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IDEALS OF COMPACT OPERATORS

ÅSVALD LIMA and EVE OJA

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Abstract

We give an example of a Banach space X such that $\mathscr{K}(X, X)$ is not an ideal in $\mathscr{K}(X, X^{**})$. We prove that if z^* is a weak* denting point in the unit ball of Z^* and if X is a closed subspace of a Banach space Y, then the set of norm-preserving extensions $HB(x^* \otimes z^*) \subseteq \mathscr{L}(Z^*, Y)^*$ of a functional $x^* \otimes z^* \in (Z \otimes X)^*$ is equal to the set $HB(x^*) \otimes \{z^*\}$. Using this result, we show that if X is an M-ideal in Y and Z is a reflexive Banach space, then $\mathscr{K}(Z, X)$ is an M-ideal in $\mathscr{K}(Z, Y)$ whenever $\mathscr{K}(Z, X)$ is an ideal in $\mathscr{K}(Z, Y)$. We also show that $\mathscr{K}(Z, X)$ is an ideal (respectively, an M-ideal) in $\mathscr{K}(Z, Y)$ for all Banach spaces Z whenever X is an ideal (respectively, an M-ideal) in Y and X^* has the compact approximation property with conjugate operators.

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1. Introduction

Let us recall that a closed subspace F of a Banach space E is an *ideal* in E if $F^\perp$, the annihilator of F in E^*, is the kernel of a norm one projection on E^*. The notion of an ideal was introduced and studied by Godefroy, Kalton, and Saphar in [8].

Let X and Y be Banach spaces. We denote by $\mathscr{L}(X, Y)$ the Banach space of bounded linear operators from X to Y, and by $\mathscr{F}(X, Y)$, $\overline{\mathscr{F}(X, Y)}$, $\mathscr{K}(X, Y)$, and $\mathscr{W}(X, Y)$ its subspaces of finite rank operators, approximable operators (that is, norm limits of finite rank operators), compact operators, and weakly compact operators.

In [15, Theorem 3] the following result was proved.

THEOREM 1.1. *$\overline{\mathscr{F}(X, X)}$ is an ideal in $\overline{\mathscr{F}(X, X^{**})}$ for every Banach space X.*

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In [15, page 455] it is stated as an open problem whether $\mathscr{K}(X, X)$ is an ideal in $\mathscr{K}(X, X^{**})$ for every Banach space X.

EXAMPLE 1.2. There exists a separable Banach space X such that

- $\mathscr{K}(X, X^{**})$ is an ideal in $\mathscr{L}(X, X^{**})$,
- $\mathscr{K}(X, X)$ is not an ideal in $\mathscr{L}(X, X)$,
- $\mathscr{K}(X, X)$ is not an ideal in $\mathscr{K}(X, X^{**})$.

PROOF. Let $X = (\sum_{n=1}^{\infty} \oplus (Z^{**}, |\cdot|_n))_2$ be the space defined (relying on the famous example due to Willis [32]) and studied by Casazza and Jarchow in [3, Theorem 1]. Recall that Z^{**} is a separable Banach space, $|\cdot|_n$ is an equivalent norm on Z^{**}, the space X fails the metric compact approximation property, but its dual space X^* has the metric compact approximation property. Since Z^{**} is separable, the spaces $(Z^{**}, |\cdot|_n)$ have the Radon-Nikodým property. Thus X has the Radon-Nikodým property. (The fact that the Radon-Nikodým property is preserved under ℓ_p-direct sums $(1 \le p < \infty)$ is mentioned in [6, page 219]. It can be proved following the idea of the proof in [6, pages 64–65] that a Banach space with a boundedly complete basis has the Radon-Nikodým property.)

Since X^* has the metric compact approximation property, by a well-known result due to Johnson [12], $\mathscr{K}(X^*, X^*)$ is an ideal in $\mathscr{L}(X^*, X^*)$. Thus $\mathscr{K}(X, X^{**})$ is an ideal in $\mathscr{L}(X, X^{**})$. Since X has the Radon-Nikodým property but fails to have the metric compact approximation property, by [15, Theorem 14], $\mathscr{K}(X, X)$ is not an ideal in $\mathscr{L}(X, X)$. Hence $\mathscr{K}(X, X)$ is not an ideal in $\mathscr{L}(X, X^{**})$. (General properties of ideals that we are using here and below in this paper are immediate from the local formulation of ideals (see Lemma 2.1, (ii), in Section 2).) If $\mathscr{K}(X, X)$ were an ideal in $\mathscr{K}(X, X^{**})$, then $\mathscr{K}(X, X)$ would be an ideal in $\mathscr{L}(X, X^{**})$ (because $\mathscr{K}(X, X^{**})$ is an ideal in $\mathscr{L}(X, X^{**})$) which is impossible. □

In [15, page 471] it is stated as an open problem if $\mathscr{K}(X, X)$ is an ideal in $\mathscr{L}(X, X)$ whenever $\mathscr{K}(X^*, X^*)$ is an ideal in $\mathscr{L}(X^*, X^*)$. Example 1.2 also solves this problem in the negative.

Every Banach space X is an ideal in its bidual X^{**} (with respect to the canonical projection of X^{***} onto X^*). Let us further assume that X is a closed subspace of a Banach space Y. In Section 2 we make a preliminary study about $\mathscr{K}(Z, X)$ being an ideal in $\mathscr{K}(Z, Y)$ for a Banach space Z. In Proposition 2.4 we prove that if $\mathscr{K}(Z, X)$ is an ideal in $\mathscr{K}(Z, Y)$ for some $Z \neq \{0\}$, then X is an ideal in Y. And in Proposition 2.5 we show that if X is an ideal in Y and $\mathscr{K}(Z, X)$ is an ideal in $\mathscr{K}(Z, X^{**})$, then $\mathscr{K}(Z, X)$ is an ideal in $\mathscr{K}(Z, Y)$. We also prove that the last property is separably determined (see Theorems 2.6, 2.7, and 2.8). In particular (see

Theorem 2.8), $\mathscr{K}(X, X)$ is an ideal in $\mathscr{K}(X, X^{**})$ whenever $\mathscr{K}(E, E)$ is an ideal in $\mathscr{K}(E, E^{**})$ for every separable ideal E in X.

The main lemma in this paper is Lemma 3.1. This is a general result describing all norm-preserving extensions of certain important functionals on operator spaces. We use Lemma 3.1, in particular, in Theorem 3.4 to prove that if Z is a reflexive Banach space, if X is an M-ideal in Y, and if $\mathscr{K}(Z, X)$ is an ideal in $\mathscr{K}(Z, Y)$, then $\mathscr{K}(Z, X)$ is an M-ideal in $\mathscr{K}(Z, Y)$. Similar results hold for u-ideals (see Theorems 3.5 and 3.6).

In Example 1.2 the dual space X^* fails to have the compact approximation property with conjugate operators (although X^* has the metric compact approximation property). The main conclusions in this paper, Corollaries 4.7 and 4.8 in Section 4, show that $\mathscr{K}(Z, X)$ is an ideal (respectively, an M-ideal) in $\mathscr{K}(Z, Y)$ for all Banach spaces Z whenever X is an ideal (respectively, an M-ideal) in Y and X^* has the compact approximation property with conjugate operators. The proofs rely on results from Section 3, the description of the dual space of compact operators due to Feder and Saphar [7], and the uniform isometric version of the Davis-Figiel-Johnson-Pełczyński factorization theorem due to Lima, Nygaard, and Oja [17].

The notation we use is standard (see [21]). We consider Banach spaces over the real field $\mathbb{R}$. The closure of a set A is denoted by $\overline{A}$, its linear span by span A, and its convex hull by conv A. The closed unit ball of a Banach space E is denoted by B_E and the identity operator on E by I_E.

2. Ideals of compact operators

Let X be a closed subspace of a Banach space Y. In this section we make a preliminary study about $\mathscr{K}(Z, X)$ being an ideal in $\mathscr{K}(Z, Y)$ for a Banach space Z. A first basic result, Proposition 2.4, says that if $\mathscr{K}(Z, X)$ is an ideal in $\mathscr{K}(Z, Y)$ for some Banach space $Z \neq \{0\}$, then X is an ideal in Y. In Proposition 2.5 we show that the converse is true whenever $\mathscr{K}(Z, X)$ is an ideal in $\mathscr{K}(Z, X^{**})$. And we prove that the last property is separably determined (see Theorems 2.6–2.8).

In Proposition 2.10 we show that if $\mathscr{K}(Z, X)$ is an M-ideal (respectively, a u-ideal) in $\mathscr{K}(Z, Y)$ for some Banach space $Z \neq \{0\}$, then X is an M-ideal (respectively, a u-ideal) in Y.

Let F be a subspace of a Banach space E. A linear operator $\Phi : F^* \to E^*$ is called a *Hahn-Banach extension operator* if $(\Phi x^*)(x) = x^*(x)$ and $\|\Phi x^*\| = \|x^*\|$ for all $x \in F$ and all $x^* \in F^*$. The next result is well known. A proof can be found in [15].

LEMMA 2.1. *Let F be a closed subspace of a Banach space E. The following statements are equivalent.*

(i) *F is an ideal in E.*

(ii) *F is locally 1-complemented in E, that is, for every finite dimensional subspace G of E and for all $\varepsilon > 0$, there is an operator $U : G \to F$ such that $\|U\| \leq 1 + \varepsilon$ and $Ux = x$ for all $x \in G \cap F$.*

(iii) *There exists a Hahn-Banach extension operator $\Phi : F^* \to E^*$.*

In [31] (see also [11, page 282]) Werner has shown that if X is an M-ideal in Y, then the injective tensor product $X\hat{\otimes}_\epsilon Z$ is an M-ideal in $Y\hat{\otimes}_\epsilon Z$ for any Banach space Z. Rao extended this to ideals in [28]. We have a short proof of this result.

PROPOSITION 2.2. *Let X be an ideal in Y and let Z be an ideal in W. Then $X\hat{\otimes}_\epsilon Z$ is an ideal in $Y\hat{\otimes}_\epsilon W$.*

PROOF. Let $\phi : X^* \to Y^*$ and $\psi : Z^* \to W^*$ be Hahn-Banach extension operators. Let $Q : Z^{***} \to Z^*$ be the canonical projection. We shall use the identifications $(X\hat{\otimes}_\epsilon Z)^* = I(X, Z^*)$ and $(Y\hat{\otimes}_\epsilon W)^* = I(Y, W^*)$ (see, for example, [6]). Since $\phi^* x = x$, $x \in X$, and $\psi^* z = z$, $z \in Z$, the map $\Phi : I(X, Z^*) \to I(Y, W^*)$ defined by $\Phi(T) = \psi \circ Q \circ T^{**} \circ \phi^*|_Y$ is clearly a Hahn-Banach extension operator. □

Theorem 1.1 has an easy generalization.

COROLLARY 2.3. *Let X be a closed subspace of a Banach space Y. The following statements are equivalent.*

(i) *X is an ideal in Y.*

(ii) *$\overline{\mathcal{F}(Z, X)}$ is an ideal in $\overline{\mathcal{F}(Z, Y)}$ for all Banach spaces Z.*

(iii) *$\overline{\mathcal{F}(Z, X)}$ is an ideal in $\overline{\mathcal{F}(Z, Y)}$ for some Banach space $Z \neq \{0\}$.*

In particular, $\overline{\mathcal{F}(Z, X)}$ is an ideal in $\overline{\mathcal{F}(Z, X^{**})}$ for all Banach spaces X and Z.

PROOF. (i) $\Rightarrow$ (ii) is immediate from Proposition 2.2 because $\overline{\mathcal{F}(Z, X)}$ and $\overline{\mathcal{F}(Z, Y)}$ can be canonically identified with $Z^*\hat{\otimes}_\epsilon X$ and $Z^*\hat{\otimes}_\epsilon Y$.
(ii) $\Rightarrow$ (iii) is trivial.
(iii) $\Rightarrow$ (i). Suppose $\overline{\mathcal{F}(Z, X)}$ is an ideal in $\overline{\mathcal{F}(Z, Y)}$. Let F be a finite dimensional subspace of Y. Let $z \in Z$ and $z^* \in Z^*$ be such that $\|z\| = \|z^*\| = z^*(z) = 1$. Denote $G = \{z^* \otimes y : y \in F\} \subseteq \mathcal{F}(Z, Y)$. Let $\epsilon > 0$ and let $V : G \to \overline{\mathcal{F}(Z, X)}$ be an operator such that $\|V\| \leq 1 + \epsilon$ and $V(S) = S$ for all $S \in G \cap \overline{\mathcal{F}(Z, X)}$. Now define a map $U : F \to X$ by $Uy = (V(z^* \otimes y))z$. Then U 'locally 1-complements' X in Y. □

Example 1.2 shows that the implication (i) $\Rightarrow$ (ii) of Corollary 2.3 fails if we consider compact operators instead of approximable operators. However, by the proof of the implication (iii) $\Rightarrow$ (i), we also have the similar result in the case of compact operators.

PROPOSITION 2.4. *Let X be a closed subspace of a Banach space Y. If $\mathscr{K}(Z,X)$ is an ideal in $\mathscr{K}(Z,Y)$ for some Banach space $Z \neq \{0\}$, then X is an ideal in Y.*

PROPOSITION 2.5. *Let X be a closed subspace of Y and assume that $\mathscr{K}(Z,X)$ is an ideal in $\mathscr{K}(Z,X^{**})$ for some Banach space $Z \neq \{0\}$. Then X is an ideal in Y if and only if $\mathscr{K}(Z,X)$ is an ideal in $\mathscr{K}(Z,Y)$.*

PROOF. In view of Proposition 2.4 we only need to prove the 'only if' part. Let $\phi : X^* \to Y^*$ and $\Phi : \mathscr{K}(Z,X)^* \to \mathscr{K}(Z,X^{**})^*$ be Hahn-Banach extension operators. Define $\Psi : \mathscr{K}(Z,X)^* \to \mathscr{K}(Z,Y)^*$ by

$$(\Psi f)(T) = (\Phi f)(\phi^*|_Y \circ T), \quad f \in \mathscr{K}(Z,X)^*, \quad T \in \mathscr{K}(Z,Y).$$

Then Ψ is linear and $\|\Psi\| \leq 1$. Since $\phi^* x = x$, $x \in X$, we have $\phi^*|_Y \circ T = T$ whenever $T \in \mathscr{K}(Z,X)$. Consequently, for any $T \in \mathscr{K}(Z,X)$ and any $f \in \mathscr{K}(Z,X)^*$, $(\Psi f)(T) = (\Phi f)(T) = f(T)$ meaning that Ψf is an extension of f. Hence Ψ is a Hahn-Banach extension operator. □

From the last proposition it follows that it is important to decide when $\mathscr{K}(Z,X)$ is an ideal in $\mathscr{K}(Z,X^{**})$. Example 1.2 in the Introduction shows that this is not true for all X and all Z. If X is the range of a norm one projection in X^{**}, then $\mathscr{K}(Z,X)$ is an ideal in $\mathscr{K}(Z,X^{**})$ for all Z. (If P is the projection, then $T \to P \circ T$ is a norm one projection from $\mathscr{K}(Z,X^{**})$ onto $\mathscr{K}(Z,X)$.)

We shall prove in Theorem 4.2 that $\mathscr{K}(Z,X)$ is an ideal in $\mathscr{K}(Z,X^{**})$ for all Banach spaces Z if this is true for all separable reflexive Banach spaces Z.

The next three results show that the question about being an ideal, can be reduced to the case of separable Banach spaces.

THEOREM 2.6. *Let X and Z be Banach spaces. If $\mathscr{K}(Z,E)$ is an ideal in $\mathscr{K}(Z,E^{**})$ for every separable ideal E in X, then $\mathscr{K}(Z,X)$ is an ideal in $\mathscr{K}(Z,X^{**})$.*

PROOF. Let $F \subseteq \mathscr{K}(Z,X^{**})$ be a finite dimensional subspace and let $\epsilon > 0$. Since $G = \{Tz : z \in Z,\ T \in F \cap \mathscr{K}(Z,X)\}$ is contained in a separable subspace of X, by a result of Sims and Yost [29] (see also [11, page 138]), there exists a separable ideal $E \subseteq X$ such that $G \subseteq E$. Let $\phi : E^* \to X^*$ be a Hahn-Banach extension operator. Then $\phi^* : X^{**} \to E^{**}$ and $F_\phi = \{\phi^* \circ T : T \in F\}$ is a finite dimensional subspace of $\mathscr{K}(Z,E^{**})$. Let $U : F_\phi \to \mathscr{K}(Z,E)$ be a linear operator such that $\|U\| \leq 1+\epsilon$ and $U(\phi^* \circ T) = \phi^* \circ T$ for all $T \in F \cap \mathscr{K}(Z,X)$ (note that if $T \in F \cap \mathscr{K}(Z,X)$, then $T(Z) \subseteq E$ and $\phi^* \circ T \in \mathscr{K}(Z,E)$). Now $V : F \to \mathscr{K}(Z,X)$ defined by $V(T) = U(\phi^* \circ T)$ 'locally 1-complements' $\mathscr{K}(Z,X)$ in $\mathscr{K}(Z,X^{**})$. □

REMARK 2.1. The assertion of Theorem 2.6 is not reversible: $\mathcal{K}(X, X^{**})$ is always an ideal in $\mathcal{K}(X, X^{****})$ (because X^{**} is the range of a norm one projection in X^{****}) but, for the separable Banach space X (a separable ideal in X^{**}) described in Example 1.2, $\mathcal{K}(X, X)$ is not an ideal in $\mathcal{K}(X, X^{**})$.

THEOREM 2.7. *Let X, Y, and Z be Banach spaces and assume X is a subspace of Y. Then $\mathcal{K}(Z, X)$ is an ideal in $\mathcal{K}(Z, Y)$ if and only if $\mathcal{K}(W, X)$ is an ideal in $\mathcal{K}(W, Y)$ for every separable ideal W in Z.*

PROOF. Assume that $\mathcal{K}(Z, X)$ is an ideal in $\mathcal{K}(Z, Y)$ and let W be an ideal in Z. Let $\Phi : \mathcal{K}(Z, X)^* \to \mathcal{K}(Z, Y)^*$ and $\phi : W^* \to Z^*$ be Hahn-Banach extension operators. For $f \in \mathcal{K}(W, X)^*$ and $S \in \mathcal{K}(Z, X)$, define $\hat{f}(S) = f(S|_W)$. Finally, define an operator $\Psi : \mathcal{K}(W, X)^* \to \mathcal{K}(W, Y)^*$ by

$$(\Psi f)(T) = (\Phi \hat{f})(T^{**} \circ \phi^*|_Z), \quad f \in \mathcal{K}(W, X)^*, \quad T \in \mathcal{K}(W, Y).$$

Then Ψ is linear and $\|\Psi\| \leq 1$. For $T \in \mathcal{K}(W, X)$, we have $T^{**} \circ \phi^*|_Z \in \mathcal{K}(Z, X)$ and therefore

$$\begin{aligned}(\Psi f)(T) = (\Phi \hat{f})(T^{**} \circ \phi^*|_Z) &= \hat{f}(T^{**} \circ \phi^*|_Z) \\ &= f(T^{**} \circ \phi^*|_W) = f(T^{**}|_W) = f(T).\end{aligned}$$

Hence Ψ is a Hahn-Banach extension operator.

Conversely, assume that $\mathcal{K}(W, X)$ is an ideal in $\mathcal{K}(W, Y)$ for every separable ideal W in Z. Let $F \subseteq \mathcal{K}(Z, Y)$ be a finite dimensional subspace and let $\epsilon > 0$. The set $\{T^*y^* : T \in F, y^* \in Y^*\}$ is separable. By a theorem due to Sims and Yost [29] (see also [11, page 138]), we can find a separable ideal W in Z with a Hahn-Banach extension operator $\phi : W^* \to Z^*$ such that $\{T^*y^* : T \in F, y^* \in Y^*\} \subseteq \phi(W^*)$. Let $i : W \to Z$ be the natural embedding. Then $i^* : Z^* \to W^*$ is the restriction operator and we get $I_{\phi(W^*)} = (\phi \circ i^*)|_{\phi(W^*)}$.

Let $F_W = \{T \circ i : T \in F\} \subseteq \mathcal{K}(W, Y)$. We can find an operator $V : F_W \to \mathcal{K}(W, X)$ with $\|V\| \leq 1 + \epsilon$ such that $V(S) = S$ for every $S \in F_W \cap \mathcal{K}(W, X)$. Define an operator $U : F \to \mathcal{K}(Z, X)$ by $U(T) = (V(T \circ i))^{**} \circ \phi^*|_Z$. Then $\|U\| \leq 1 + \epsilon$. For any $z \in Z$, $y^* \in Y^*$, and $T \in F \cap \mathcal{K}(Z, X)$ we get (since $T \circ i \in F_W \cap \mathcal{K}(W, X)$ and $T^*y^* \in \phi(W^*)$)

$$\begin{aligned}y^*(U(T)z) &= y^*((V(T \circ i))^{**}(\phi^* z)) \\ &= y^*((T^{**} \circ i^{**})(\phi^* z)) \\ &= y^*((T^{**} \circ i^{**} \circ \phi^*)z) \\ &= ((\phi \circ i^* \circ T^*)(y^*))(z) \\ &= (T^*y^*)(z) = y^*(Tz).\end{aligned}$$

Thus $U(T) = T$. Hence $\mathcal{K}(Z, X)$ is an ideal in $\mathcal{K}(Z, Y)$. □

It is clear from Theorems 2.6 and 2.7 that $\mathscr{K}(X,X)$ is an ideal in $\mathscr{K}(X,X^{**})$ whenever $\mathscr{K}(E,F)$ is an ideal in $\mathscr{K}(E,F^{**})$ for all separable ideals E and F in X. However, further developing the method of proofs of those theorems we obtain the following stronger result.

THEOREM 2.8. *Let X be a Banach space. If $\mathscr{K}(E,E)$ is an ideal in $\mathscr{K}(E,E^{**})$ for every separable ideal E in X, then $\mathscr{K}(X,X)$ is an ideal in $\mathscr{K}(X,X^{**})$.*

PROOF. Assume that $\mathscr{K}(E,E)$ is an ideal in $\mathscr{K}(E,E^{**})$ for every separable ideal E in X. Let $F\subseteq\mathscr{K}(X,X^{**})$ be a finite dimensional subspace and let $\epsilon>0$. Since the sets $\{Tx : x\in X,\ T\in F\cap\mathscr{K}(X,X)\}\subseteq X$ and $\{T^*x^* : x^*\in X^*,\ T\in F\}\subseteq X^*$ are separable, we can find a separable ideal E in X together with a Hahn-Banach extension operator $\phi : E^*\to X^*$ so that $\{Tx : x\in X,\ T\in F\cap\mathscr{K}(X,X)\}\subseteq E$ and $\{T^*x^* : x^*\in X^*,\ T\in F\}\subseteq\phi(E^*)$ (again we used the Sims-Yost theorem). Let $i : E\to X$ be the natural embedding.

Denote $G=\{\phi^*\circ T\circ i : T\in F\}\subseteq\mathscr{K}(E,E^{**})$. Then there exists an operator $V : G\to\mathscr{K}(E,E)$ such that $\|V\|\le 1+\epsilon$ and $V(S)=S$ for all $S\in G\cap\mathscr{K}(E,E)$. Define $U : F\to\mathscr{K}(X,X)$ by $U(T)=(V(\phi^*\circ T\circ i))^{**}\circ\phi^*|_X$, $T\in F$. Then $\|U\|\le 1+\epsilon$. We conclude by showing that $U(T)=T$ for all $T\in F\cap\mathscr{K}(X,X)$.

Let $T\in F\cap\mathscr{K}(X,X)$. Since $Tx\in E, x\in X$, we have in fact that $T\in\mathscr{K}(X,E)$. We also have that $U(T)\in\mathscr{K}(X,E)$ (this is true for all $T\in F$). Since $T\in\mathscr{K}(X,E)$ and $\phi^*e=e, e\in E$, we have $\phi^*\circ T\circ i\in\mathscr{K}(E,E)$. Hence $V(\phi^*\circ T\circ i)=\phi^*\circ T\circ i$. For any $x\in X$ and $e^*\in E^*$, we get

$$\begin{aligned} e^*(U(T)x) &= e^*((\phi^*\circ T\circ i)^{**}(\phi^*x)) = (\phi^*x)((\phi^*\circ T\circ i)^*e^*)\\ &= (\phi^*x)((i^*\circ T^*\circ\phi)e^*) = ((\phi\circ i^*\circ T^*\circ\phi)e^*)(x)\\ &= ((\phi\circ i^*)(T^*\phi e^*))(x) = (T^*\phi e^*)(x) = (\phi e^*)(Tx) = e^*(Tx) \end{aligned}$$

since $T^*\phi e^*\in\phi(E^*)$, $\phi\circ i^*=I_{\phi(E^*)}$, and $Tx\in E$. Thus $U(T)=T$ as desired. □

REMARK 2.2. The assertion of Theorem 2.8 is not reversible: see Remark 2.1 and note that $\mathscr{K}(X^{**},X^{**})$ is always an ideal in $\mathscr{K}(X^{**},X^{****})$.

Some ideals have additional properties. Best known are probably M-ideals defined by Alfsen and Effros in [1], see also [11]. A more general type of ideals, the u-ideals, was first introduced by Casazza and Kalton in [4] and thoroughly studied by Godefroy, Kalton, and Saphar in [8].

Let us recall that a closed subspace F of a Banach space E is an *M-ideal* (respectively, a *u-ideal*) in E if there exists a linear projection P on E^* with $\ker P=F^\perp$ such that $\|f\|=\|Pf\|+\|f-Pf\|$ for all $f\in E^*$ (respectively, $\|I_{E^*}-2P\|=1$).

LEMMA 2.9 (see [15, 8]). *Let F be a closed subspace of a Banach space E. Then F is an M-ideal (respectively, a u-ideal) in E if and only if condition* (M) (*respectively, condition* (u)) *below is satisfied.*

(M) *For every finite dimensional subspace G in E and every $\epsilon > 0$, there exists a linear operator $U : G \to F$ such that $Ux = x$ for all $x \in G \cap F$ and $\|Ux + y - Uy\| \leq (1+\epsilon)\max(\|x\|, \|y\|)$ for all $x, y \in G$.*

(u) *For every finite dimensional subspace G in E and every $\epsilon > 0$, there exists a linear operator $U : G \to F$ such that $Ux = x$ for all $x \in G \cap F$ and $\|x - 2Ux\| \leq (1+\epsilon)\|x\|$ for all $x \in G$.*

PROOF. See [15, Theorem 4] and [8, Proposition 3.6]. □

In the next section we shall need the following analogue of Proposition 2.4.

PROPOSITION 2.10. *Let X be a closed subspace of a Banach space Y. If $\mathscr{K}(Z, X)$ is an M-ideal (respectively, a u-ideal) in $\mathscr{K}(Z, Y)$ for some Banach space $Z \neq \{0\}$, then X is an M-ideal (respectively, a u-ideal) in Y.*

PROOF. We argue as in the proof of Corollary 2.3, (iii) $\Rightarrow$ (i), but we use the local formulations of M-ideals and u-ideals from Lemma 2.9. □

3. Hahn-Banach extension operators

Let X be a subspace of a Banach space Y. For each $x^* \in X^*$, let $HB(x^*)$ denote the set of norm-preserving extensions of x^* to Y. Hahn-Banach extension operators $\Phi : X^* \to Y^*$ act as linear selection functions since $\Phi x^* \in HB(x^*)$ for all $x^* \in X^*$. This shows that if we can describe the sets $HB(x^*)$, then we get important information about possible Hahn-Banach extension operators.

The next lemma is fundamental for the results we obtain in this paper. It describes all norm-preserving extensions of certain important functionals on operator spaces. It also explains surprisingly well why some of those functionals have unique norm-preserving extensions: the reason is that all their norm-preserving extensions must have a special form that makes them unique.

LEMMA 3.1. *Let X, Y, and Z be Banach spaces and assume that X is a closed subspace of Y. Consider $Z \otimes X$ as a subspace of $\mathscr{L}(Z^*, Y)$. If z^* is a weak* denting point of B_{Z^*} and $x^* \in X^*$, then, for $x^* \otimes z^* \in (Z \otimes X)^*$, the equality*

$$HB(x^* \otimes z^*) = HB(x^*) \otimes \{z^*\}$$

holds.

PROOF. It is clear that $HB(x^*) \otimes \{z^*\} \subseteq HB(x^* \otimes z^*)$. For the converse, let $\phi = x^* \otimes z^*$ and let $\psi \in HB(\phi)$. We may assume that $\|x^*\| = 1$. It suffices to prove

CLAIM. $\psi \in \overline{B_{Y^*} \otimes \{z^*\}}^{w^*}$ *in* $\mathscr{L}(Z^*, Y)^*$.

Assume that the claim has been proved. Choose a net (y_α^*) in B_{Y^*} such that $y_\alpha^* \otimes z^* \underset{\alpha}{\to} \psi$ weak*. By passing to a subnet, we may assume that $y_\alpha^* \to y^* \in B_{Y^*}$ weak*. Let $z \in Z$ satisfy $z^*(z) = 1$. Then for any $x \in X$,

$$x^*(x) = x^*(x)z^*(z) = \phi(z \otimes x) = \lim_\alpha (y_\alpha^* \otimes z^*)(z \otimes x) = y^*(x)z^*(z) = y^*(x),$$

so we get that $y^*|_X = x^*$. Thus $y^* \in HB(x^*)$.

For any $T \in \mathscr{L}(Z^*, Y)$,

$$\psi(T) = \lim_\alpha (y_\alpha^* \otimes z^*)(T) = \lim_\alpha y_\alpha^*(Tz^*) = y^*(Tz^*) = (y^* \otimes z^*)(T),$$

so $\psi = y^* \otimes z^*$. □

PROOF OF THE CLAIM. Suppose for contradiction that $\psi \notin \overline{B_{Y^*} \otimes \{z^*\}}^{w^*}$. Then for some $T \in \mathscr{L}(Z^*, Y)$ with $\|T\| = 1$ and some $\epsilon > 0$, we get

$$\eta(T) < \psi(T) - 6\epsilon \quad \text{for all } \eta \in B_{Y^*} \otimes \{z^*\}.$$

By the description of denting points due to Werner [30, Lemma 2], there exist $\delta > 0$ and $z \in Z$ such that $z^*(z) = 1$, $\|z\| \le 1 + \delta\epsilon$, and

$$(\|w^*\| \le 1 \text{ and } w^*(z) > 1 - \delta) \quad \Rightarrow \quad \|w^* - z^*\| \le \epsilon.$$

Choose $x \in X$ such that $x^*(x) = 1$ and $\|x\| \le 1 + \delta\epsilon$, and define $S = z \otimes x \in Z \otimes X$. Since $\psi(S) = 1$ and $\psi \in B_{\mathscr{L}(Z^*,Y)^*} = \overline{\text{conv}}^{w^*}(B_{Y^*} \otimes B_{Z^*})$ (the last equality being clear from the bipolar theorem), we can find $\psi_\epsilon \in \text{conv}\,(B_{Y^*} \otimes B_{Z^*})$ such that

$$\psi_\epsilon(S) > 1 - \delta^2\epsilon^2, \quad |\psi_\epsilon(T) - \psi(T)| < \epsilon.$$

Let us write $\psi_\epsilon = \sum_{i=1}^m \lambda_i y_i^* \otimes z_i^*$, where $z_i^* \in B_{Z^*}$, $y_i^* \in B_{Y^*}$, $\lambda_i > 0$, and $\sum_{i=1}^m \lambda_i = 1$. We may suppose that $y_i^*(x) \ge 0$ for all i. Let $J = \{i : z_i^*(z) > 1 - \delta\}$. Then we get

$$\begin{aligned} 1 - \delta^2\epsilon^2 < \psi_\epsilon(S) &= \sum_{i=1}^m \lambda_i y_i^*(x) z_i^*(z) \\ &= \sum_{i \in J} \lambda_i y_i^*(x) z_i^*(z) + \sum_{i \notin J} \lambda_i y_i^*(x) z_i^*(z) \end{aligned}$$

$$\leq \sum_{i\in J}\lambda_i(1+\delta\epsilon)^2 + \sum_{i\notin J}\lambda_i(1+\delta\epsilon)(1-\delta)$$
$$= (1+\delta\epsilon)\left[(1+\delta\epsilon)\sum_{i\in J}\lambda_i + (1-\delta)\sum_{i\notin J}\lambda_i\right]$$
$$= (1+\delta\epsilon)\left[1+\delta\epsilon\sum_{i\in J}\lambda_i - \delta\sum_{i\notin J}\lambda_i\right]$$
$$\leq (1+\delta\epsilon)\left[1+\delta\epsilon-\delta\sum_{i\notin J}\lambda_i\right].$$

Thus $1-\delta\epsilon < 1+\delta\epsilon-\delta\sum_{i\notin J}\lambda_i$, meaning that $\sum_{i\notin J}\lambda_i < 2\epsilon$. Let $\eta = \sum_{i=1}^m \lambda_i y_i^*\otimes z^*$. Then we get

$$\|\psi_\epsilon - \eta\| = \left\|\sum_{i=1}^m \lambda_i y_i^*\otimes(z_i^*-z^*)\right\|$$
$$\leq \sum_{i\in J}\lambda_i\|z_i^*-z^*\| + \sum_{i\notin J}\lambda_i\|z_i^*-z^*\| < \epsilon+4\epsilon = 5\epsilon.$$

But $\eta \in B_{Y^*}\otimes\{z^*\}$, so

$$\psi_\epsilon(T) - 5\epsilon \leq \eta(T) < \psi(T) - 6\epsilon.$$

Thus $\epsilon < \psi(T) - \psi_\epsilon(T) < \epsilon$, a contradiction. □

REMARK 3.1. The particular case of Lemma 3.1 with $X = Y$ is precisely Lemma 4.3 in [18] stating that $x^*\otimes z^*$ has a unique norm-preserving extension to $\mathscr{L}(Z^*, X)$. Therefore, the proof of Lemma 3.1 provides, in particular, a new and simpler proof to Lemma 4.3 in [18] and its earlier versions [16, Lemma 3.4], [20, Theorem 3.7], [15, Lemma 11], and [15, Lemma 12]. The uniqueness of norm-preserving extensions have been used to obtain some main results in [2, 15, 16, 18, 20].

Let us point out the particular case $Y = X^{**}$ of Lemma 3.1.

COROLLARY 3.2. *Let X and Z be Banach spaces. Consider $Z\otimes X$ as a subspace of $\mathscr{L}(Z^*, X^{**})$. If z^* is a weak* denting point of B_{Z^*} and $x^*\in X^*$, then, for $x^*\otimes z^*\in (Z\otimes X)^*$, the equality $HB(x^*\otimes z^*) = HB(x^*)\otimes\{z^*\}$ holds.*

We shall say that a closed subspace X of a Banach space Y has the *unique ideal property* in Y if there is at most one ideal projection, this is, at most one norm one projection π on Y^* with $\ker\pi = X^\perp$. From the relation between ideal projections and

Hahn-Banach extension operators it is clear that X has the unique extension property in Y if and only if there is at most one Hahn-Banach extension operator $\phi : X^* \to Y^*$.

An obvious example of subspaces having the unique ideal property is presented by subspaces having property U: X is said to have *property U* in Y if every $x^* \in X^*$ has a unique norm-preserving extension to Y (this notion is due to Phelps [27]; for a recent study of such subspaces see [25] and [26]). It is well known that M-ideals (more generally, semi M-ideals and HB-subspaces) have property U and therefore they also have the unique ideal property (for a study of u-ideals having property U see [22]).

In the case when $Y = X^{**}$, let us note that $\phi \in \mathscr{L}(X^*, X^{***})$ is a Hahn-Banach extension operator if and only if $\phi^*|_{X^{**}} \in \mathscr{L}(X^{**}, X^{**})$ has norm one and $\phi^*|_X = I_X$. Thus the unique ideal property of X in X^{**} is the same as the unique extension property of X introduced and deeply studied by Godefroy and Saphar in [9] (using the term 'X is uniquely decomposed') and [10]. Let us recall that X is said to have the *unique extension property* if the only operator $T \in \mathscr{L}(X^{**}, X^{**})$ such that $\|T\| \leq 1$ and $T|_X = I_X$ is $T = I_{X^{**}}$.

In particular, the following Banach spaces have the unique extension property (see [10]): spaces which have property U in their bidual (Hahn-Banach smooth spaces), those with a Fréchet-differentiable norm, separable polyhedral Lindenstrauss spaces, spaces of compact operators $\mathscr{K}(Z, X)$ for reflexive Z and X.

COROLLARY 3.3. *Let X, Y, and $Z \neq \{0\}$ be Banach spaces and assume that X is a subspace of Y having the unique ideal property in Y. If $\mathscr{K}(Z^*, X)$ is an ideal in $\mathscr{K}(Z^*, Y)$ with an ideal projection P, then X is an ideal in Y and its unique ideal projection $\pi : Y^* \to Y^*$ satisfies, for all weak* denting points $z^* \in B_{Z^*}$ and all $y^* \in Y^*$, the equality $P(y^* \otimes z^*) = (\pi y^*) \otimes z^*$.*

PROOF. Proposition 2.4 yields that X is an ideal in Y. Let $\pi : Y^* \to Y^*$ denote its unique ideal projection and let $\phi : X^* \to Y^*$ be the unique Hahn-Banach extension operator. Then $\pi = \phi j^*$, where $j : X \to Y$ is the natural embedding.

Let $\Phi : \mathscr{K}(Z^*, X)^* \to \mathscr{K}(Z^*, Y)^*$ be a Hahn-Banach extension operator satisfying

$$Pf = \Phi(f|_{\mathscr{K}(Z^*,X)}), \quad f \in \mathscr{K}(Z^*, Y)^*.$$

For any weak* denting point z^* of B_{Z^*}, by Lemma 3.1, we have

$$\Phi(x^* \otimes z^*) \in HB(x^*) \otimes \{z^*\}, \quad x^* \in X^*.$$

By the linearity of Φ, it is straightforward that the map $\phi_{z^*} : X^* \to Y^*$ defined by

$$\Phi(x^* \otimes z^*) = (\phi_{z^*} x^*) \otimes z^*, \quad \phi_{z^*} x^* \in HB(x^*), \quad x^* \in X^*,$$

is linear and therefore it is a Hahn-Banach extension operator. Thus $\phi_{z^*} = \phi$. And for any $y^* \in Y^*$ we get

$$P(y^* \otimes z^*) = \Phi((j^* y^*) \otimes z^*) = (\phi j^* y^*) \otimes z^* = (\pi y^*) \otimes z^*$$

as desired. □

In [31] Werner proved that if X is an M-ideal in Y and Z is a Banach space and if Z^* or Y has the approximation property, then $\mathscr{K}(Z, X)$ is an M-ideal in $\mathscr{K}(Z, Y)$. The next result shows that, for reflexive Z, if $\mathscr{K}(Z, X)$ is an ideal in $\mathscr{K}(Z, Y)$, then it is already an M-ideal without any requirement about the approximation property.

THEOREM 3.4. *Let X be a closed subspace of a Banach space Y and let $Z \neq \{0\}$ be a reflexive Banach space. Then $\mathscr{K}(Z, X)$ is an M-ideal in $\mathscr{K}(Z, Y)$ if and only if X is an M-ideal in Y and $\mathscr{K}(Z, X)$ is an ideal in $\mathscr{K}(Z, Y)$.*

PROOF. If $\mathscr{K}(Z, X)$ is an M-ideal in $\mathscr{K}(Z, Y)$, then X is an M-ideal in Y by Proposition 2.10.

Assume that X is an M-ideal in Y and $\mathscr{K}(Z, X)$ is an ideal in $\mathscr{K}(Z, Y)$. Let P be a norm one projection on $\mathscr{K}(Z, Y)^*$ with $\ker P = \mathscr{K}(Z, X)^\perp$. Let π denote an M-ideal projection on Y^* with $\ker \pi = X^\perp$. By the uniqueness of Hahn-Banach extensions in the case of M-ideals, π is a unique ideal projection. Hence, for all $y^* \in Y^*$ and all weak* denting points $z \in B_Z$, by Corollary 3.3,

$$P(y^* \otimes z) = (\pi y^*) \otimes z.$$

Since $Z = \overline{\operatorname{span}}(w^*\text{-dent } B_Z)$, we get that

$$P(y^* \otimes z) = (\pi y^*) \otimes z, \quad y^* \in Y^*,\ z \in Z.$$

Consider now any $f \in \mathscr{K}(Z, Y)^*$. It suffices to prove that

$$\|Pf\| + \|f - Pf\| \leq \|f\|.$$

By the description of $\mathscr{K}(Z, Y)^*$ due to Feder and Saphar [7, Theorem 1] (here we use once more that Z is reflexive), there exists an element u in the projective tensor product $Y^* \hat{\otimes}_\pi Z$ such that

$$f(T) = \operatorname{trace}(Tu), \quad T \in \mathscr{K}(Z, Y),$$

and $\|f\| = \|u\|_\pi$. For any $\epsilon > 0$, let u be represented as $u = \sum_{n=1}^\infty y_n^* \otimes z_n$ so that

$$\|u\|_\pi + \epsilon \geq \sum_{n=1}^\infty \|y_n^*\| \|z_n\|.$$

Define $g \in \mathscr{K}(Z,Y)^*$ by

$$g(T) = \sum_{n=1}^{\infty}((\pi y_n^* \otimes z_n)(T)), \quad T \in \mathscr{K}(Z,Y).$$

Since

$$f(T) = \sum_{n=1}^{\infty}((y_n^* \otimes z_n)(T)), \quad T \in \mathscr{K}(Z,Y),$$

it is clear that $g = Pf$ and

$$(f-g)(T) = \sum_{n=1}^{\infty}(((y_n^* - \pi y_n^*) \otimes z_n)(T)), \quad T \in \mathscr{K}(Z,Y).$$

Therefore,

$$\begin{aligned}\|f\| + \epsilon &\geq \sum_{n=1}^{\infty}\|y_n^*\|\|z_n\| = \sum_{n=1}^{\infty}\|\pi y_n^*\|\|z_n\| + \sum_{n=1}^{\infty}\|y_n^* - \pi y_n^*\|\|z_n\| \\ &\geq \|g\| + \|f-g\| = \|Pf\| + \|f - Pf\|\end{aligned}$$

as desired. □

A closed subspace F of a Banach space E is called a *semi M-ideal* (see [14] or [11, page 43]) if there is a (nonlinear) projection P from E^* onto $F^\perp$ such that $P(\lambda f + Pg) = \lambda Pf + Pg$ and $\|f\| = \|Pf\| + \|f - Pf\|$ for all $f, g \in E^*$ and all scalars λ.

REMARK 3.2. Theorem 3.4 remains true if one replaces 'X is an M-ideal in Y' by the weaker condition 'X is a semi M-ideal in Y'. This is clear from the fact that X is an M-ideal in Y if and only if X is an ideal in Y and X is a semi M-ideal in Y (see, for example, [11, page 43]). In particular, X is an M-ideal in its bidual X^{**} whenever X is a semi M-ideal in X^{**}. We do not know whether $\mathscr{K}(Z,X)$ is an M-ideal in $\mathscr{K}(Z,X^{**})$ whenever $\mathscr{K}(Z,X)$ is a semi M-ideal in $\mathscr{K}(Z,X^{**})$.

The method of proof of Theorem 3.4 enables us to extend the theorem from M-ideals to more general classes of ideals (for example, to ideals F in E with respect to an ideal projection P satisfying $\|af + bPf\| + c\|Pf\| \leq \|f\|$ for given numbers a, b, c, and for all $f \in E^*$; these ideals were recently studied in [23] and [24]) under the assumption that X has the unique ideal property in Y. The corresponding result on u-ideals reads as follows.

THEOREM 3.5. *Let X be a closed subspace of a Banach space Y having the unique ideal property in Y and let $Z \neq \{0\}$ be a reflexive Banach space. Then $\mathscr{K}(Z,X)$ is*

a u-ideal in $\mathscr{K}(Z,Y)$ if and only if X is a u-ideal in Y and $\mathscr{K}(Z,X)$ is an ideal in $\mathscr{K}(Z,Y)$.

A closed subspace F of a Banach space E is called a *strict u-ideal* in E if there exists a linear projection P on E^* with $\ker P = F^\perp$ such that $\|I_{E^*} - 2P\| = 1$ and the range $\operatorname{ran} P$ is a norming subspace of E^*. This notion was introduced and deeply studied by Godefroy, Kalton, and Saphar [8].

THEOREM 3.6. *Let X be either a separable Banach space or a Banach space containing no copy of ℓ_1. If X is a strict u-ideal in X^{**} and $\mathscr{K}(Z,X)$ is an ideal in $\mathscr{K}(Z,X^{**})$ for a reflexive Banach space Z, then $\mathscr{K}(Z,X)$ is a strict u-ideal in $\mathscr{K}(Z,X^{**})$.*

PROOF. Let $\pi : X^{***} \to X^{***}$ be the projection from the definition of a strict u-ideal and let P denote the ideal projection on $\mathscr{K}(Z,X^{**})^*$. It follows from [8, Propositions 5.2 and 2.7] that X^* does not contain any proper norming closed subspace. But then X has the unique extension property (see [10, Proposition 2.5]) and we can apply Theorem 3.5 to conclude that $\mathscr{K}(Z,X)$ is a u-ideal in $\mathscr{K}(Z,X^{**})$. Moreover, the proof of Theorem 3.4 shows that P is the desired u-ideal projection and

$$P(x^{***} \otimes z) = (\pi x^{***}) \otimes z, \quad x^{***} \in X^{***},\ z \in Z.$$

In view of the last equality $\operatorname{ran} P$ contains the functionals $x^{***} \otimes z$ with $x^{***} \in \operatorname{ran} \pi$ and $z \in Z$. But these functionals give the norm of any $T \in \mathscr{K}(Z,X^{**})$ (by $\|T\| = \sup\{|x^{***}(Tz)| : x^{***} \in B_{\operatorname{ran}\pi},\ z \in B_Z\}$) because $\operatorname{ran}\pi$ is a norming subspace (for X^{**}) in X^{***} (in fact, $\operatorname{ran}\pi = X^*$ (see [8])). □

4. Ideals of compact operators and the compact approximation property

Let X be a closed subspace of a Banach space Y. In this section we shall prove that $\mathscr{K}(Z,X)$ is an ideal (respectively, an M-ideal) in $\mathscr{K}(Z,Y)$ for all Banach spaces Z whenever X is an ideal (respectively, an M-ideal) in Y and X^* has the compact approximation property with conjugate operators. We begin by showing that if results about $\mathscr{K}(Z,X)$ being an ideal (from a given class of ideals) in $\mathscr{K}(Z,Y)$ or $\mathscr{W}(Z,Y)$ are true for all reflexive Banach spaces Z, then they are true for all Banach spaces Z. The method of proof is based on the following version of a factorization result for weakly compact operators by Lima, Nygaard, and Oja in [17, Corollary 2.4].

LEMMA 4.1. *Let Y and Z be Banach spaces and let G be a finite dimensional subspace of $\mathscr{W}(Z,Y)$. Then there exist a reflexive Banach space W, a norm one operator $J : Z \to W$, and a linear isometry $\Phi : G \to \mathscr{W}(W,Y)$ such that $T =$*

$\Phi(T) \circ J$ and $\overline{\operatorname{ran} T} = \overline{\operatorname{ran} \Phi(T)}$ in Y for all $T \in G$. Moreover, T is compact if and only if $\Phi(T)$ is compact. In particular, if $G \subseteq \mathscr{K}(Z, Y)$, then W is separable.

PROOF. Applying [17, Theorem 2.3], as in the proof of [17, Corollary 2.4], we can find a reflexive Banach space W, a norm one operator $I : W^* \to Z^*$, and a linear isometry $\Phi : G \to \mathscr{W}(W, Y)$ so that $\overline{\operatorname{ran} I^*} = W$, $T^{**} = \Phi(T) \circ I^*$ for all $T \in G$, and T is compact if and only if $\Phi(T)$ is compact. As in [17, Corollary 2.4], putting $J = I^*|_Z : Z \to W$, we have $\|J\| = 1$ and $T = \Phi(T) \circ J$ for all $T \in G$. Clearly $\operatorname{ran} T \subseteq \operatorname{ran} \Phi(T)$. On the other hand,

$$\begin{aligned} \operatorname{ran} \Phi(T) &= (\Phi(T))(W) = (\Phi(T))(\overline{I^*(Z^{**})}) \\ &\subseteq \overline{(\Phi(T))(I^*(Z^{**}))} = \overline{\operatorname{ran} T^{**}} \subseteq \overline{\operatorname{ran} T} \end{aligned}$$

because T is weakly compact. Hence $\overline{\operatorname{ran} T} = \overline{\operatorname{ran} \Phi(T)}$.

In particular, if $G \subseteq \mathscr{K}(Z, Y)$, then by the proof of [17, Theorem 2.3] and [5, Lemma 1, (xi)] W^* is separable, hence W is also separable. □

THEOREM 4.2. *Let X be a closed subspace of a Banach space Y. Then $\mathscr{K}(Z, X)$ is an ideal in $\mathscr{K}(Z, Y)$ for all Banach spaces Z if and only if $\mathscr{K}(W, X)$ is an ideal in $\mathscr{K}(W, Y)$ for all separable reflexive Banach spaces W.*

PROOF. The proof is similar to the proof of [17, Theorem 3.1]. Let $\mathscr{K}(W, X)$ be an ideal in $\mathscr{K}(W, Y)$ for all separable reflexive spaces W. For a Banach space Z, let G be a finite dimensional subspace of $\mathscr{K}(Z, Y)$ and let $\epsilon > 0$. By Lemma 4.1, we can find a separable reflexive space W, a norm one operator $J : Z \to W$, and a linear isometry Φ mapping G into $\mathscr{K}(W, Y)$ such that $T = \Phi(T) \circ J$ for all $T \in G$. If $U : \Phi(G) \to \mathscr{K}(W, X)$ is an operator from the local formulation of the notion of an ideal (see Lemma 2.1), then the operator $V : G \to \mathscr{K}(Z, X)$ defined by $V(T) = U(\Phi(T)) \circ J$, $T \in G$, has the same local properties as U. In particular, if $T \in G \cap \mathscr{K}(Z, X)$, then $\Phi(T) \in \mathscr{K}(W, X)$ because $\operatorname{ran} \Phi(T) \subseteq \overline{\operatorname{ran} T} \subseteq X$. Hence $V(T) = \Phi(T) \circ J = T$. Thus, $\mathscr{K}(Z, X)$ is an ideal in $\mathscr{K}(Z, Y)$. □

A similar result is true for special classes of (semi) ideals.

THEOREM 4.3. *Let X be a closed subspace of a Banach space Y. Then $\mathscr{K}(Z, X)$ is an M-ideal (respectively, a u-ideal or a semi M-ideal) in $\mathscr{K}(Z, Y)$ for all Banach spaces Z if and only if $\mathscr{K}(W, X)$ is an M-ideal (respectively, a u-ideal or a semi M-ideal) in $\mathscr{K}(W, Y)$ for all separable reflexive Banach spaces W.*

PROOF. The proof for M-ideals and u-ideals is similar to the proof of Theorem 4.2. Instead of the local formulation of ideals, it uses the local formulations of M-ideals and u-ideals from Lemma 2.9.

The proof for semi M-ideals will use the following characterization due to Lima [14]: a closed subspace F of a Banach space E is a semi M-ideal in E if and only if for all $x \in B_E$, all $y \in B_F$, and all $\epsilon > 0$, there exists $z \in F$ satisfying

$$\|x \pm y - z\| \leq 1 + \epsilon.$$

Let $\mathscr{K}(W, X)$ be a semi M-ideal in $\mathscr{K}(W, Y)$ for all separable reflexive Banach spaces W. For a Banach space Z, let $T \in B_{\mathscr{K}(Z,Y)}$, $S \in B_{\mathscr{K}(Z,X)}$, and $\epsilon > 0$. Put $G = \operatorname{span}\{S, T\} \subseteq \mathscr{K}(Z, Y)$ and let W, J, and Φ be as in Lemma 4.1. Note that W is separable and $\Phi(S) \in \mathscr{K}(W, X)$. Since $\mathscr{K}(W, X)$ is a semi M-ideal in $\mathscr{K}(W, Y)$, there exists $U \in \mathscr{K}(W, X)$ such that $\|\Phi(T) \pm \Phi(S) - U\| \leq 1 + \epsilon$. But then $\|T \pm S - U \circ J\| \leq 1 + \epsilon$, and $U \circ J \in \mathscr{K}(Z, X)$. This shows that $\mathscr{K}(Z, X)$ is a semi M-ideal in $\mathscr{K}(Z, Y)$. □

By the same reasoning as in the proofs of Theorem 4.2 and Theorem 4.3, we can prove the following result.

THEOREM 4.4. *Let X be a closed subspace of a Banach space Y. Then $\mathscr{K}(Z, X)$ is an ideal (respectively, an M-ideal, a u-ideal, or a semi M-ideal) in $\mathscr{W}(Z, Y)$ for all Banach spaces Z if and only if $\mathscr{K}(W, X)$ is an ideal (respectively, an M-ideal, a u-ideal, or a semi M-ideal) in $\mathscr{W}(W, Y)$ for all reflexive Banach spaces W.*

REMARK 4.1. The particular case of Theorem 4.4 for ideals and for $X = Y$ was proved in [17, Theorem 3.1].

Let us point out the following quite surprising observation.

COROLLARY 4.5. *Let X be a semi M-ideal (respectively, a u-ideal having the unique ideal property) in a Banach space Y. If $\mathscr{K}(W, X)$ is an ideal in $\mathscr{K}(W, Y)$ for all separable reflexive Banach spaces W, then $\mathscr{K}(Z, X)$ is an M-ideal (respectively, a u-ideal) in $\mathscr{K}(Z, Y)$ for all Banach spaces Z.*

PROOF. The proof is immediate from Theorem 3.4 together with Remark 3.2 (respectively, Theorem 3.5) and Theorem 4.3. □

We conclude by showing that Corollary 4.5 applies if X^* has the *compact approximation property with conjugate operators*, that is, there exists a net (K_α) in $\mathscr{K}(X, X)$ such that (K_α^*) converges to I_{X^*} uniformly on compact subsets of X^*.

THEOREM 4.6. *Let X be an ideal in a Banach space Y with an ideal projection π and let Z be a reflexive Banach space. If X^* has the compact approximation property with conjugate operators, then $\mathscr{K}(Z, X)$ is an ideal in $\mathscr{L}(Z, Y)$ with an ideal projection P satisfying $P(y^* \otimes z) = (\pi y^*) \otimes z$ for all $y^* \in Y^*$ and all $z \in Z$.*

PROOF. Let $\phi : X^* \to Y^*$ be a Hahn-Banach extension operator satisfying $\pi = \phi j^*$, where $j : X \to Y$ is the natural embedding. We shall use the description of $\mathscr{K}(Z, X)^*$ due to Feder and Saphar [7, Theorem 1] (we can use it because Z is reflexive). For any $g \in \mathscr{K}(Z, X)^*$, there exists $u \in X^* \hat{\otimes}_\pi Z$ such that

$$g(S) = \operatorname{trace}(Su), \quad S \in \mathscr{K}(Z, X),$$

and $\|g\| = \|u\|_\pi$. Let this $u = \sum_{n=1}^\infty x_n^* \otimes z_n$ with $\|x_n^*\| \to 0$ and $\sum_{n=1}^\infty \|z_n\| < \infty$. We assume that a net (K_α^*) with $K_\alpha \in \mathscr{K}(X, X)$ converges to I_{X^*} uniformly on compact subsets of X^*. If $T \in \mathscr{L}(Z, Y)$, then $K_\alpha^{**} \circ \phi^* \circ T^{**}|_Z \in \mathscr{K}(Z, X)$ and

$$\begin{aligned}
&\left|\operatorname{trace}(T(\phi u)) - g(K_\alpha^{**} \circ \phi^* \circ T^{**}|_Z)\right| \\
&\quad = \left|\operatorname{trace}(T(\phi u)) - \operatorname{trace}(K_\alpha^{**} \circ \phi^* \circ T^{**}|_Z)u\right| \\
&\quad = \left|\sum_{n=1}^\infty (\phi x_n^*)(Tz_n) - \sum_{n=1}^\infty x_n^*(K_\alpha^{**}\phi^* T z_n)\right| \\
&\quad = \left|\sum_{n=1}^\infty (\phi^* T z_n)(x_n^* - K_\alpha^* x_n^*)\right| \\
&\quad \le \sup_n \|(I_{X^*} - K_\alpha^*)(x_n^*)\| \|T\| \sum_{n=1}^\infty \|z_n\| \underset{\alpha}{\to} 0
\end{aligned}$$

because $\{0, x_1^*, x_2^*, \dots\}$ is a compact subset of X^*.

Let $\Phi : \mathscr{K}(Z, X)^* \to \mathscr{L}(Z, Y)^*$ be defined by

$$\begin{aligned}
(\Phi g)(T) &= \lim_\alpha g(K_\alpha^{**} \circ \phi^* \circ T^{**}|_Z) \\
&= \operatorname{trace}(T(\phi u)), \quad g \in \mathscr{K}(Z, X)^*,\ T \in \mathscr{L}(Z, Y).
\end{aligned}$$

The existence of the limit implies the linearity of Φg for all $g \in \mathscr{K}(Z, X)^*$ and of Φ. Moreover, $\|\Phi g\| \le \|g\|$ for all $g \in \mathscr{K}(Z, X)^*$ because

$$|\operatorname{trace}(T(\phi u))| \le \|T(\phi u)\|_\pi \le \|T\|\|\phi u\|_\pi \le \|T\|\|\phi\|\|u\|_\pi = \|T\|\|g\|.$$

Since, for any $g \in \mathscr{K}(Z, X)^*$ and $S \in \mathscr{K}(Z, X)$, we have

$$\begin{aligned}
(\Phi g)(S) &= \operatorname{trace}(S(\phi u)) = \operatorname{trace}\left(\sum_{n=1}^\infty \phi x_n^* \otimes Sz_n\right) \\
&= \sum_{n=1}^\infty x_n^*(Sz_n) = \operatorname{trace}(Su) = g(S),
\end{aligned}$$

meaning that Φg extends g, we conclude that Φ is a Hahn-Banach extension operator from $\mathscr{K}(Z, X)^*$ to $\mathscr{L}(Z, Y)^*$. Thus, $\mathscr{K}(Z, X)$ is an ideal in $\mathscr{L}(Z, Y)$.

Let P be the ideal projection on $\mathscr{L}(Z,Y)^*$ defined by Φ, that is,

$$Pf = \Phi(f|_{\mathscr{K}(Z,X)}), \quad f \in \mathscr{L}(Z,Y)^*.$$

Considering $x^* \otimes z \in \mathscr{K}(Z,X)^*$ with $x^* \in X^*$ and $z \in Z$, we get, for any $T \in \mathscr{L}(Z,Y)$, that $(\Phi(x^* \otimes z))(T) = \operatorname{trace}(T(\phi(x^* \otimes z))) = (\phi x^*)(Tz) = ((\phi x^*) \otimes z)(T)$. Hence

$$\Phi(x^* \otimes z) = (\phi x^*) \otimes z, \quad x^* \in X^*,\ z \in Z,$$

and therefore, for all $y^* \in Y^*$ and $z \in Z$,

$$P(y^* \otimes z) = \Phi((j^* y^*) \otimes z) = (\phi j^* y^*) \otimes z = (\pi y^*) \otimes z,$$

as desired. □

COROLLARY 4.7. *Let X be an ideal in a Banach space Y. If X^* has the compact approximation property with conjugate operators, then $\mathscr{K}(Z,X)$ is an ideal in $\mathscr{W}(Z,Y)$ (and therefore also in $\mathscr{K}(Z,Y)$) for all Banach spaces Z.*

PROOF. The proof is immediate from Theorem 4.6 and Theorem 4.4. □

REMARK 4.2. Example 1.2 shows that the assumption 'X^* has the compact approximation property with conjugate operators' is essential in Corollary 4.7 and Theorem 4.6 (recall that X is always an ideal in X^{**}) and cannot be replaced by the assumption 'X^* has the metric compact approximation property'.

COROLLARY 4.8. *Let X be an M-ideal (respectively, a u-ideal having the unique ideal property) in a Banach space Y. If X^* has the compact approximation property with conjugate operators, then $\mathscr{K}(Z,X)$ is an M-ideal (respectively, a u-ideal) in $\mathscr{K}(Z,Y)$ for all Banach spaces Z.*

PROOF. By Corollary 4.7, $\mathscr{K}(Z,X)$ is an ideal in $\mathscr{K}(Z,Y)$ for all Banach spaces Z and therefore Corollary 4.5 applies to obtain the desired conclusion. □

REMARK 4.3. The assumption 'X^* has the compact approximation property with conjugate operators' is also essential in Corollary 4.8 (see Remark 4.2). Namely, if X is the closed subspace of c_0 constructed by Johnson and Schechtman (see [13, Corollary JS]), then X is an M-ideal in X^{**}, X has a basis, and X^* does not have the approximation property. Moreover, as it will be shown in a forthcoming paper of the authors, based on the present article and [19], there exists a separable reflexive Banach space Z such that $\mathscr{K}(Z,X)$ is not an ideal in $\mathscr{K}(Z,X^{**})$.

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Department of Mathematics
Agder College
Gimlemoen 25J
Serviceboks 422
4604 Kristiansand
Norway
e-mail: asvald.lima@hia.no

Faculty of Mathematics
Tartu University
Liivi 2-606
EE-50409 Tartu
Estonia
e-mail: eveoja@math.ut.ee