

AN UNCERTAINTY PRINCIPLE LIKE HARDY'S THEOREM FOR NILPOTENT LIE GROUPS

AJAY KUMAR and CHET RAJ BHATTA

Dedicated to Eberhard Kaniuth on his 65th birthday

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Abstract

We extend an uncertainty principle due to Cowling and Price to threadlike nilpotent Lie groups. This uncertainty principle is a generalization of a classical result due to Hardy. We are thus extending earlier work on \mathbb{R}^n and Heisenberg groups.

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Introduction

A classical theorem of Hardy [6] on Fourier transform pairs says that a non zero function f on the real line \mathbb{R} and its Fourier transform \hat{f} cannot both be very rapidly decreasing. More precisely, let the Fourier transform be defined by

$$\hat{f}(y) = \int_{\mathbb{R}} f(x)e^{-2\pi ixy} dx, \quad y \in \mathbb{R}.$$

Hardy's theorem says that if $|f(x)| \leq Ce^{-\alpha\pi x^2}$ for all $x \in \mathbb{R}$ and $|\hat{f}(y)| \leq Ce^{-\beta\pi y^2}$ for all $y \in \mathbb{R}$ with $\alpha\beta > 1$ then $f = 0$ a.e. For a proof see [6] or [4, Theorem 3.2]. The following is a generalization of this theorem due to Cowling and Price [3].

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THEOREM (Cowling and Price). *Let $f : \mathbb{R} \rightarrow \mathbb{C}$ be measurable and*

- (i) $\|e_a f\|_{L^p(\mathbb{R})} < \infty$,
- (ii) $\|e_b \hat{f}\|_{L^q(\mathbb{R})} < \infty$,

where $a, b > 0$, $e_k(x) = e^{k\pi x^2}$ and $1 \leq \min(p, q) < \infty$. If $ab \geq 1$, then $f = 0$ almost everywhere. If $ab < 1$, then there exist infinitely many linearly independent functions satisfying (i) and (ii).

An analogue of the Cowling-Price Theorem has been proved in [1] for Euclidean spaces, the Heisenberg group \mathbb{H}_n and the Euclidean motion group of the plane. In this paper we concern ourselves with results of this kind on certain nilpotent Lie groups, thereby considerably extending the results for \mathbb{R}^n and \mathbb{H}_n .

Threadlike nilpotent Lie groups

For $n \geq 3$, let \mathfrak{g}_n be the n -dimensional real nilpotent Lie algebra with basis X_1, X_2, \dots, X_n and non trivial Lie brackets $[X_n, X_{n-1}] = X_{n-2}, \dots, [X_n, X_2] = X_1$. Here \mathfrak{g}_n is a $(n-1)$ -step nilpotent and is a semi-direct product of $\mathbb{R}X_n$ and the abelian ideal $\sum_{j=1}^{n-1} \mathbb{R}X_j$. Note that \mathfrak{g}_3 is the Heisenberg Lie algebra. Let $G_n = \exp \mathfrak{g}_n$.

For $\xi = \sum_{j=1}^{n-1} \xi_j X_j^* \in \mathfrak{g}_n^*$, the coadjoint action of G_n is given by

$$Ad^*(e^{tX_n})\xi = \sum_{j=1}^{n-1} P_j(\xi, t)X_j^*,$$

where, for $1 \leq j \leq n-1$, $P_j(\xi, t)$ is the polynomial in t defined by

$$P_j(\xi, t) = \sum_{k=1}^{j-1} (1/k!) (-1)^k t^k \xi_{j-k}.$$

The orbit of ξ is generic with respect to the basis $\{X_1^*, X_2^*, \dots, X_n^*\}$ if and only if $\xi_1 \neq 0$, and the jumping indices are 2 to n ; see [2] for details. The cross section X_{ξ_1} for the set of generic orbits is given by

$$X_{\xi_1} = \{ \xi = (\xi_1, 0, \xi_3, \dots, \xi_{n-1}, 0) : \xi_i \in \mathbb{R}, \xi_1 \neq 0 \}.$$

For $\xi \in \mathfrak{g}_n^*$, let π_ξ denote the irreducible representation of G_n associated with ξ . Then the mapping $\xi \rightarrow \pi_\xi$ is bijection of X_{ξ_1} and the set of all generic irreducible representations. Plancherel measure on \widehat{G}_n is supported by these π_ξ .

Denoting by \mathcal{F} the Fourier transform on \mathbb{R}^{n-1} , it follows that the Hilbert-Schmidt norm of the operator $\pi_\xi(f)$, $f \in L^1 \cap L^2(G_n)$ is given by

$$\|\pi_\xi(f)\|_{HS}^2 = \int_{\mathbb{R}^2} \left| \mathcal{F} f(P_1(\xi, t), \dots, P_{n-1}(\xi, t), t-s) \right|^2 ds dt$$

(for details see [2] and [5]).

Given a function $f : G_n \rightarrow \mathbb{C}$ and $y = (y_2, \dots, y_n) \in \mathbb{R}^{n-1}$, let $f_y, f_y^* : \mathbb{R} \rightarrow \mathbb{C}$ be defined by

$$f_y(x_1) = f\left(e^{x_1 X_1 + \sum_{j=2}^n y_j X_j}\right),$$

and $f_y^*(x_1) = \overline{f_y(-x_1)}$.

The following lemma is proved in [7, Section 2 and Section 3].

LEMMA 1. *Let $f : G_n \rightarrow \mathbb{C}$ be a measurable function such that $|f(x)| \leq ce^{-a\pi\|x\|^2}$ for some $a, c > 0$ and all $x \in G_n$. Let $g : \mathbb{R} \rightarrow \mathbb{C}$ be defined by*

$$g(x_1) = \int_{\mathbb{R}^{n-1}} f_y * f_y^*(x_1) dy.$$

Then $|g(x_1)| \leq Ce^{-a\pi x_1^2/2}$ for some $C > 0$ and all $x_1 \in \mathbb{R}$ and

$$(*) \quad \hat{g}(\xi_1) = |\xi_1| \int_{\mathbb{R}^{n-3}} \|\pi_\xi(f)\|_{HS}^2 d\xi_3 \cdots d\xi_{n-1}.$$

THEOREM 2. *Let a, b and q be real numbers such that $a, b > 0$ and $q \geq 2$. Let $f : G_n \rightarrow \mathbb{C}$ be a measurable function and suppose that f satisfies:*

- (i) $|f(x)| \leq Ce^{-a\pi\|x\|^2}$ for some $C > 0$ and all $x \in G_n$.
- (ii) $\int_{\mathbb{R}^{n-2}} |\xi_1| e^{bq\pi\|\xi\|^2} \|\pi_\xi(f)\|_{HS}^q d\xi_1 d\xi_3 \cdots d\xi_{n-1} < \infty$.

Then the following hold:

- (1) If $q = 2$ and $ab \geq 1$, then $f = 0$ a.e.
- (2) If $q > 2$ and $ab > 1$, then $f = 0$ a.e.

PROOF. For $\alpha \in \mathbb{R}$, let $e_\alpha : \mathbb{R} \rightarrow \mathbb{R}$ denote the function $e_\alpha(t) = e^{\alpha\pi t^2}$. Let $g : \mathbb{R} \rightarrow \mathbb{C}$ be defined as in Lemma 1. We apply the Cowling-Price Theorem [3] to conclude that $g = 0$. Then Lemma 1 shows that $\pi_\xi(f) = 0$ for almost all $\xi \in \mathbb{R}^{n-2}$, whence $f = 0$ a.e.

For $q = 2$ by hypothesis (ii),

$$\|e_{2b}\hat{g}\|_1 = \int_{\mathbb{R}} e_{2b}(\xi_1) \left(\int_{\mathbb{R}^{n-3}} |\xi_1| \|\pi_\xi(f)\|_{HS}^2 d\xi_3 \cdots d\xi_{n-1} \right) d\xi_1 < \infty.$$

Since $|g(x_1)| \leq Ce^{-a\pi x_1^2/2}$ by Lemma 1 and $ab \geq 1$ so the Cowling-Price Theorem yields $g = 0$.

For $q > 2$ and $ab > 1$, choose $\epsilon > 0$ such that $ab' > 1$, $b' = b - \epsilon$. Then for $\xi' = (\xi_3, \dots, \xi_{n-1})$, we have

$$\begin{aligned} \|e_{2b'}\hat{g}\|_{q/2}^{q/2} &= \int_{\mathbb{R}} e_{b'q}(\xi_1)|\hat{g}(\xi_1)|^{q/2}d\xi_1 \\ &= \int_{\mathbb{R}} e_{b'q}(\xi_1) \left(\int_{\mathbb{R}^{n-3}} |\xi_1| \|\pi_{\xi}(f)\|_{HS}^2 d\xi' \right)^{q/2} d\xi_1 \\ &\leq \int_{\mathbb{R}} e_{b'q}(\xi_1)|\xi_1|^{q/2} \left(\int_{\mathbb{R}^{n-3}} e_{2b'}(\|\xi'\|) \|\pi_{\xi}(f)\|_{HS}^2 d\xi' \right)^{q/2} d\xi_1 \\ &= \int_{\mathbb{R}} e_{b'q}(\xi_1)|\xi_1|^{q/2} \left(\int_{\mathbb{R}^{n-3}} e_{2b}(\|\xi'\|) \|\pi_{\xi}(f)\|_{HS}^2 e_{-2\epsilon}(\|\xi'\|) d\xi' \right)^{q/2} d\xi_1. \end{aligned}$$

Applying Hölder's inequality with $q/2$ and $q/(q-2)$ we obtain

$$\begin{aligned} \|e_{2b'}\hat{g}\|_{q/2}^{q/2} &\leq \int_{\mathbb{R}} \left(e_{b'q}(\xi_1)|\xi_1|^{q/2} \left(\int_{\mathbb{R}^{n-3}} e_{-(2\epsilon q)/(q-2)}(\|\xi'\|) d\xi' \right)^{(q/2)-1} \right. \\ &\quad \left. \times \int_{\mathbb{R}^{n-3}} e_{bq}(\|\xi'\|) \|\pi_{\xi}(f)\|_{HS}^q d\xi' \right) d\xi_1 \\ &\leq K_1 \int_{\mathbb{R}} \left(e_{qb}(\xi_1) e_{-q\epsilon}(\xi_1) |\xi_1|^{(q/2)-1} \right) |\xi_1| \\ &\quad \times \left(\int_{\mathbb{R}^{n-3}} e_{bq}(\|\xi'\|) \|\pi_{\xi}(f)\|_{HS}^q d\xi' \right) d\xi_1 \\ &\leq K \int_{\mathbb{R}^{n-2}} e_{bq}(\|\xi\|) \|\pi_{\xi}(f)\|_{HS}^q |\xi_1| d\xi < \infty, \end{aligned}$$

for certain positive constants K_1 and K . Thus $g = 0$ by the Cowling-Price Theorem. \square

REMARK 3. If the formula (*) in Lemma 1 reduces to $\hat{g}(\xi_1) = |\xi_1| \|\pi_{\xi}(f)\|_{HS}^2$ for some G_n , then for $1 \leq q < 2$ and $ab \geq 2$ along with the hypothesis in Theorem 2 implies that $f = 0$ a.e. The proof can be given as in [1, Theorem 2.1]. The above condition is satisfied if $G_n = G_3, G_{5,1}, G_{5,3}$ and $G_{5,6}$; see [9] for the definitions and structure of these groups.

THEOREM 4. Let a and b be positive real numbers and $1 \leq \min(p, q) < \infty$. Suppose that $f \in L^1(G_n) \cap L^2(G_n)$ satisfies the following conditions:

- (i) $\int_{G_n} e^{pa\pi\|x\|^2} |f(x)|^p dx < \infty$,
- (ii) $\int_{\mathbb{R}^{n-2}} |\xi_1| e^{b\pi q\|\xi\|^2} \|\pi_{\xi}(f)\|_{HS}^q d\xi < \infty$.

If $q \geq 2$ and $ab > 1$ then $f = 0$ a.e.

PROOF. Easy computations show that when, as before, identifying G_n as a set with \mathbb{R}^n , the product of two elements $y = (y_1, \dots, y_n)$ and $x = (x_1, \dots, x_n)$ of G_n is given by $yx = y + x + \sum_{j=1}^{n-2} (1/j!) y_n^j (x_{j+1}, \dots, x_{n-1}, 0, \dots, 0)$. For $\|x\| \geq 1$, this implies

$$\|yx\| \geq \|x\| - \|y\| - \|x\| \sum_{j=1}^{n-2} \frac{1}{j!} |y_n|^j \geq \|x\| \left(1 - \|y\| - \sum_{j=1}^{n-2} \frac{1}{j!} \|y\|^j \right).$$

Define $\varphi : (0, \infty) \rightarrow \mathbb{R}$ by $\varphi(\epsilon) = 1 - \epsilon - \sum_{j=1}^{n-2} (\epsilon^j/j!)$. Thus $\|yx\| \geq \|x\|\varphi(\epsilon)$, whenever $\|x\| \geq 1$ and $\|y\| \leq \epsilon$.

Let g be a continuous function on G_n such that $g(y) = g(y^{-1})$ for all $y \in G_n$ and $g(y) = 0$ for all y such that $\|y\| \geq \epsilon$. Since G_n is unimodular, for $x \in G_n$ such that $\|x\| \geq 1$,

$$\begin{aligned} (|g| * e_a |f|)(x) &= \int_{G_n} |g(y)| e_a(\|yx\|) |f(yx)| dy \\ &\geq \int_{G_n} |g(y)| e_a(\|x\|\varphi(\epsilon)) |f(y^{-1}x)| dy \\ &= e_a(\|x\|\varphi(\epsilon)) (|g| * |f|)(x). \end{aligned}$$

By (i) $e_a |f|$ is an L^p -function and $|g|$ is an $L^{p'}$ function ($1/p + 1/p' = 1$), so $g * e_a |f|$ is an L^∞ function. Thus with $C = \||g| * e_a |f|\|_\infty < \infty$, we have

$$|g * f(x)| \leq |g| * |f|(x) \leq C e_{-a}(\|x\|\varphi(\epsilon))$$

for all $x \in G_n$ such that $\|x\| \geq 1$. Since $g * f$ is continuous, it follows that for some constant $C > 0$, $|g * f(x)| \leq C e_{-a}(\|x\|\varphi(\epsilon))$ for all $x \in G_n$. In addition,

$$\|\pi_\xi(g * f)\|_{HS} \leq \|\pi_\xi(g)\| \cdot \|\pi_\xi(f)\|_{HS} \leq \|g\|_1 \|\pi_\xi(f)\|_{HS}$$

and hence, by hypothesis

$$\int_{\mathbb{R}^{n-2}} |\xi_1| e_{bq}(\|\xi\|) \|\pi_\xi(g * f)\|_{HS}^q d\xi \leq \|g\|_1^q \int_{\mathbb{R}^{n-2}} |\xi_1| e_{bq}(\|\xi\|) \|\pi_\xi(f)\|_{HS}^q d\xi < \infty.$$

Now for $\epsilon > 0$ sufficiently small, $ab\varphi(\epsilon) > 1$ so by Theorem 2 it follows that $g * f = 0$. Taking for g an approximate identity, we conclude that $f = 0$ a.e. \square

The following result follows from Theorem 2, Remark 3 and Theorem 4.

THEOREM 5. *If $G_n = G_3, G_{5,1}, G_{5,3}$ or $G_{5,6}$ and $a, b > 0$. Suppose that p and q are such that $1 \leq \min(p, q) < \infty$ and $f \in L^1 \cap L^2(G_n)$ satisfies*

- (i) $\int_{\mathbb{R}^n} e^{pa\pi\|x\|^2} |f(x)|^p dx < \infty$ if $p < \infty$ and $|f(x)| \leq Ce^{-a\pi\|x\|^2}$ if $p = \infty$,
(ii) $\int_{\mathbb{R}^{n-2}} |\xi_1| e^{b\pi q\|\xi\|^2} \|\pi_\xi(f)\|_{HS}^q d\xi < \infty$ if $q < \infty$ and $\|\pi_\xi(f)\|_{HS} \leq Ce^{-b\pi\|\xi\|^2}$ if $q = \infty$.

Then the following hold:

- (1) If $q \geq 2$ and $ab > 1$, then $f = 0$ a.e.
- (2) If $1 \leq q < 2$ and $ab > 2$, then $f = 0$ a.e.

Let $G = \exp \mathfrak{g}$ be a simply connected nilpotent Lie group. Let U denote the Zariski open subset of \mathfrak{g}^* consisting of all elements in generic orbits with respect to the basis $\{X_1^*, \dots, X_n^*\}$ [2, Section 3.1, Theorem 3.1.9]. Let S be the set of jump indices, and set $T = \{1, 2, \dots, n\} \setminus S$ and $\mathfrak{g}_T^* = \sum_{j \in T} \mathbb{R} X_j^*$.

Then $X = U \cap \mathfrak{g}_T^*$ is a cross-section for the generic orbits and $\{\pi_\xi : \xi \in X\}$ supports the Plancherel measure on \widehat{G} .

The following is a generalization of Morgan's Theorem [8] which can be proved using [7, Lemma 2].

THEOREM 6. *Let $G = \exp \mathfrak{g}$ be a simply connected nilpotent Lie group. Let α, β and C be positive real numbers and suppose that $f : G \rightarrow \mathbb{C}$ is a measurable function such that*

- (i) $|f(x)| \leq Ce^{-\alpha\pi\|x\|^p}$,
- (ii) $\|\pi_\xi(f)\|_{HS} \leq Ce^{-\beta\pi\|\xi\|^q}$ for all $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in X$,

where $p \geq 2$, $1/p + 1/q = 1$. If $(\alpha p)^{1/p} (\beta q)^{1/q} > 2$ then $f = 0$ a.e.

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Department of Mathematics
Rajdhani College
(University of Delhi)
Raja Garden, New Delhi - 110 015
India
e-mail: ajaykr@bol.net.in

Department of Mathematics
University of Delhi
Delhi - 110 007
India
e-mail: crbhatta@yahoo.com