q)$ consisting of all inner derivations on Q and let d and δ be two non-zero derivations on R. By [10, Theorem 2] we have the following result (see also [13, Theorem 1]):

FACT 1. *Let R be a prime ring of characteristic different from 2, if d and δ are C-linearly independent modulo D_{int} and $\Phi({}^{\Delta_j}x_i)$ is a differential identity on R, where Δ_j are derivations words of the following form δ, d, δ^2, δd, d^2, then $\Phi(y_{ji})$ is a generalized polynomial identity on R, where y_{ji} are distinct indeterminates.*

As a particular case, we have:

FACT 2. *If d is a non-zero derivation on R and*

$$\Phi(x_1, \dots, x_n, {}^{d}x_1, \dots, {}^{d}x_n, {}^{d^2}x_1, \dots, {}^{d^2}x_n)$$

is a differential identity on R, then one of the following holds

(i) *either $d \in D_{\text{int}}$*

(ii) *or R satisfies the generalized polynomial identity*

$$\Phi(x_1, \dots, x_n, y_1, \dots, y_n, z_1, \dots, z_n).$$

We study now the case when δ and d are both Q-inner derivations:

LEMMA 2. *If δ and d are both Q-inner non-zero derivations, then $f(x_1, \dots, x_n)$ is central-valued on R.*

PROOF. Let δ be the inner derivation induced by the element $a \in Q$, and d the one induced by $b \in Q$. Trivially a and b are not in the extended centroid C, which is the center of Q. These assumptions say that R satisfies the generalized polynomial identity $[a, [b, f(x_1, \dots, x_n)]_2]$ which is explicitely:

$$\begin{aligned} &abf^2(x_1, \dots, x_n) + af^2(x_1, \dots, x_n)b - 2af(x_1, \dots, x_n)bf(x_1, \dots, x_n) \\ &\quad - bf^2(x_1, \dots, x_n)a - f^2(x_1, \dots, x_n)ba + 2f(x_1, \dots, x_n)bf(x_1, \dots, x_n)a. \end{aligned}$$

By a theorem due to Beidar [1, Theorem 2] this generalized polynomial identity is also satisfied by Q. In case C is infinite, we have $[a, [b, f(r_1, \dots, r_n)]_2] = 0$ for all $r_1, \dots, r_n \in Q \bigotimes_C \overline{C}$, where $\overline{C}$ is the algebraic closure of C. Since both Q and $Q \bigotimes_C \overline{C}$ are centrally closed [8, Theorems 2.5 and 3.5], we may replace R by Q or

$Q \bigotimes_C \overline{C}$ according as C is finite or infinite. Thus we may assume that R is centrally closed over C which is either finite or algebraically closed and

$$[a, [b, f(r_1, \ldots, r_n)]_2] = 0, \quad \text{for all } r_1, \ldots, r_n \in R.$$

By Martindale's theorem [16], R is a primitive ring having a non-zero socle with C as the associated division ring. In light of Jacobson's theorem [9, page 75] R is isomorphic to a dense ring of linear transformations on some vector space V over C.

Assume first that V is finite-dimensional over C. Then the density of R on V implies that $R \cong M_k(C)$, the ring of all $k \times k$ matrices over C. In this case the conclusion follows by Lemma 1.

Assume next that V is infinite-dimensional over C. We will prove that in this case we get a contradiction. Since V is infinite dimensional over C then, as in Lemma 2 in [18], the set $f(R)$ is dense on R and so from $[a, [b, f(r_1, \ldots, r_n)]_2] = 0$, for all $r_1, \ldots, r_n \in R$, we have $[a, [b, r]_2] = 0$, for all $r \in R$. As a consequence a falls in to the centralizer of the set $\{[b, x]_2 : x \in R\}$. By main result in [4] the set $\{[b, x]_2 : x \in R\}$ contains a non-zero right ideal of R and so its centralizer coincides with the center of R; that is $a \in C$, which is a contradiction. □

We need the following lemma:

Lemma 3. *Let R be a prime K-algebra of characteristic different from 2 and $f(x_1, \ldots, x_n)$ a multilinear polynomial over K. If, for any $i = 1, \ldots, n$,*

$$[f(r_1, \ldots, z_i, \ldots, r_n), f(r_1, \ldots, r_n)] \in Z(R)$$

for all $z_i, r_1, \ldots, r_n \in R$, then the polynomial $f(x_1, \ldots, x_n)$ is central-valued on R.

Proof. Let $s \in R$, then by assumption

$$[s, f(r_1, \ldots, r_n)]_2 = \left[\sum_i f(r_1, \ldots, [s, r_i], \ldots, r_n), f(r_1, \ldots, r_n)\right] \in Z(R).$$

Hence, $[s, f(r_1, \ldots, r_n]_3 = [[s, f(r_1, \ldots, r_n)]_2, f(r_1, \ldots, r_n)] = 0$ and the result follows by [12, Theorem]. □

Now we are ready to prove our main result.

Theorem 1. *Let K be a commutative ring with unity, R a prime K-algebra of characteristic different from 2, d and δ non-zero derivations of R, $f(x_1, \ldots, x_n)$ a multilinear polynomial over K. If $\delta([d(f(r_1, \ldots, r_n)), f(r_1, \ldots, r_n)]) = 0$ for all $r_1, \ldots, r_n \in R$, then $f(x_1, \ldots, x_n)$ is central-valued on R.*

PROOF. Since $f(x_1,\dots,x_n)$ a multilinear polynomial, we can write

$$f(x_1,\dots,x_n) = x_1x_2\cdots x_n + \sum_{\sigma\in S_n,\sigma\neq\mathrm{id}} \alpha_\sigma x_{\sigma(1)}\cdots x_{\sigma(n)}$$

where S_n is the permutation group over n elements and any $\alpha_\sigma \in C$.

In all that follows we denote by $f^d(x_1,\dots,x_n)$, $f^{d\delta}(x_1,\dots,x_n)$ the polynomials obtained from $f(x_1,\dots,x_n)$ replacing each coefficient α_σ with $d(\alpha_\sigma)$ and $\delta(d(\alpha_\sigma))$ respectively. In this way we have

$$d(f(r_1,\dots,r_n)) = f^d(r_1,\dots,r_n) + \sum_i f(r_1,\dots,d(r_i),\dots,r_n)$$

and similarly for $\delta(d(f(r_1,\dots,r_n))$.

First suppose that δ and d are C-independent modulo D_{int}. By assumption, for all $r_1,\dots,r_n\in R$, $\delta([d(f(r_1,\dots,r_n)),\ f(r_1,\dots,r_n)])=0$, that is, R satisfies the differential identity

$$\begin{aligned}&\Bigg[f^{d\delta}(x_1,\dots,x_n) + \sum_{i\geq 1} f^d(x_1,\dots,{}^{\delta}x_i,\dots,x_n) + \sum_{i\geq 1} f(x_1,\dots,{}^{\delta d}x_i,\dots,x_n)\\ &\quad + \sum_{i\neq j} f(x_1,\dots,{}^{\delta}x_i,\dots,{}^{d}x_j,\dots,x_n),\ f(x_1,\dots,x_n)\Bigg]\\ &\quad + \Bigg[f^d(x_1,\dots,x_n) + \sum_{i\geq 1} f(x_1,\dots,{}^{d}x_i,\dots,x_n),\ f^{\delta}(x_1,\dots,x_n)\\ &\quad + \sum_{i\geq 1} f(x_1,\dots,{}^{\delta}x_i,\dots,x_n)\Bigg].\end{aligned}$$

By Kharchenko's theorem [10] R satisfies the polynomial identity

$$\begin{aligned}&\Bigg[f^{d\delta}(x_1,\dots,x_n) + \sum_{i\geq 1} f^d(x_1,\dots,y_i,\dots,x_n) + \sum_{i\geq 1} f(x_1,\dots,z_i,\dots,x_n)\\ &\quad + \sum_{i\neq j} f(x_1,\dots,y_i,\dots,t_j,\dots,x_n),\ f(x_1,\dots,x_n)\Bigg]\\ &\quad + \Bigg[f^d(x_1,\dots,x_n) + \sum_{i\geq 1} f(x_1,\dots,t_i,\dots,x_n),\ f^{\delta}(x_1,\dots,x_n)\\ &\quad + \sum_{i\geq 1} f(x_1,\dots,y_i,\dots,x_n)\Bigg].\end{aligned}$$

In particular, R satisfies any blended component

$$[f(x_1,\dots,z_i,\dots,x_n),\ f(x_1,\dots,x_n)]$$

in the indeterminates $x_1,\dots,x_n,z_i$ for all $i \geq 1$, which implies that $f(x_1,\dots,x_n)$ is central-valued on R by Lemma 3.

Let now δ and d C-dependent modulo D_{int}. There exist $\gamma_1,\gamma_2 \in C$, such that $\gamma_1\delta + \gamma_2 d \in D_{\mathrm{int}}$, and, by Lemma 2, it is clear that at most one of the two derivations can be inner.

Suppose $\gamma_1 = 0$ and $\gamma_2 \neq 0$; then, for some non-central element $q \in Q$, $d = d_q$ is the inner derivation induced by q and δ is an outer derivation. By the assumptions, $\delta([q,\ f(r_1,\dots,r_n)]_2) = 0$, for all $r_1,\dots,r_n \in R$, that is,

$$\begin{aligned}0 &= [\delta(q),\ f(r_1,\dots,r_n)]_2\\ &\quad+\left[\left[q,\ f^{\delta}(r_1,\dots,r_n)+\sum_i f(r_1,\dots,\delta(r_i),\dots,r_n)\right],\ f(r_1,\dots,r_n)\right]\\ &\quad+\left[[q,\ f(r_1,\dots,r_n)],\sum_i f(r_1,\dots,\delta(r_i),\dots,r_n)+f^{\delta}(r_1,\dots,r_n)\right].\end{aligned}$$

As above, by Kharchenko's result, R satisfies the generalized polynomial identity

$$\begin{aligned}&[\delta(q),\ f(x_1,\dots,x_n)]_2\\ &\quad+\left[\left[q,\ f^{\delta}(x_1,\dots,x_n)+\sum_i f(x_1,\dots,y_i,\dots,x_n)\right],\ f(x_1,\dots,x_n)\right]\\ &\quad+\left[[q,\ f(x_1,\dots,x_n)],\sum_i f(x_1,\dots,y_i,\dots,x_n)+f^{\delta}(x_1,\dots,x_n)\right].\end{aligned}$$

In particular, R satisfies the blended component in the indeterminates $x_1,\dots,x_n,y_1$, that is,

$$[[q,\ f(y_1,x_2,\dots,x_n)],\ f(x_1,\dots,x_n)] + [[q,\ f(x_1,\dots,x_n)],\ f(y_1,x_2,\dots,x_n)].$$

Hence $2[q,\ f(r_1,\dots,r_n)]_2 = 0$ for all $r_1,\dots,r_n \in R$. Since $q \notin C$, this implies that $f(x_1,\dots,x_n)$ is central-valued on R [12, Theorem].

Suppose now $\gamma_2 = 0$ and $\gamma_1 \neq 0$; then, for some non-central element $q \in Q$, $\delta = d_q$ is the inner derivation induced by q and d is an outer derivation.

In this case, for all $r_1,\dots,r_n \in R$, we have:

$$\begin{aligned}0 &= [q,[d(f(r_1,\dots,r_n)),\ f(r_1,\dots,r_n)]]\\ &= \left[q,[f^{d}(r_1,\dots,r_n)+\sum_i f(r_1,\dots,d(r_i),\dots,r_n),\ f(r_1,\dots,r_n)]\right]\end{aligned}$$

and, as above using the Kharchenko's theorem, R satisfies the following generalized polynomial identities

$$[q, [f(x_1, \dots, y_i, \dots, x_n),\ f(x_1, \dots, x_n)]] \quad \forall i = 1, \dots, n.$$

By [5] either q centralizes a noncentral Lie ideal of R or the polynomials

$$[f(x_1, \dots, y_i, \dots, x_n),\ f(x_1, \dots, x_n)]$$

are central-valued on R, for all $i = 1, \dots, n$. In the first case, it is well know that q is a central element of R (see [3, Lemma 2]), and this is a contradiction. It follows that the polynomials $[f(x_1, \dots, y_i, \dots, x_n),\ f(x_1, \dots, x_n)]$ are central-valued on R, for all $i = 1, \dots, n$; and this implies again that $f(x_1, \dots, x_n)$ is central-valued on R by Lemma 3.

Finally, we may assume that both γ_1 and γ_2 are non-zero. So $\delta = \gamma d + d_q$, with $0 \neq \gamma \in C$ and $q \in Q$.

Therefore, for all $r_1, \dots, r_n \in R$

$$\begin{aligned}(\gamma d + d_q)[d(f(r_1, \dots, r_n)),\ f(r_1, \dots, r_n)] \\ = \gamma d[d(f(r_1, \dots, r_n)),\ f(r_1, \dots, r_n)] \\ + [q, [d(f(r_1, \dots, r_n)),\ f(r_1, \dots, r_n)]] = 0.\end{aligned}$$

Suppose that d is an outer derivation. In this case R satisfies the differential identity

$$\begin{aligned}\gamma \Bigg[& f^{d^2}(x_1, \dots, x_n) + \sum_{i \geq 1} f^d(x_1, \dots, {}^d x_i, \dots, x_n) + \sum_{j \geq 1} f(x_1, \dots, {}^{d^2} x_j, \dots, x_n) \\ & + \sum_{i \neq j} f(x_1, \dots, {}^d x_i, \dots, {}^d x_j, \dots, x_n),\ f(x_1, \dots, x_n) \Bigg] \\ & + \left[q, \left[f^d(x_1, \dots, x_n) + \sum_{r \geq 1} f(x_1, \dots, {}^d x_r, \dots, x_n),\ f(x_1, \dots, x_n) \right] \right]\end{aligned}$$

and so the Kharchenko's theorem provides that

$$\begin{aligned}\gamma \Bigg[& f^{d^2}(x_1, \dots, x_n) + \sum_{i \geq 1} f^d(x_1, \dots, y_i, \dots, x_n) + \sum_{j \geq 1} f(x_1, \dots, z_j, \dots, x_n) \\ & + \sum_{i \neq j} f(x_1, \dots, y_i, \dots, y_j, \dots, x_n),\ f(x_1, \dots, x_n) \Bigg] \\ & + \left[q, \left[f^d(x_1, \dots, x_n) + \sum_{r \geq 1} f(x_1, \dots, y_r, \dots, x_n),\ f(x_1, \dots, x_n) \right] \right]\end{aligned}$$

is a polynomial identity on R. Hence R satisfies the blended components

$$[f(x_1,\ldots,z_j,\ldots,x_n),\ f(x_1,\ldots,x_n)] \quad \forall j=1,\ldots,n.$$

and this implies that $f(x_1,\ldots,x_n)$ is central-valued on R by Lemma 3.

Finally, if d is Q-inner, then δ is also Q-inner and we end up by Lemma 2. □

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