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INVOLUTIONS ON FINITE-DIMENSIONAL ALGEBRAS OVER REAL CLOSED FIELDS

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Abstract

It is shown that the following conditions on a finite-dimensional algebra A over a real closed field or an algebraically closed field of characteristic zero are equivalent: (i) A admits a special involution, in the sense of Easdown and Munn, (ii) A admits a proper involution, (iii) A is semisimple.

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1. Introduction

A field is termed *formally real* if -1 cannot be expressed in it as a sum of squares and *real closed* if it is a formally real field that has no formally real proper algebraic extension. Many real closed fields exist; examples include the real field $\mathbb{R}$ and the field of all real algebraic numbers.

It is clear that a real closed field F has characteristic zero. Less obvious is the fact that F is totally ordered by the rule that $a \leq b$ if and only if $b - a = c^2$ for some c [5, Section 70, Theorem 1]. We shall make use of this total ordering without further comment.

Recall that an involution on a ring R is a mapping $^* : R \to R$ such that

$$(\forall\, a, b \in R) \quad (a+b)^* = a^* + b^*, \quad (ab)^* = b^* a^*, \quad a^{**} = a.$$

Now let A be an algebra over a real closed field F. An *involution* on A is defined to be an involution * on the ring $(A, +, \cdot)$ with the additional property that

$$(\forall\, a \in A)(\forall\, \lambda \in F) \quad (\lambda a)^* = \lambda a^*.$$

Two types of involution * on A concern us here. We say that * is

(i) *proper* if $aa^* = 0$ implies $a = 0$ for all $a \in A$,

(ii) *special* if, for every nonempty finite subset T of A,

$$(\exists\, t \in T)(\forall\, u, v \in T) \quad tt^* = uv^* \;\Rightarrow\; u = v.$$

Note that, in each case, the defining condition is on the multiplicative semigroup of A. The notion of a special involution was introduced in [2]. It is perhaps surprising that many naturally occurring involutions are special: for instance, transposition on the algebra of all real $n \times n$ matrices and conjugation on the algebra of all real quaternions [2], the mapping

$$\sum_{x \in S} \alpha_x x \mapsto \sum_{x \in S} \alpha_x x^{-1}$$

on the semigroup algebra $\mathbb{R}[S]$ of an inverse semigroup S over $\mathbb{R}$ (in particular, on the group algebra $\mathbb{R}[S]$, where S is a group) [3], and the mapping

$$\sum_{w \in M} \alpha_w w \mapsto \sum_{w \in M} \alpha_w \overleftarrow{w}$$

on $\mathbb{R}[M]$, where M is a free monoid of arbitrary rank and $\overleftarrow{w}$ denotes the reverse of the word w in M [1].

The two properties are not independent: in fact, every special involution is proper, as we now show. Let * be a special involution on an algebra A over a real closed field and let $a \in A$ be such that $aa^* = 0$. Take $T := \{a, 0\}$. Then there exists $t \in T$ such that, for all $u, v \in T$, $tt^* = uv^*$ implies $u = v$. For each possibility, $tt^* = 0 = 0a^*$ and so $a = 0$. Thus * is proper. However, a simple example demonstrates that not every proper involution is special. Let A denote the group algebra $\mathbb{R}[G]$, where G is the cyclic group of order 4. Since A is commutative, the identity mapping on A is an involution; moreover, since A is semisimple then, for $a \in A$, $a^2 = 0$ implies $a = 0$. Thus the identity mapping is proper. However, if g denotes a generator of G then, by taking $T := G$ and noting the equations $(g^0)^2 = gg^3 = (g^2)^2$ and $g^2 = g^0 g^2 = (g^3)^2$, we see that the identity mapping is not special.

The aim of the present paper is to show that, on a finite-dimensional algebra A over a real closed field, the following conditions are equivalent: (i) A admits a special involution, (ii) A admits a proper involution, (iii) A is semisimple. (In the previous paragraph, we have an example of an involution on a real finite-dimensional semisimple algebra that is proper but not special; however, a different involution—namely that induced by inversion in the group—is special.) With a natural adjustment to the definition of an involution, a similar result follows for a finite-dimensional algebra over an algebraically closed field of characteristic zero. These results, for the real and complex fields, were announced in [4]. In each case, the author's proof used a result in representation theory. The direct proofs given here are extensions of the argument in [2, Example 4].

2. Finite-dimensional algebras over real closed fields

We begin by considering division algebras. A classical theorem of Frobenius states that, to within isomorphism, the only finite-dimensional division algebras over $\mathbb{R}$ are $\mathbb{R}$ itself, the complex field $\mathbb{C}$ and the algebra $\mathbb{H}$ of real quaternions [5, Section 131, pages 201–202]. The proof of this theorem applies also to the case where $\mathbb{R}$ is replaced by any real closed field F, for it depends only on two particular properties [5, Section 70, Theorems 1 and 3]:

(i) F is totally ordered and contains a square root of each non-negative element,

(ii) the field obtained from F by adjoining a root of the irreducible polynomial $x^2 + 1$ is algebraically closed.

All the details of the lemma below now follow routinely.

LEMMA 2.1. *Every finite-dimensional division algebra D over a real closed field F admits an involution ${}^c : D \to D, d \mapsto d^c$ ('conjugation') such that*

$$(\forall\, d \in D) \quad dd^c \in F, \quad dd^c \geq 0, \quad dd^c = 0 \;\Rightarrow\; d = 0,$$

according to the cases

(i) $D = F : \alpha^c = \alpha \ (\alpha \in F)$;

(ii) $D = F[i]$, *where* $i^2 = -1 : (\alpha + \beta i)^c = \alpha - \beta i \ (\alpha, \beta \in F)$;

(iii) $D = F[i, j, k]$, *where* $i^2 = j^2 = k^2 = -1$, $ij = k = -ji$, $jk = i = -kj$; $ki = j = -ik$:

$$(\alpha + \beta i + \gamma j + \delta k)^c = \alpha - \beta i - \gamma j - \delta k \quad (\alpha, \beta, \gamma, \delta \in F).$$

Observe that the lemma states, in particular, that every finite-dimensional division algebra over a real closed field admits a proper involution.

THEOREM 2.2. *The following conditions on a finite-dimensional algebra A over a real closed field are equivalent*:

(i) *A admits a special involution,*

(ii) *A admits a proper involution,*

(iii) *A is semisimple.*

PROOF. Since, as remarked in Section 1, every special involution is a proper involution, (i) implies (ii).

A standard short argument, which we include for completeness, shows that (ii) implies (iii). Let * be a proper involution on A and let N denote the radical of A. Suppose that $a \in N \backslash 0$. Then $aa^* \in N$ and so $(aa^*)^m = 0$ for some least positive integer m. Since * is proper, $m \geq 2$. Write $b := (aa^*)^{m-1}$. Then $b = b^*$ and so $bb^* = b^2 = (aa^*)^{2m-2} = 0$, since $2m-2 \geq m$. Thus $b = 0$, contrary to the minimality of m. Hence $N = 0$ and so A is semisimple.

We complete the proof by showing that (iii) implies (i). Denote the ground field of A by F. Consider first the algebra $M_n(D)$ of all $n \times n$ matrices over a finite-dimensional division algebra D over F. Define $^\dagger : M_n(D) \to M_n(D)$, $a \mapsto a^\dagger$, by writing $a^\dagger := (a^c)^T$, where a^c denotes the matrix obtained from a by replacing each entry a_{ij} by ${a_{ij}}^c$ (with ^{c} as in Lemma 2.1) and ^{T} denotes transposition. Since ^{c} is an involution on D, it follows easily that † is an involution on $M_n(D)$. Now denote the trace of $a \in M_n(D)$ by $\tau(a)$. Then, for all $a = [a_{ij}] \in M_n(D)$, we have that $\tau(aa^\dagger) = \sum_{i,j=1}^n a_{ij}{a_{ij}}^c$ and so, by Lemma 2.1,

$$(1) \qquad (\forall\, a \in M_n(D)) \quad \tau(aa^\dagger) \in F, \quad \tau(aa^\dagger) \geq 0, \quad \tau(aa^\dagger) = 0 \Rightarrow a = 0.$$

Let A be semisimple. By Wedderburn's theorem, we may assume, without loss of generality, that A is the external direct sum of algebras A_i $(i = 1, \ldots, k)$, where, for each i, $A_i = M_{n_i}(D_i)$ for some positive integer n_i and some finite-dimensional division algebra D_i over F. No confusion should arise from the use of the same symbol † to denote the involution defined on each $M_{n_i}(D_i)$ as in the previous paragraph. For all $a \in A$, denote the A_i-component of a by a_i $(i = 1, \ldots, k)$. Then $^* : A \to A$, $a \mapsto a^*$ defined by $(a^*)_i = {a_i}^\dagger$ $(i = 1, \ldots, k)$ is readily seen to be an involution on A. We show that it is special.

Let T be a nonempty finite subset of A. Choose $t \in T$ such that

$$(2) \qquad \sum_{i=1}^k \tau(t_i{t_i}^\dagger) = \max\left\{\sum_{i=1}^k \tau(w_i{w_i}^\dagger) : w \in T\right\}.$$

Suppose that $tt^* = uv^*$, for some $u, v \in T$. Then, for each i,

$$t_i{t_i}^\dagger = u_i{v_i}^\dagger = (u_i{v_i}^\dagger)^\dagger = v_i{u_i}^\dagger$$

and so

$$(u_i - v_i)(u_i - v_i)^\dagger = u_i{u_i}^\dagger + v_i{v_i}^\dagger - 2t_i{t_i}^\dagger.$$

Thus, by (1) and (2),

$$\begin{aligned} 0 &\leq \sum_{i=1}^k \tau((u_i - v_i)(u_i - v_i)^\dagger) \\ &= \sum_{i=1}^k \tau(u_i{u_i}^\dagger) + \sum_{i=1}^k \tau(v_i{v_i}^\dagger) - 2\sum_{i=1}^k \tau(t_i{t_i}^\dagger) \leq 0. \end{aligned}$$

Hence $\sum_{i=1}^{k} \tau((u_i - v_i)(u_i - v_i)^\dagger) = 0$. Thus, by (1), $u_i = v_i$ for each i. Consequently, $u = v$. This shows that * is special. □

An analogous result (Corollary 2.3) holds for algebras over algebraically closed fields of characteristic zero. As noted earlier, if we adjoin to a real closed field a root of $x^2 + 1$ then the resulting field is algebraically closed. In fact, all algebraically closed fields arise in this way. A statement, with a proof for the countable case, is given in [5, Section 71]; the general case is a simple application of Zorn's lemma. Let A be an algebra over an algebraically closed field F of characteristic zero. Then $F = R[i]$, where R is a maximal formally real subfield and $i^2 = -1$. Define ^{c} ('conjugation') on F by taking $(\xi + \eta i)^c = \xi - \eta i$ $(\xi, \eta \in R)$. By an *involution* * on A we now mean a ring involution such that

$$(3) \qquad (\forall\, a \in A)(\forall\, \lambda \in F) \quad (\lambda a)^* = \lambda^c a^*.$$

Proper and *special* involutions are defined as before. Suppose that A is semisimple. Since F is algebraically closed, the only finite-dimensional division algebra over F is F itself and so A is isomorphic to a direct sum of full matrix algebras over F. Now regard A as an algebra over R. Then $^* : A \to A$, constructed from ^{c} as in the proof that (iii) implies (i) in Theorem 2.2, is a special involution; moreover, it satisfies (3). Hence * is a special involution on A as an algebra over F. Thus we obtain the result below:

COROLLARY 2.3. *The following conditions on a finite-dimensional algebra A over an algebraically closed field of characteristic zero are equivalent: A admits a special involution, A admits a proper involution, A is semisimple.*

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References

[1] M. J. Crabb, C. M. McGregor, W. D. Munn and S. Wassermann, 'On the algebra of a free monoid', *Proc. Roy. Soc. Edinburgh Sect. A* **126** (1996), 939–945.

[2] D. Easdown and W. D. Munn, 'On semigroups with involution', *Bull. Austral. Math. Soc.* **48** (1993), 103–110.

[3] ———, 'Trace functions on inverse semigroup algebras', *Bull. Austral. Math. Soc.* **52** (1995), 359–372.

[4] W. D. Munn, ‘Special involutions’, in: *Semigroup theory and its applications (New Orleans, LA, 1994)* (eds. K. H. Hofmann and M. W. Mislove), London Math. Soc. Lecture Note Ser. 231 (Cambridge University Press, Cambridge, 1996) pp. 157–165.

[5] B. L. Van der Waerden, *Modern algebra*, English edition (Ungar, New York, 1949 (Vol. I) and 1950 (Vol. II)).

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