

ONE-SIDED IDEALS AND APPROXIMATE IDENTITIES IN OPERATOR ALGEBRAS

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(Received 5 October 2001; revised 5 May 2003)

Communicated by G. Willis

Abstract

A left ideal of any C^* -algebra is an example of an operator algebra with a right contractive approximate identity (r.c.a.i.). Indeed, left ideals in C^* -algebras may be characterized as the class of such operator algebras, which happen also to be triple systems. Conversely, we show here and in a sequel to this paper, that operator algebras with r.c.a.i. should be studied in terms of a certain left ideal of a C^* -algebra. We study left ideals from the perspective of ‘Hamana theory’ and using the multiplier algebras of an operator space studied elsewhere by the author. More generally, we develop some general theory for operator algebras which have a 1-sided identity or approximate identity, including a Banach-Stone theorem for these algebras, and an analysis of the ‘multiplier operator algebra’.

2000 *Mathematics subject classification*: primary 46L05, 46L07, 47L30; secondary 46H10, 47L75.

1. Introduction and notation

A norm closed left ideal of any C^* -algebra is an example of an operator algebra with a right contractive approximate identity. More is true; indeed left ideals in C^* -algebras may be characterized as the class of nonselfadjoint operator algebras with a right contractive approximate identity, which happen also to be ‘triple systems’ (see Theorem 2.6). This suggests that left ideals in C^* -algebras may profitably be studied using machinery that exploits *both* the ‘operator algebra’ and the ‘triple’ structure, and indeed we take this approach here. For example, ‘morphisms’ of left ideals in C^* -algebras will be what we call ‘ideal homomorphisms’ below, namely homomorphisms which are also ‘triple morphisms’.

This research was supported by a grant from the National Science Foundation.

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A (concrete) operator algebra is a closed subalgebra of $B(H)$, for some Hilbert space H . More abstractly, an operator algebra will be an algebra A with a complete norm defined on the space $M_n(A)$ of $n \times n$ matrices with entries in A , for each $n \in \mathbb{N}$, such that there exists a *completely isometric* homomorphism $A \rightarrow B(H)$ for some Hilbert space H . (We recall that a map $T : X \rightarrow Y$ is completely isometric if $[x_{ij}] \mapsto [T(x_{ij})]$ is isometric on $M_n(X)$ for all $n \in \mathbb{N}$.) An operator algebra is *unital* if it has a two-sided contractive identity. Unital operator algebras were characterized abstractly in [11]. However the class of nonselfadjoint operator algebras which is perhaps of most interest to C^* -algebraists or those interested in noncommutative geometry is the class of one-sided ideals in a C^* -algebra, which as we said possess only a one-sided approximate identity. Unfortunately, there seems to be no general results in the literature on operator algebras with a one-sided approximate identity, and thus part of the purpose of this note is to collect together some general theory of such algebras. Indeed, we show amongst other things that such algebras have an abstract characterization, Banach-Stone type theorems, reasonable multiplier algebras (which are operator algebras with two sided identity of norm 1), and they have an operator space predual if and only if they are ‘dual operator algebras’ in the usual strong sense of that term (see Theorem 4.6). Also, this subject becomes a little more interesting with a certain ‘transference principle’ in mind. This principle (which was proved first in the sequel [8]), allows one to deduce many general results about operator algebras with one-sided approximate identity, from results about left ideals in a C^* -algebra. Namely, there is an important left ideal $\mathfrak{J}_e(A)$ of a C^* -algebra $\mathcal{E}(A)$, which is associated to any such operator algebra A . We call $\mathfrak{J}_e(A)$ the ‘left ideal envelope’ of A . This is analogous to what happens in the case of operator algebras with two-sided identities, which are largely studied these days in terms of a certain C^* -algebra, namely the C^* -envelope.

We now describe the layout of the paper. In Section 2 we discuss one-sided ideals in C^* -algebras. In Section 3 we study a technical condition which commonly encountered operator algebras with a one-sided approximate identity possess. In Section 4 we assemble a collection of general results about operator algebras with a one-sided approximate identity. The principal tools used here are the multiplier algebra of an operator space studied in [6, 10, 7, 30], the ‘left ideal envelope’ of the last paragraph, and the facts from Section 2. In Section 5 we look at Banach-Stone type theorems. The classical Banach-Stone theorem (see, for example, [14, IV.2]) may be stated in the following form: if $C(K_1) \cong C(K_2)$ linearly isometrically, then they are $*$ -isomorphic (from which it is clear that the compact spaces K_1 and K_2 are homeomorphic). Indeed, the usual proofs show that the linear isometry equals a $*$ -isomorphism $C(K_1) \rightarrow C(K_2)$ multiplied by a fixed unitary in $C(K_2)$. There are numerous noncommutative versions of this, the most well known due to Kadison [20], where the $C(K)$ spaces are replaced by C^* -algebras. In Section 5 we examine such

theorems for maps between one-sided ideals in a C^* -algebra or between operator algebras with one-sided identities or contractive approximate identities. In Section 6 we study the ‘left multiplier operator algebra’ $LM(A)$ of an operator algebra A with a left contractive approximate identity.

We end the introduction with some more notation, and some background results which will be useful in various places. We reserve the letters H , K for Hilbert spaces, and J for a left ideal of a C^* -algebra. We will make the blanket convention that all ideals, left or otherwise, are assumed to be closed, that is, complete.

We shall abbreviate ‘right (respectively, left) contractive approximate identity’ to ‘r.c.a.i.’ (respectively, ‘l.c.a.i.’). For additional information on one-sided contractive approximate identities in general Banach algebras we refer the interested reader to the works of P. G. Dixon (see [23] for references), G. A. Willis (see [34] and references therein), and the general texts [13, 23]. If A is an algebra then we write $\lambda : A \rightarrow \text{Lin}(A)$ for the canonical ‘left regular representation’ of A on itself. By a ‘representation’ $\pi : A \rightarrow B(H)$ of an operator algebra A we shall mean a completely contractive homomorphism. If A has r.c.a.i. and if we say that π is *nondegenerate*, then at the very least we mean that $\pi(A)H$ is dense in H . Note that this last condition does not imply in general that $\pi(e_\alpha)\zeta \rightarrow \zeta$ for $\zeta \in H$, where $\{e_\alpha\}$ is the r.c.a.i., as one is used to in the two-sided case. One also cannot appeal to Cohen’s factorization theorem in its usual form (see, however, [23, Section 5.2]).

We will use without comment several very basic facts from C^* -algebra theory (see, for example, [26]), such as the basic definitions of the left multiplier algebra $LM(A)$ of a C^* -algebra, and the multiplier algebra $M(A)$.

As a general reference for operator spaces the reader might consult [17, 25, 27] or the forthcoming [32]. We write $CB(X)$ for the operator space of completely bounded maps $X \rightarrow X$. We write $\hat{} : X \rightarrow X^{**}$ for the canonical map, this is a complete isometry if X is an operator space, and is a homomorphism if X is an operator algebra (giving the second dual the Arens product [13]). It follows from [13, 28.7] that if A is an operator algebra with r.c.a.i. then A^{**} is an operator algebra with right identity of norm 1. If A has a right identity e , then \hat{e} is the right identity of A^{**} . If A is an operator algebra with two right identities e and f of norm 1, then since e and f are orthogonal projections, we have $e = ef = e^* = fe = f$. Thus an operator algebra has at most one right identity of norm 1.

It will be helpful throughout the paper to keep in mind the basic examples C_n (respectively, R_n); namely the $n \times n$ matrices ‘supported on’ the first column (respectively, row). This is a left (respectively, right) ideal of M_n , and has the projection E_{11} as the 1-sided identity. We write $C_n(X)$ for the first column on $M_n(X)$, that is $M_{n,1}(X)$. If X is an operator space, then so is $C_n(X)$.

If X and Y are subsets of an operator algebra we usually write XY for the *norm closure* of the set of finite sums of products xy of a term in X and a term in Y . For

example, if J is a left ideal of a C^* -algebra A , then with this convention J^*J and JJ^* are norm closed C^* -algebras. This convention extends to three sets, thus $JJ^*J = J$ for a left ideal of a C^* -algebra as is well known (or use the proof of Lemma 2.1 below to see this). We recall more generally that a TRO (*ternary ring of operators*) is a (norm closed for this paper) subspace X of $B(K, H)$ such that $XX^*X \subset X$. It is well known (copy the proof of Lemma 2.1 below) that in this case $XX^*X = X$. Then XX^* and X^*X are C^* -algebras, which we will call the left and right C^* -algebras of X respectively, and X is a $(XX^*) - (X^*X)$ -bimodule. A linear map $T : X \rightarrow Y$ between TRO's is a *triple morphism* if $T(xy^*z) = T(x)T(y)^*T(z)$ for all $x, y, z \in X$. TRO's are operator spaces, and triple morphisms are completely contractive, and indeed are completely isometric if they are 1-1 (see, for example, [19], this is related to results of Harris and Kaup). A completely isometric surjection between TRO's is a triple morphism. This last result might date back to around 1986, to Hamana, Kirchberg, and Ruan's PhD thesis independently. See [19] or [6, A.5] for a proof.

We will say that an operator space X is an *abstract triple system* if it is linearly completely isometrically isomorphic to a TRO Z . Note that then one may pull back the triple product on Z to a triple product $\{\cdot, \cdot, \cdot\}$ on X , and by the just mentioned result of Hamana, Kirchberg and Ruan, this triple product on X is unique, that is, independent of Z . That is, this triple product is completely determined by the 'operator space structure' or matrix norms on X .

Often it is convenient to state only the 'right-handed' version of a result. For example, Theorem 4.6 is a result about operator algebras with r.c.a.i. Of course by symmetry there will be a matching 'left-handed' version, in our example it will be about operator algebras with l.c.a.i. If we want to invoke this 'left-handed' version, we will refer to the 'other-handed version of Theorem 4.6', for example.

2. One-sided ideals in C^* -algebras

We begin by reviewing some background facts.

LEMMA 2.1 (Classical). *A norm closed left ideal J in a C^* -algebra is an operator algebra with a positive right contractive approximate identity. Also $J \cap J^* = J^*J \subset J \subset JJ^*$, so that J is a left ideal of the C^* -algebra JJ^* .*

PROOF. A left ideal J in a C^* -algebra A is clearly a subalgebra of A . Also JJ^* and J^*J are C^* -subalgebras of A . So J^*J has a positive c.a.i. $\{e_\alpha\}$; and for $x \in J$,

$$\|xe_\alpha - x\|^2 = \|e_\alpha x^* x e_\alpha - x^* x e_\alpha - e_\alpha x^* x + x^* x\| \rightarrow 0.$$

The remaining assertions follow immediately from this; for example if $x \in J \cap J^*$ then $x^* = \lim x^* e_\alpha$, so that $x \in J^*J$. □

- LEMMA 2.2.** (1) Suppose that $a \in B(H, K)$, and $\{e_\alpha\}$ is a net of contractions in $B(H)$ such that $ae_\alpha \rightarrow a$. Then $ae_\alpha e_\alpha^* \rightarrow a$, $ae_\alpha^* e_\alpha \rightarrow a$, and $ae_\alpha^* \rightarrow a$.
- (2) If J is a left ideal of a C^* -algebra, and if $\{e_\alpha\}$ is a r.c.a.i. for J , then $\{e_\alpha^* e_\alpha\}$ is a nonnegative right contractive approximate identity for J (and indeed also is a 2-sided c.a.i. for the C^* -subalgebra $J \cap J^* = J^* J$).
- (3) Any r.c.a.i. for a C^* -algebra is a l.c.a.i. too.

PROOF. (1) We use a technique from [9]. If $ae_\alpha \rightarrow a$ then $ae_\alpha e_\alpha^* a^* \rightarrow aa^*$, so that $0 \leq a(I - e_\alpha e_\alpha^*)a^* \rightarrow 0$. Thus by the C^* -identity, $a\sqrt{I - e_\alpha e_\alpha^*} \rightarrow 0$. Multiplying by $\sqrt{I - e_\alpha e_\alpha^*}$ we see that $a(I - e_\alpha e_\alpha^*) \rightarrow 0$ as required for the first assertion. Also,

$$\|ae_\alpha^* - a\| \leq \|ae_\alpha^* - ae_\alpha e_\alpha^*\| + \|ae_\alpha e_\alpha^* - a\| \rightarrow 0$$

since $\|ae_\alpha^* - ae_\alpha e_\alpha^*\| \leq \|a - ae_\alpha\| \rightarrow 0$. Finally,

$$\|ae_\alpha^* e_\alpha - a\| \leq \|ae_\alpha^* e_\alpha - ae_\alpha\| + \|ae_\alpha - a\| \leq \|ae_\alpha^* - a\| + \|ae_\alpha - a\| \rightarrow 0$$

by what we just proved.

Items (2) and (3) are clear from (1), but in any case are well known. \square

The next lemma concerns ‘principal ideals’. By a ‘principal ideal’ in a C^* -algebra A , we mean by analogy with pure algebra, an ideal of the form Ax for some $x \in A$. We are not taking the norm closure here, $Ax = \{ax : a \in A\}$ for some $x \in A$; however in view of the importance of closed ideals in C^* -algebra theory, below we only consider principal ideals which are already norm closed.

PROPOSITION 2.3. Let A be a C^* -algebra, and $x \in A$ (respectively, $x \in M(A)$), and suppose that $J = Ax$ is uniformly closed. Then $J = Ae$, where e is an orthogonal projection in J (respectively, in $M(A)$).

PROOF. Since J is the range of an adjointable map on A , J is orthogonally complemented in the sense of C^* -module theory, by [29, 15.3.9]. This implies that $J = Ae$ where e is an orthogonal projection in $M(A)$. This proves the very last assertion. Also, if A is unital we are done, and note that in this case Ae has a right identity of norm 1. However in any case, if $x \in A$, then $Ax = M(A)x$ (clearly $Ax \subset M(A)x$, but if $T \in M(A)$ then $Tx = \lim T e_\alpha x \in Ax$). Thus applying the above we see that J has a right identity f of norm 1, and $f \in J \subset A$. Hence $J = Af$. \square

If J is a left ideal in a C^* -algebra, then we define an *ideal representation* or *ideal homomorphism* of J to be a restriction of a $*$ -representation $\theta : J J^* \rightarrow B(H)$ to J . Clearly such a map is completely contractive.

PROPOSITION 2.4. *Let J be a left ideal of a C^* -algebra, and let $\pi : J \rightarrow B(H)$ be a function. Then π is the restriction of a $*$ -representation $\theta : JJ^* \rightarrow B(H)$ if and only if π is a homomorphism and a triple morphism. Moreover such π is completely isometric if and only if π is 1-1, and if and only if θ is 1-1.*

PROOF. If π is the restriction of a $*$ -representation then it is evident that π is a homomorphism and a triple morphism. Conversely, it is well known (see [19, 2.1]), that if π is a triple morphism, then there is an associated $*$ -homomorphism $\theta : JJ^* \rightarrow B(H)$ with the property that $\theta(xy^*) = \pi(x)\pi(y)^*$ for all $x, y \in J$. If in addition π is a homomorphism, and $\{e_\alpha\}$ is a positive r.c.a.i. for J , then $\{\pi(e_\alpha)\}$ is a positive r.c.a.i. for $\pi(J)$, and so for $x \in J$ we have by Lemma 2.2 that

$$\theta(x) = \lim \theta(xe_\alpha) = \lim \pi(x)\pi(e_\alpha)^* = \pi(x).$$

If further π is 1-1, then it is shown in [19] that θ is 1-1. □

The following result is a simple consequence of the fact that $JJ^*J = J$:

LEMMA 2.5. *Let J be a left ideal of a C^* -algebra, and let $\theta : JJ^* \rightarrow B(H)$ be a $*$ -homomorphism. If π is the restriction of θ to J then θ is nondegenerate if and only if $\pi(J)H$ is dense in H .*

THEOREM 2.6. *Let A be an abstract operator algebra which is also an abstract triple system (we are assuming the underlying matrix norms for both structures coincide). Then A has a r.c.a.i. for the algebra product if and only if there exists a left ideal J in a C^* -algebra, and a surjective complete isometry $A \rightarrow J$ which is both a homomorphism (that is, multiplicative), and a triple morphism.*

PROOF. The one direction is clear. For the other, we appeal to Theorem 4.4 below to obtain a completely isometric homomorphism j from A into a left ideal J of a certain C^* -algebra. Since J happens to be a triple envelope of A , and since there is a surjective complete isometry π from A onto a TRO, the universal property of the triple envelope applied to π forces j to be surjective. □

REMARKS. (1) Neal and Russo have a striking recent ‘matrix norm’ characterization of abstract triple systems [22]. Putting such a result together with our last theorem, and together with a characterization of operator algebras with right contractive approximate identity (r.c.a.i.) (see Theorem 4.3), will give a ‘completely abstract’ characterization of left ideals in C^* -algebras.

It would be interesting if, in the spirit of [22], one could give a purely linear characterization of left ideals in C^* -algebras. There is such a result in [7], but it makes reference to the containing C^* -algebra in the hypotheses.

(2) A slight modification of Theorem 2.6 also gives a characterization of C^* -algebras, by replacing ‘r.c.a.i.’ by ‘c.a.i.’. We are grateful to Bernie Russo for pointing out a recent paper [18] which gives such a characterization, but without needing the matrix norms.

We end this section with a ‘1-sided version’ of Sakai’s theorem characterizing von Neumann algebras. This result may be known to experts (certainly most of it is contained in a result from [31] (see also [15])).

THEOREM 2.7. *Let J be a left ideal in a C^* -algebra, and suppose that J possesses a Banach space predual. Then $M(JJ^*)$ is a W^* -algebra containing J as a weak*-closed principal left ideal.*

PROOF. By [31], the multiplier algebra $M(JJ^*)$ is a W^* -algebra and J is a dual operator space. By [5, Theorem 2.5], J has a right identity e . From this one sees that $J = Je^*e \subset JJ^*e \subset J$, so that $J = JJ^*e \subset M(JJ^*)e = M(JJ^*)e^2 \subset JJ^*e = J$. Thus $J = M(JJ^*)e$. \square

3. Properties (\mathcal{R}) and (\mathcal{L})

For a left ideal J in a C^* -algebra, it follows from the proof in Lemma 2.1 that J^*J also equals $\{x \in J : e_\alpha x \rightarrow x\}$, where $\{e_\alpha\}$ is the c.a.i. for J mentioned above. This is part of our motivation for the next definition.

DEFINITION 3.1. We say that an operator algebra A with r.c.a.i. (respectively, l.c.a.i.) has property (\mathcal{R}) (respectively (\mathcal{L})) if an r.c.a.i. (respectively l.c.a.i.) $\{e_\alpha\}$ exists for A such that $e_\alpha e_{\alpha'} \rightarrow e_{\alpha'}$ (respectively, $e_{\alpha'} e_\alpha \rightarrow e_{\alpha'}$) for each fixed $e_{\alpha'}$ in the net. In this case we define $\mathcal{R}(A) = \{x \in A : e_\alpha x \rightarrow x\}$ (respectively $\mathcal{L}(A) = \{x \in A : x e_\alpha \rightarrow x\}$).

REMARK. We note that a left ideal of a C^* -algebra has property (\mathcal{R}) , and in this case $\mathcal{R}(A) = J^*J$. More generally a subalgebra of a C^* -algebra with a self-adjoint right c.a.i. has property (\mathcal{R}) , since in this case $(e_\alpha e_{\alpha'})^* = e_{\alpha'} e_\alpha \rightarrow e_{\alpha'} = e_{\alpha'}^*$. An operator algebra with two-sided c.a.i. obviously has property (\mathcal{R}) , and in this case $\mathcal{R}(A) = A$. Certainly every operator algebra with a right identity of norm 1 has property (\mathcal{R}) .

Open question. Are there any operator algebras with r.c.a.i. which do not have property (\mathcal{R}) ?

PROPOSITION 3.2. *If an operator algebra A with r.c.a.i. has property (\mathcal{R}) , then $\mathcal{R}(A)$ is a norm closed right ideal of A (and hence is an operator algebra) with two sided c.a.i. Moreover, $\mathcal{R}(A)$ does not depend on the particular c.a.i. $\{e_\alpha\}$ considered. Also, $A\mathcal{R}(A) = A$ and $\mathcal{R}(A)A = \mathcal{R}(A)$. Similar results hold for property (\mathcal{L}) .*

PROOF. The first assertion we leave as a simple exercise. Suppose that A has property (\mathcal{R}) with respect to one r.c.a.i. $\{e_\alpha\}$, and let $\{f_\beta\}$ be another r.c.a.i. such that $f_\beta f_{\beta'} \rightarrow f_{\beta'}$ for every fixed β' . Let $B = \{a \in A : f_\beta a \rightarrow a\}$, another right ideal of A with two sided c.a.i. Note that $\mathcal{R}(A)B = \mathcal{R}(A)$ and $B\mathcal{R}(A) = B$. Thus by (the other-handed version of) [9, Theorem 4.15], $B = \mathcal{R}(A)$. The remaining assertions are left to the reader. \square

EXAMPLE 3.3. Let B be a unital operator algebra, a unital subalgebra of a W^* -algebra N , and define $M_\infty(N)$ to be the von Neumann algebra $B(\ell^2) \bar{\otimes} N$, thought of as infinite matrices $[b_{ij}]$ with entries b_{ij} indexed over $i, j \in \mathbb{N}$. We let $M_\infty(B)$ be the subset of $M_\infty(N)$ consisting of those matrices with entries b_{ij} in B . Often $M_\infty(B)$ is not an operator algebra, however there are several operator algebras inside $M_\infty(B)$ which occasionally play a role. To construct one, let $C_\infty^w(B)$ be the ‘first column’ of $M_\infty(B)$, and let $R_\infty(B)$ be the space of row vectors $[b_1 b_2 \cdots]$ with entries $b_i \in B$, such that $\sum_k b_k b_k^*$ converges in norm. We may then consider the closed subspace $A = C_\infty^w(B)R_\infty(B)$ of $M_\infty(B)$; those familiar with operator space theory will have no trouble verifying that A is a subalgebra of $M_\infty(N)$, that A has a nonnegative r.c.a.i., and indeed if $B = N$ then A is a left ideal of $M_\infty(B)$. In fact, A contains the C^* -algebra $\mathbb{K}_\infty(B)$, namely the spatial tensor product $\mathbb{K}(\ell_2) \otimes B$ (which in the language of C^* -modules equals $\mathbb{K}(C_\infty(A))$), and the usual c.a.i. for this C^* -algebra, namely $I_n \otimes 1_B$, is a r.c.a.i. for A . Thus A has property (\mathcal{R}) . It is easily verified that $\mathbb{K}_\infty(B)$ is a right ideal in A , and in fact $\mathcal{R}(A) = \mathbb{K}_\infty(B)$.

If A has left identity e of norm 1, then A clearly has property (\mathcal{L}) of Definition 3.1, and this identity is the 2-sided identity of $\mathcal{L}(A) = Ae$. Moreover, the map $A \rightarrow \mathcal{L}(A)$ taking $a \mapsto ae$, is a completely contractive homomorphism, and also is a complete quotient map and indeed is a projection onto $\mathcal{L}(A)$. On the other hand, if A has a l.c.a.i. and property (\mathcal{L}) , then by passing to the second dual A^{**} we can make similar assertions: there is a completely contractive homomorphism $A^{**} \rightarrow \mathcal{L}(A)^{**}$, which is a complete quotient map and indeed a projection. This is the map $F \mapsto FE$, where E is a weak* limit point of the c.a.i. of $\mathcal{L}(A)$. We use this in the next result.

PROPOSITION 3.4. *Suppose that A is an operator algebra with l.c.a.i. and property (\mathcal{L}) of Definition 3.1. Let $\pi : A \rightarrow B(H)$ be a completely contractive representation (respectively, and also $\pi(A)H$ is dense in H). Then $\pi|_{\mathcal{L}(A)} : \mathcal{L}(A) \rightarrow B(H)$ is a completely contractive homomorphism (respectively, and also such that $\pi(\mathcal{L}(A))H$ is dense in H). Conversely, if $\theta : \mathcal{L}(A) \rightarrow B(H)$ is a completely contractive homomorphism, then there exists a completely contractive homomorphism $\pi : A \rightarrow B(H)$ extending θ . If further $\theta(A)H$ is dense in H then π is unique, and $\pi(A)H$ is dense in H . Finally,*

$$\{T \in B(H) : T\pi(A) \subset \pi(A)\} = \{T \in B(H) : T\pi(\mathcal{L}(A)) \subset \pi(\mathcal{L}(A))\}.$$

PROOF. The first statements are simple exercises. For the converse, given such $\theta : \mathcal{L}(A) \rightarrow B(H)$, consider the series of completely contractive homomorphisms

$$A \hookrightarrow A^{**} \rightarrow \mathcal{L}(A)^{**} \xrightarrow{\theta^{**}} B(H)^{**} \rightarrow B(H).$$

The homomorphism $A^{**} \rightarrow \mathcal{L}(A)^{**}$ is the map described above the Proposition, and the other maps are the canonical ones. The composition of these homomorphisms is the desired π . We leave it to the reader to check the details. Since $\pi(a)\theta(b)\zeta = \pi(ab)\zeta = \theta(ab)\zeta$ for $a \in A, b \in \mathcal{L}(A), \zeta \in H$ we see that π is unique if $\pi(A)H$ is dense.

Finally, using the ‘other-handed version’ of the last assertion of Proposition 3.2, we see, for example, that if $T\pi(A) \subset \pi(A)$ then

$$T\pi(\mathcal{L}(A)) = T\pi(A)\pi(\mathcal{L}(A)) \subset \pi(A)\pi(\mathcal{L}(A)) = \pi(\mathcal{L}(A)).$$

The other direction is similar. □

The previous result shows that A and $\mathcal{L}(A)$ have the same representation theory. Thus the following definition which plays a role in the last section is somewhat natural: we say that a nondegenerate representation $\pi : A \rightarrow B(H)$ is completely ‘ \mathcal{L} -isometric’, if $\pi|_{\mathcal{L}(A)}$ is completely isometric on $\mathcal{L}(A)$.

REMARK. If A has a left identity of norm 1 but no right identity, and if $\pi : A \rightarrow B(H)$ is a nondegenerate isometric representation, then $\pi(e) = \text{Id}$, so that $\pi(ae) = \pi(a)$, so that $ae = a$ for all $a \in A$. This is a contradiction. Thus there is in general little point in seeking nondegenerate isometric representations of algebras with l.c.a.i. This is why we study \mathcal{L} -isometric representations.

4. A collection of general results

As this title indicates, this section is somewhat of a miscellany. The major tool needed is the left multiplier algebra $\mathcal{M}_\ell(X)$ of an operator space X . This is a unital operator algebra, which is a subalgebra of $CB(X)$ containing Id_X , but with a different (bigger in general) norm. There are several equivalent definitions of $\mathcal{M}_\ell(X)$ given in [6, 7, 10]; however the reader may take the definition of $\mathcal{M}_\ell(X)$ from the following result from [7]:

THEOREM 4.1. *A linear $T : X \rightarrow X$ on an operator space is in $\text{Ball}(\mathcal{M}_\ell(X))$ if and only if $T \oplus \text{Id}$ is a complete contraction $C_2(X) \rightarrow C_2(X)$.*

The matrix norms on $\mathcal{M}_\ell(X)$ may be described via the natural isomorphism $M_n(\mathcal{M}_\ell(X)) \cong \mathcal{M}_\ell(M_n(X))$. That is, the norm of a matrix $[T_{ij}]$ of multipliers may be taken to be the norm in $\mathcal{M}_\ell(M_n(X))$ of the map $[x_{ij}] \mapsto [\sum_k T_{ik}(x_{kj})]$.

LEMMA 4.2. *Let A be an operator algebra with a r.c.a.i. Then the canonical ‘left regular representation’ of A on itself yields completely contractive embeddings (that is, 1-1 homomorphisms) $A \rightarrow \mathcal{M}_\ell(A) \hookrightarrow CB(A)$, and the first of these embeddings, and their composition, are completely isometric.*

PROOF. Let $\lambda : A \rightarrow CB(A)$ be the left regular representation. This map is certainly completely contractive, however since $\lambda(a)(e_\alpha) = ae_\alpha \rightarrow a$ it is clear that λ is a complete isometry. Suppose that $a \in \text{Ball}(A)$, and that $y = [y_{ij}]$ and $y' = [y'_{ij}]$ are in $M_m(A)$. Then

$$\left\| \begin{bmatrix} ay_{ij} \\ y'_{ij} \end{bmatrix} \right\| = \left\| \begin{bmatrix} a \otimes I_m & 0 \\ 0 & \text{Id} \otimes I_m \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} y \\ y' \end{bmatrix} \right\|.$$

Here Id may be regarded as I_H for a particular representation of A . Thus $\lambda(a)$ satisfies the criterion of Theorem 4.1, so that $\lambda(a) \in \text{Ball}(\mathcal{M}_\ell(A))$. A similar argument works at the matrix level. Thus λ factors through $\mathcal{M}_\ell(A)$ via the two completely contractive homomorphisms above. Since λ is completely isometric, so is the first embedding. \square

We now turn to characterizations of operator algebras, which was our main original motivation for introducing multipliers of operator spaces in [6]. We pointed out in [6, Section 5] that in order to prove the characterization of operator algebras of [7] say, it is clearly only necessary to check that the ‘left regular representation’ $\lambda : A \rightarrow CB(A)$, is a complete isometry into the operator algebra $\mathcal{M}_\ell(A)$. But this is immediate from a theorem such as 4.1 above—see the simple proof of the next result, which is a variant of [5, 1.11].

THEOREM 4.3. *Let A be an operator space which is an algebra with a right identity of norm 1 or r.c.a.i. Then A is completely isometrically isomorphic to a concrete operator algebra (via a homomorphism of course), if and only if we have $\|(x \oplus \text{Id}_n)y\| \leq 1$ for all $n \in \mathbb{N}$ and $x \in \text{Ball}(M_n(A))$, $y \in \text{Ball}(M_{2n,n}(A))$.*

To explain the notation of the theorem, we have written Id for a formal identity, thus the expression $(x \oplus \text{Id}_n)y$ above means that the upper $n \times n$ -submatrix of y is left multiplied by x , and the lower submatrix is left alone.

PROOF. This is identical to the proof of Lemma 4.2 above, except when proving the analogue of the displayed equation—there one needs to use the hypothesis of our theorem. See the remarks above. \square

The following theorem, first proved in [8], is one of our main tools to deduce results about operator algebras with r.c.a.i., from results about left ideals in a C^* -algebra.

Let A be an operator algebra with r.c.a.i., and suppose that $i : A \rightarrow B$ is a completely isometric homomorphism into a C^* -algebra. Let J be the ‘TRO generated

by $i(A)$ ': the span in B of expressions of the form $i(a_1)i(a_2)^*i(a_3)i(a_4)^*\cdots i(a_{2n+1})$, for $a_i \in A$. By Lemma 2.2 (1) it is clear that $J \subset JJ^*$, so that JJ^* is a C^* -algebra which has J as a left ideal. In fact clearly JJ^* is the C^* -subalgebra of B generated by $i(A)$. We say that a pair (J, i) consisting of a left ideal J in a C^* -algebra, and a completely isometric homomorphism $i : A \rightarrow J$, is a *left ideal extension* of A if J is the 'TRO generated by $i(A)$ ' in the sense above. In this case $\{i(e_\alpha)\}$ is a r.c.a.i. for J if $\{e_\alpha\}$ is a r.c.a.i. for A .

THEOREM 4.4 ([8]). *Let A be an operator algebra with r.c.a.i. Then there exists a left ideal extension $(\mathfrak{J}_e(A), j)$ of A , with $\mathfrak{J}_e(A)$ a left ideal in a C^* -algebra $\mathcal{E}(A)$, such that for any other left ideal extension (J, i) of A , there exists a (necessarily unique and surjective) ideal homomorphism (see Proposition 2.4) $\tau : J \rightarrow \mathfrak{J}_e(A)$ such that $\tau \circ i = j$. Thus $\mathfrak{J}_e(A)/(\text{Ker } \tau) \cong J$ completely isometrically homomorphically (that is, as operator algebras) too. Moreover $(\mathfrak{J}_e(A), j)$ is unique in the following sense: given any other (J', j') with this universal property, then there exists a surjective completely isometric homomorphism $\theta : \mathfrak{J}_e(A) \rightarrow J'$ such that $\theta \circ j = j'$.*

Finally, $(\mathfrak{J}_e(A), j)$ is a triple envelope for A in the sense of [19].

We call $(\mathfrak{J}_e(A), j)$ the *left ideal envelope* of A , and set $\mathcal{E}(A) = \mathfrak{J}_e(A)\mathfrak{J}_e(A)^*$, a C^* -algebra. The map j will be called the *Shilov embedding homomorphism*. From the last assertion of the theorem, and the first definition of $\mathcal{M}_\ell(A)$ given in [6, Section 4], we may identify $\mathcal{M}_\ell(A)$ with $\{R \in LM(\mathcal{E}(A)) : Rj(A) \subset j(A)\}$.

COROLLARY 4.5. *Let A be an operator algebra with r.c.a.i., and λ the usual left regular representation of A . Any $T \in \mathcal{M}_\ell(A)$, regarded as a map on A , satisfies $T\lambda(A) \subset \lambda(A)$. Thus elements of $\mathcal{M}_\ell(A)$, considered as maps on A , are right A -module maps. That is, $\mathcal{M}_\ell(A) \subset CB_A(A)$ as sets. Also, $\mathcal{M}_r(A) \subset {}_A CB(A)$ as sets.*

PROOF. The first assertion follows from the remark before the statement of the Corollary, together with the fact that j is a homomorphism. For if $a \in A$, then the map $b \mapsto ab$ on A , corresponds to the map $j(b) \mapsto j(a)j(b)$ on $j(A)$. Thus if the left multiplier T corresponds to an $R \in LM(\mathcal{E}(A))$ with $Rj(a) = j(T(a))$ then

$$j(T(ab)) = Rj(ab) = Rj(a)j(b) = j(T(a))j(b) = j(T(a)b)$$

for any $b \in A$. This amounts to the first assertion, and also yields the second assertion immediately. The third is similar. \square

Corollary 4.5 allows us to generalize the main result of [5] (see also [21]) to algebras with one-sided c.a.i.:

THEOREM 4.6. *Let A be an operator algebra with r.c.a.i., which has a predual operator space. Then A has a right identity e of norm 1. Also A is a ‘dual operator algebra’, which means that the product on A is separately weak* continuous, and there exists a completely isometric homomorphism, which is also a homeomorphism with respect to the weak* topologies, of A onto a σ -weakly (that is, weak*-) closed subalgebra B of some $B(H)$.*

PROOF. The first assertion appears in [5, Theorem 2.5] (indeed for this part we only need a predual Banach space). From [5, Theorem 3.2], $\mathcal{M}_\ell(A)$ is a dual operator algebra. We saw in Lemma 4.2 and Corollary 4.5 that $\lambda : A \rightarrow \mathcal{M}_\ell(A)$ is a completely isometric homomorphism onto a left ideal of $\mathcal{M}_\ell(A)$. Hence $\lambda(A) = \mathcal{M}_\ell(A)\lambda(e)$. Thus $\lambda(A)$ is a weak* closed subalgebra of $\mathcal{M}_\ell(A)$, and so $B = \lambda(A)$ is a dual operator algebra. If we take a bounded net $\lambda(a_i) \rightarrow \lambda(a)$ weak* in $\lambda(A)$, then by definition of the weak* topology on $\mathcal{M}_\ell(A)$ from [5, 3.2], $a_i e = a_i \rightarrow a e = a$ weak* in A . Thus λ^{-1} is weak* continuous, so that by the Krein-Smulian theorem (see [5, Lemma 1.5]) λ is weak* continuous. \square

Results such as Theorem 4.4 are useful for deducing results about general operator algebras with r.c.a.i., from results about left ideals in C^* -algebras. For example, here is a sample application of this ‘transference principle’ (other examples will be given later):

COROLLARY 4.7. *Let A be an operator algebra with a right contractive approximate identity, and also a right identity. Then A has a right identity of norm 1, which is the limit in norm of the r.c.a.i.*

PROOF. First suppose that $A = J$ is a left ideal of a C^* -algebra, and suppose that J has a right identity. Then J is a principal left ideal and so by Proposition 2.3, J has a right identity e of norm 1. So $e = e^* \in J \cap J^* = J^*J$. If $\{e_\alpha\}$ is a r.c.a.i. for J then $\{e_\alpha^* e_\alpha\}$ is a 2-sided c.a.i. for J^*J (see Lemma 2.2 (ii)), thus $e_\alpha^* e_\alpha = e_\alpha^* e_\alpha e \rightarrow e$. Finally, $\|e_\alpha - e\|^2 = \|e_\alpha^* e_\alpha - e_\alpha^* e - e e_\alpha + e\| \rightarrow 0$.

If A is nonselfadjoint, and if $\{e_\alpha\}$ is the r.c.a.i. for A , then $\{j(e_\alpha)\}$ is a r.c.a.i. for the left ideal envelope $\mathfrak{J}_e(A)$. Similarly $\mathfrak{J}_e(A)$ and A have a common right identity. Hence by the last paragraph, our r.c.a.i. converges in norm. \square

5. The Banach-Stone theorem

We prove several stages, or cases, of this theorem, which asserts essentially that linear surjective complete isometries between left ideals of C^* -algebras (respectively, between operator algebras with r.c.a.i.), are characterized as a composition of a translation by a partial isometry u , and a surjective completely isometric homomorphism

onto another right ideal (respectively, operator algebra with r.c.a.i.) which is a translate of one of the original ideals (respectively, algebras) by u^* . To see that the ‘translate by a partial isometry’ is not artificial, consider an infinite dimensional Hilbert space H and S the shift operator. Set $I = B(H)$ and $J = B(H)S$. These ideals are clearly linearly completely isometric, but there is no homomorphism of I onto J (since J has no 2-sided identity). This example shows that the following theorem (which comprises Case (1)) is best possible:

THEOREM 5.1. *Let I and J be principal left ideals in C^* -algebras A and B ; thus $I = Ae$ and $J = Bf$, say, for orthogonal projections e, f in I, J respectively. Suppose also that $\varphi : I \rightarrow J$ is a linear surjective complete isometry. Then there exists a partial isometry u in B with initial projection f , and a completely isometric surjective ideal homomorphism (see Proposition 2.4) $\pi : I \rightarrow J_1$ such that $\varphi = \pi(\cdot)u$ and $\pi = \varphi(\cdot)u^*$. Here $J_1 = Bu^* = Ju^* = Buu^* \subset B$ is another left ideal of B with right identity uu^* .*

Conversely, if J is a left ideal of a C^ -algebra B , and if u is a partial isometry in B with initial projection a right identity for J , then $Ju^* = Bu^* = Buu^*$ is a left ideal J_1 of B with right identity uu^* of norm 1, and J_1 is linearly completely isometrically isomorphic to J via right multiplication by u^* . Hence the composition of right multiplication by u^* , with any completely isometric surjective homomorphism $I \rightarrow J_1$, is a linear completely isometric isomorphism $I \rightarrow J$.*

Finally, if $\varphi : I \rightarrow J$ is a linear surjective complete isometry, and if $\varphi(e) = f$, then $u = u^ = f$ and $J_1 = J$ in the above; and φ is a homomorphism. Conversely, if φ is a homomorphism, then necessarily $\varphi(e) = f$.*

PROOF. Recall from the introduction that a completely isometric surjection between TRO's is a triple morphism. Hence φ is a triple isomorphism. Therefore if $u = \varphi(e)$ then it is easy to check that $\pi(\cdot) = \varphi(\cdot)u^*$ is a homomorphism onto Ju^* . Similar considerations show that $p = uu^*$ is an idempotent, which is an orthogonal projection since it is selfadjoint. Thus u is a partial isometry. We claim that $u^*u = f$. To see this note that u^*u is an orthogonal projection, and that for any $\varphi(x) \in J$ we have $\varphi(x)u^*u = \varphi(xe) = \varphi(x)$, using the definition of a triple morphism. Thus $fu^*u = f$. On the other hand, $u^*uf = u^*u$ since $u \in Bf$. Hence $f = u^*u$. Also, $Ju^* = Bfu^* = Bu^*uu^* = Bu^*$. Defining J_1 to be this last space we see that it is clearly a left ideal of B , and J_1 contains uu^* , which is indeed a right identity of norm 1 for J_1 since u is a partial isometry. Thus $J_1 = Buu^*$ too.

Since $\pi(\cdot) = \varphi(\cdot)u^*$ we obtain $\pi(\cdot)u = \varphi(\cdot)u^*u = \varphi(\cdot)f = \varphi(\cdot)$. It follows from this too that π is a complete isometry, and therefore also a triple morphism. Thus π is a completely isometric ideal homomorphism.

Conversely, if J, B, u are as stated, then $J = Bu^*u$ so that $Ju^* = Bu^*$ which

is also a left ideal of B . Clearly the last space equals Buu^* since $Buu^* \subset Bu^* = Bu^*uu^* \subset Buu^*$. The remainder of the converse direction is left to the reader.

The very last assertion is easy to see from the uniqueness of a contractive right identity (proved in Section 1). \square

Having thoroughly analyzed the Banach-Stone theorem in Case (1), we now move to Case (2). Here we look at linear completely isometric isomorphisms $\varphi : A \rightarrow B$ between operator algebras with a right identity of norm 1. In the assertions in the first paragraph of the statement of the next theorem, and in the proofs of these assertions, we regard A and B as having been identified with subalgebras of $\mathfrak{J}_e(A)$ and $\mathfrak{J}_e(B)$ respectively (see Theorem 4.4). Thus mention of the ‘canonical Shilov embedding homomorphisms’ j have been suppressed, and all products and adjoints in that paragraph are taken in the containing C^* -algebra $\mathcal{E}(B) = \mathfrak{J}_e(B)\mathfrak{J}_e(B)^*$.

THEOREM 5.2 (Banach-Stone for operator algebras with right identities). *Suppose that $\varphi : A \rightarrow B$ is a surjective linear completely isometric isomorphism between operator algebras with a right identity of norm 1. Then there exists a partial isometry $u \in \mathfrak{J}_e(B)$ (indeed, in B) with initial projection the right identity of B , such that the subspace $B' = Bu^*$ of $\mathcal{E}(B)$ is a subalgebra (and consequently an operator algebra) with a right identity uu^* of norm 1; and there exists a completely isometric surjective homomorphism $\pi : A \rightarrow B'$, such that $\varphi = \pi(\cdot)u$ and $\pi = \varphi(\cdot)u^*$. Also, $u^*B \subset B$.*

*Conversely, suppose we are given a partial isometry u on a Hilbert space H , such that u lies in a subalgebra $B \subset B(H)$, such that the initial projection of u is a right identity of B , and such that $u^*B \subset B$. Then $B' = Bu^*$ is an operator algebra with right identity uu^* of norm 1, and B' is linearly completely isometrically isomorphic to B via right multiplication by u . Thus the composition of right multiplication by u^* , with any completely isometric surjective homomorphism $A \rightarrow B'$, is a linear completely isometric isomorphism $A \rightarrow B$.*

PROOF. Suppose that $\varphi : A \rightarrow B$ is a linear completely isometric isomorphism, and extend φ to a linear completely isometric isomorphism $\bar{\varphi} : \mathfrak{J}_e(A) \rightarrow \mathfrak{J}_e(B)$ (such extension exists by Hamana theory ([19] or [6, Appendix A])). By Theorem 4.4, $\mathfrak{J}_e(A)$ is a left ideal of the C^* -algebra $\mathcal{E}(A)$, and $\mathfrak{J}_e(A)$ has right identity e . Similar assertions hold for $\mathfrak{J}_e(B)$. Thus by the proof of Theorem 5.1, if $u = \varphi(e) = \bar{\varphi}(e)$ then u is a partial isometry in B , with $u^* \in B^* \subset \mathcal{T}(B)^* \subset \mathcal{E}(B)$, whose initial projection is f , and $\pi = \bar{\varphi}(\cdot)u^*$ is a completely isometric surjective homomorphism $\mathfrak{J}_e(A) \rightarrow \mathfrak{J}_e(B)u^*$. The restriction of π to A maps onto the subalgebra Bu^* of $\mathcal{E}(B)$. Since u is a partial isometry, uu^* is indeed a right identity of Bu^* . Finally, since $Bu^*Bu^* \subset Bu^*$, post multiplying by u gives $Bu^*B \subset B$, so that

$$u^*B = u^*uu^*B = fu^*B \subset Bu^*B \subset B.$$

Conversely, given u as stated, then since $u^*B \subset B$ we have that Bu^* is a subalgebra of $\mathcal{E}(B)$ with right identity uu^* . The remainder of the converse direction is obvious. \square

REMARK. In Theorem 5.2, u and u^* are in $LM(B)$ in the language of [8]. Also, one can prove further that $\mathfrak{J}_e(B') = \mathfrak{J}_e(B)u^*$, and that $\mathcal{E}(B') = \mathcal{E}(B)$. We omit the details.

COROLLARY 5.3. *Suppose that $\varphi : A \rightarrow B$ is a surjective linear completely isometric isomorphism between operator algebras with right identities e and f of norm 1. Then φ is a homomorphism if and only if $\varphi(e) = f$.*

PROOF. The one direction follows from the uniqueness of a contractive right identity (proved in Section 1). The other direction follows by noting that if we follow the proof of Theorem 5.2, then $\bar{\varphi}(e) = f$, so that $\bar{\varphi}$ is a homomorphism by last assertion of Theorem 5.1. \square

COROLLARY 5.4. *Suppose that A is an operator algebra with a right identity of norm 1, and suppose that A has another product $m : A \times A \rightarrow A$ with respect to which A is completely isometrically isomorphic to an operator algebra with a right identity of norm 1. Then there is a partial isometry $u \in \mathfrak{J}_e(A)$ (and, indeed, in A) such that $m(x, y) = xu^*y$ for all $x, y \in A$. Indeed u is the right identity for m , and u^*u is the right identity for the first product.*

We now turn to Case (3) of the Banach-Stone theorem. We only state the ‘forward implication’; the (tidier) converse we leave as an exercise.

THEOREM 5.5 (Banach-Stone theorem for left ideals in C^* -algebras). *Consider a surjective linear complete isometry $\varphi : I \rightarrow J$ between arbitrary left ideals of C^* -algebras. Let $\mathcal{E} = JJ^*$, and let \mathcal{M} be the von Neumann algebra $(JJ^*)^{**}$. Then there exists another left ideal J_1 of \mathcal{E} , with $J_1J_1^* = \mathcal{E}$, and a surjective completely isometric ideal homomorphism (see Proposition 2.4) $\pi : I \rightarrow J_1$. Moreover there exists a partial isometry $u \in \mathcal{M}$ such that the initial projection of u is the right identity of J^{**} (indeed of $RM(J)$ —see Section 4), and such that $J_1 = Ju^*$, $J = J_1u$, and such that $\varphi = \pi(\cdot)u$, and $\pi = \varphi(\cdot)u^*$.*

PROOF. Consider the second dual $\varphi^{**} : I^{**} \rightarrow J^{**} \subset \mathcal{M}$, and now we are back in Case (1). For if I is a left ideal of a C^* -algebra A , then I^{**} is a left ideal of A^{**} , but now I^{**} has a right identity e of norm 1, which may be taken to be a weak*-accumulation point of the r.c.a.i. of I (by [13, 28.7]). Thus by Case (1) we have that $u = \varphi^{**}(e)$ is a partial isometry in $J^{**} \subset \mathcal{M}$, and the initial projection of u is the matching right identity of J^{**} . Moreover $\pi = \varphi^{**}(\cdot)u^*$ is a completely isometric homomorphism and

so on. Restricting π to I gives a completely isometric homomorphism π' onto the subalgebra $J_0 = Ju^*$ of \mathcal{M} , and φ is the composition of π' with a right translation by u . Moreover, π' is easily seen to be a triple morphism:

$$\pi'(x)\pi'(y)^*\pi'(z) = \varphi^{**}(\hat{x})u^*u\varphi^{**}(\hat{y})^*\varphi^{**}(\hat{z})u = \varphi(x)\varphi(y)^*\varphi(z)u = \varphi(xy^*z)u,$$

which is simply $\pi'(xy^*z)$, for $x, y, z \in I$. Thus π' is a completely isometric ideal homomorphism. Therefore, by Proposition 2.4, π' is the restriction of a surjective 1-1 *-homomorphism $II^* \rightarrow J_0J_0^*$. Thus $J_0J_0^* = Ju^*uJ^* = \mathcal{E}$ contains J_0 as a left ideal; or to be more precise, $\hat{\mathcal{E}}$ contains J_0 . Thus we may regard π' as a completely isometric homomorphism $\pi : I \rightarrow J_1$ onto a right ideal J_1 of \mathcal{E} (note $\hat{J}_1 = J_0$). The rest is clear. \square

We briefly discuss Case (4) of the Banach-Stone theorem, the case of a surjective linear complete isometry between arbitrary operator algebras with r.c.a.i. Again it is clear that by passing to the second dual and using Case (2) in the way we tackled Case (3) using Case (1), or using Case (3) in the way we tackled Case (2) using Case (1), will give a result resembling Theorems 5.1, 5.2, and 5.5. We leave the details to the reader.

COROLLARY 5.6. *Let $\varphi : A \rightarrow B$ be a surjective linear complete isometry between left ideals of C^* -algebras, or between operator algebras with r.c.a.i. Then φ is a homomorphism if and only if there exists a r.c.a.i. $\{e_\alpha\}$ for A such that $\{\varphi(e_\alpha)\}$ is a r.c.a.i. for B .*

PROOF. If the latter condition holds then $\varphi^{**} : A^{**} \rightarrow B^{**}$ is a surjective linear complete isometry. Let E be a weak* limit point of $\{e_\alpha\}$ in A^{**} , and since φ^{**} is weak*-continuous, $\varphi^{**}(E)$ is a weak* limit point of $\{\varphi(e_\alpha)\}$. So we are in the situation of Corollary 5.3 (with the algebras replaced by their second duals), so that φ^{**} and consequently φ is a homomorphism. The converse direction is easier. \square

REMARK. Banach-Stone theorems for unital operator algebras or operator algebras with two-sided approximate identities may be found in [1, 2, 3, 16] and [6, Appendix B.1].

6. $LM(A)$ for an algebra with left contractive approximate identity

In this section we develop the ‘left multiplier operator algebra’ $LM(A)$ of an operator algebra with l.c.a.i. Since this follows closely the essentially known theory for the case of a two-sided c.a.i. (see [28, 24, 9, 4, 6]) we will try to be brief. The left multiplier operator algebra of an operator algebra with r.c.a.i. turns out to have a quite

different theory, which is studied in the sequel [8], and which we will not mention again in the present paper. On the other hand, $RM(A)$ for an operator algebra with r.c.a.i. is the ‘other-handed version’ of what we do below.

If A is an algebra, then a *left multiplier* of A is a right A -module map $T : A \rightarrow A$. The *left multiplier algebra* is the unital algebra of left multipliers of A , together with the left regular representation (which maps A into the left multiplier algebra of A). If A is a Banach algebra which has a one-sided approximate identity, then it follows from the closed graph theorem and a variant on Cohen’s factorization theorem that any left multiplier is bounded [23, 5.2.6]. Thus the left multiplier algebra equals $B_A(A)$, the unital Banach algebra of bounded right A -module maps. If A is an operator algebra with l.c.a.i., then it follows more or less immediately from the relation $T(a) = \lim_{\alpha} T(e_{\alpha})a$ which clearly holds for all $T \in B_A(A)$, $a \in A$, that $B_A(A) = CB_A(A)$ isometrically. Here $CB_A(A)$ is the set of completely bounded right A -module maps. One would wish the left multiplier algebra of an operator algebra to be a unital operator algebra, and fortunately it turns out that $CB_A(A)$ with its usual matrix norms is an abstract operator algebra. This is seen in the next theorem. Thus we define the left multiplier operator algebra of an operator algebra with l.c.a.i., to be the pair $(CB_A(A), \lambda)$, where λ is the left regular representation of A .

More generally, we consider pairs (D, μ) consisting of a unital operator algebra D and a completely contractive homomorphism $\mu : A \rightarrow D$, such that $D\mu(A) \subset \mu(A)$. Sometimes we write μ_A to indicate the dependence on A . We say that two such pairs (D, μ) and (D', μ') are completely isometrically *A-isomorphic* if there exists a completely isometric surjective homomorphism $\theta : D \rightarrow D'$ such that $\theta \circ \mu = \mu'$. This is an equivalence relation. We will also use the term ‘left multiplier operator algebra of A ’ for any pair (D, μ) as above which is completely isometrically A -isomorphic to $(CB_A(A), \lambda)$.

THEOREM 6.1. *Let A be an operator algebra with l.c.a.i. Then the following operator algebras are all completely isometrically isomorphic*

- (1) $\{x \in A^{**} : x\hat{A} \subset \hat{A}\} / \text{Ker } q$ where q is the canonical homomorphism into $CB(A)$,
- (2) $\mathcal{M}_{\ell}(A)$ (see Section 4),
- (3) $CB_A(A)$,

and in particular, $CB_A(A)$ is an operator algebra. If A satisfies condition (\mathcal{L}) of Definition 3.1 (for example, if A has a left identity of norm 1, or a two-sided c.a.i., or if A is a right ideal of a C^ -algebra), then the algebras above are completely isometrically isomorphic to*

- (4) $\{T \in B(H) : T\pi(A) \subset \pi(A)\}$, for any completely \mathcal{L} -isometric nondegenerate representation π of A (see definition after Proposition 3.4),

(5) $LM(B)$ where $B = \mathcal{L}(A)$ (see Definition 3.1),

(6) $\{x \in B^{**} : x\hat{A} \subset \hat{A}\} \subset A^{**}$, where $B = \mathcal{L}(A)$.

If A has a two-sided c.a.i., then $\text{Ker } q = (0)$ in (1).

PROOF. We first observe that for any operator algebra A there are natural completely contractive homomorphisms $\{x \in A^{**} : x\hat{A} \subset \hat{A}\} \rightarrow \mathcal{M}_\ell(A) \rightarrow CB(A)$. Let us write σ for the first homomorphism, and θ for the second. From the ‘left handed variant’ of Corollary 4.5, the image of θ lies in $CB_A(A)$. Next note that given $S \in CB_A(A)$, then one may let F be a weak* accumulation point of $S(e_\alpha)$ in A^{**} , for the l.c.a.i. $\{e_\alpha\}$ for A . Clearly $\|F\| \leq \|S\|$. For $a \in A$, we have

$$S(a) = \lim_{\alpha} S(e_\alpha a) = \lim_{\alpha} S(e_\alpha)a = Fa.$$

Hence $q(F) = S$, where $q = \theta \circ \sigma$. Thus q is a quotient map, and similarly it is a complete quotient map. Thus σ is also a complete quotient map, and $\text{Ker } \sigma = \text{Ker } q$ since θ is 1-1. This proves the completely isometric isomorphism between (1) and (3), and also between (1) and (2). Thus $\mathcal{M}_\ell(A) \cong CB_A(A)$ completely isometrically, which also shows that $CB_A(A)$ is a unital operator algebra (or this fact may be proved directly).

Now suppose that A has property (\mathcal{L}) , and set $B = \mathcal{L}(A)$ as in Definition 3.1. Then $B^{**} \subset A^{**}$. Examining the proof of (1) = (3) above, we see easily that the terms $S(e_\alpha)$ actually lie in B . Hence the F there lies in $\{x \in B^{**} : x\hat{A} \subset \hat{A}\}$. Thus the map q mentioned above, restricted to the last set, is a complete quotient map too. Therefore it is a complete isometry if we can show that it is 1-1. To see this suppose that F is in the set in (6) and $q(F) = 0$. Then $F\hat{e}_\alpha = 0$. This implies that $F = 0$, using the fact from [13, Section 28] that a weak* limit point of the \hat{e}_α is a 2-sided identity for B^{**} , and the fact that the multiplication in a dual operator algebra is separately weak* continuous. Thus we have that (3) = (6) completely isometrically. Note too that if A is an operator algebra with 2-sided c.a.i. then this shows that $\text{Ker } q = (0)$ in (1). Note that if F is in the set in (6), then $FB \subset B$ quite clearly. Conversely if $FB \subset B$ then for $a \in A$ we have $Fa = \lim F e_\alpha a \in A$ since $F e_\alpha a \in Ba \subset A$. This shows that (6) = (5).

Finally, to prove that (4) = (5), we may without loss of generality, by the definition after Proposition 3.4 and the last assertion of that proposition, assume that $B = A$ is an operator algebra with 2-sided c.a.i., and that $\pi : A \rightarrow B(H)$ is a nondegenerate completely isometric homomorphism. This case is no doubt well known by now (but first done in [24] perhaps); briefly, one way to see it is as follows. If we write $LM(\pi)$ for the algebra in (4), then there is a natural map $\rho : LM(\pi) \rightarrow CB_A(A)$, namely $\rho(T)(a) = \pi^{-1}(T\pi(a))$ for $a \in A$. If $[T_{ij}] \in M_n(LM(\pi))$, then

$$\|[\rho(T_{ij})]\|_n = \sup\{\|[\rho(T_{ij})(a_{kl})]\|_{nm} = \sup\{\|[T_{ij}\pi(a_{kl})]\|_{nm} \leq \|[T_{ij}]\|_n,$$

where the supremum is taken over matrices $[a_{kl}]$ of norm ≤ 1 . Thus ρ is completely contractive. To see that ρ is completely isometric we take the $[a_{kl}]$ above to be the 1×1 matrix e_α . Given $\epsilon > 0$, choose a vector $\vec{\zeta} \in \text{Ball}(H^{(n)})$ such that

$$\|[T_{ij}]\|_n \leq \|[T_{ij}]\vec{\zeta}\|_{H^{(n)}} + \epsilon.$$

Then

$$\|[T_{ij}]\|_n \leq \lim_{\alpha} \|[T_{ij}\pi(e_\alpha)]\vec{\zeta}\|_{H^{(n)}} + \epsilon \leq \sup_{\alpha} \|[T_{ij}\pi(e_\alpha)]\| + \epsilon.$$

However this last quantity is dominated by $\|[\rho(T_{ij})]\|_n$, by the third last displayed equation. Thus ρ is completely isometric.

To see that ρ is onto, suppose that $R \in B_A(A)$. We obtain a related map $T \in B(H)$ which may be defined by $T\pi(a)\zeta = \pi(Ta)\zeta$, for $a \in A, \zeta \in H$. Another way to see this quickly is by using the well known fact that in this case, $H \cong A \hat{\otimes}_A H$. We omit the simple details, which as we said at the beginning of this section, are essentially well known to experts. \square

REMARKS. (1) Let A be an operator algebra with left identity e of norm 1. Then one may show that $LM(A) = Ae$, which is a unital subalgebra of A . It is also a unital subalgebra of $\mathcal{E}(A)$, and $\mathcal{E}(A)$ is a unital C^* -algebra.

To see all this, note that in this case $\mathcal{L}(A) = Ae$, which is a unital algebra. Thus the first assertion of the remark follows from (5) of Theorem 6.1. We saw in Theorem 4.4 that $J = \mathfrak{J}_e(A)$ is a right ideal of a C^* -algebra, and that J has a left identity e . Thus $\mathcal{E}(A) = JJ^*$ has e as a 2-sided identity. Finally, $Ae \subset JJ^* = \mathcal{E}(A)$.

(2) Suppose that A is an operator algebra with l.c.a.i., and that $\pi : A \rightarrow B(H)$ is a completely isometric representation. Define $LM(\pi) = \{T \in B(H) : T\pi(A) \subset \pi(A)\}$, the left idealizer of $\pi(A)$ in $B(H)$. Then it is straightforward to exhibit a completely contractive homomorphism $\sigma : LM(\pi) \rightarrow LM(A) = CB_A(A)$. Conversely, given $T \in CB_A(A)$, taking a weak operator limit point S of $\pi(T(e_\alpha))$ gives $S \in LM(\pi)$. This is really saying that $LM(A) \cong LM(\pi)/\text{Ker } \sigma$ completely isometrically isomorphically. One may view this observation as an attempt to remove the use of property (\mathcal{L}) in (4).

It is interesting to note that if π is the usual representation of R_2 , then $LM(\pi)$ is a 3-dimensional operator algebra (this was pointed out to me by M. Kaneda). Note that $LM(\pi)$ is highly dependent on π , to see this consider R_2 again; the natural representation π has $LM(\pi)$ 3-dimensional. However, if $\sigma = \pi \oplus \epsilon$, where ϵ is the projection onto the 1-1 coordinate, then $LM(\sigma)$ is strictly larger. It would be interesting to see if there is a nonrestrictive condition under which one obtains ‘independence from the particular π used’.

One may think of each of the six equivalent algebras in Theorem 6.1 as a pair (D, μ_A) , where $\mu_A : A \rightarrow D$ is a completely contractive homomorphism. Let us

spell out what the map μ_A is in each case. In (1), it is the map $a \mapsto \hat{a} + \text{Ker } q$; in (2) and (3) it is the left regular representation λ ; in (4) it is π ; in (5) it is the natural left representation of A on its left ideal $\mathcal{L}(A)$; and in (6) the map μ_A is $a \mapsto \hat{a}E$, where E is as in the remark before Proposition 3.4. All these maps are completely contractive homomorphisms.

COROLLARY 6.2. *Each of the first three (and indeed all six, if A has property (\mathcal{L})) operator algebras in the previous theorem, together with its associated map μ_A discussed above, is a left multiplier operator algebra of A . That is, they are each completely isometrically A -isomorphic to $(CB_A(A), \lambda)$.*

We leave these assertions to the reader.

We now turn to the notion which in the C^* -algebra literature is referred to as ‘essential homomorphisms’ or sometimes ‘nondegenerate homomorphisms’. For our purpose we shall use the name ‘ A -nondegenerate morphism’. For us this shall mean a completely contractive homomorphism $\pi : A \rightarrow LM(B)$ satisfying the following equivalent conditions:

THEOREM 6.3. *Let A and B be two operator algebras with l.c.a.i.’s, and let $\pi : A \rightarrow LM(B)$ be a completely contractive homomorphism. The following are equivalent.*

- (i) *There exists a l.c.a.i. $\{e_\alpha\}$ for A such that $\pi(e_\alpha)b \rightarrow b$ for all $b \in B$.*
- (ii) *For every l.c.a.i. $\{e_\alpha\}$ for A , we have $\pi(e_\alpha)b \rightarrow b$.*
- (iii) *B is a nondegenerate left A -module via π ,*
- (iv) *Any $b \in B$ may be written $b = \pi(a)b'$ for some $a \in A, b' \in B$.*

If these conditions hold, then there exists a completely contractive unital homomorphism $\hat{\pi} : LM(A) \rightarrow LM(B)$ such that $\hat{\pi} \circ \mu_A = \mu_B$, and this homomorphism may be defined by $\hat{\pi}(x)(\pi(a)b) = \pi(xa)b$ for $x \in LM(A), a \in A, b \in B$. Finally, $\hat{\pi}$ is completely isometric if π is completely isometric.

PROOF. Clearly (i) implies that the span of terms $\pi(a)b$ is dense in B , which is what we mean by nondegenerate. So (i) implies (iii). Clearly (iii) implies (ii), and (ii) implies (i), and (iv) implies (iii). That (iii) implies (iv) follows from [23, Section 5.2].

If these conditions hold, view $LM(A)$ and $LM(B)$ as in Theorem 6.1 (3). We may follow the proof of Theorem 6.2 in [4]. The main difference is that we ignore the element e mentioned there, which we can get away with by taking d there to be the l.c.a.i. from A . One also needs to use [23, 5.2.2], and the matrix version of it, in order to show that $\hat{\pi} : LM(A) \rightarrow CB_B(B)$, and that $\hat{\pi}$ is a complete contraction.

It remains to prove the last assertion. Supposing that π is completely isometric, we

have for $T \in LM(A)$ that

$$\|\hat{\pi}(T)\|_{cb} \geq \|[\hat{\pi}(T)(\pi(a_{ij})b_{kl})]\| = \|\pi(Ta_{ij})b_{kl}\|$$

providing that $\|[a_{ij}]\|, \|[b_{kl}]\| \leq 1$. Taking the supremum over all such $[b_{kl}] \in M_m(B)$, gives that $\|\hat{\pi}(T)\|_{cb} \geq \|\pi(Ta_{ij})\| = \|[Ta_{ij}]\|$. Taking the supremum over all such $[a_{ij}] \in M_n(A)$ gives that $\|\hat{\pi}(T)\|_{cb} \geq \|T\|_{cb}$. So $\hat{\pi}$ is isometric, and similarly it is completely isometric. \square

REMARK. The canonical map $\mu_A : A \rightarrow LM(A)$ is an A -nondegenerate morphism.

COROLLARY 6.4. *Let A be a closed subalgebra of an operator algebra B , and suppose that A contains a l.c.a.i. for B . Then $LM(A) \hookrightarrow LM(B)$ completely isometrically as a subalgebra.*

REMARK. The one ‘drawback’ of our left multiplier algebra $LM(A)$ above is that it does not contain the algebra itself in general; but this is no surprise to anyone who has looked at the ‘multiplier’ or ‘centralizer’ theory of nonunital Banach algebras. Indeed if one insists that the left multiplier algebra of A be a pair (B, ν) consisting of an operator algebra B and a completely isometric homomorphism $\nu : A \rightarrow B$ with $B\nu(A) \subset \nu(A)$, then unfortunately one must lose the useful ‘essential’ condition (namely that $x\nu(A) = 0$ implies $x = 0$). This departs from the classical ‘multiplier’/‘centralizer’ framework from Banach algebra theory ([23, Section 1.2], for example), where a multiplier which annihilates A must be the zero multiplier. Also it seems that one cannot hope for conditions like (1)–(3) of Theorem 6.1.

Note added in proof. Some of our motivation for the present work was to solve some questions which arose in our work on one-sided M -ideals [7, 33]. In the latter paper we give some additional results on one-sided ideals in operator algebras.

Acknowledgements

We thank M. Kaneda for catching several misprints, and for several conversations related to the present paper and [8]. We also thank the referee for his direction.

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