

# WAVELET DECOMPOSITION OF CALDERÓN-ZYGMUND OPERATORS ON FUNCTION SPACES

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## Abstract

We make use of the Beylkin-Coifman-Rokhlin wavelet decomposition algorithm on the Calderón-Zygmund kernel to obtain some fine estimates on the operator and prove the  $T(1)$  theorem on Besov and Triebel-Lizorkin spaces. This extends previous results of Frazier *et al.*, and Han and Hofmann.

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## 1. Introduction

In recent years, there has been significant progress on the problem of boundedness of generalized Calderón-Zygmund operators on various function spaces. The operators in question can be described as follows. Let  $K(x, y)$  be a continuous function defined on  $(\mathbb{R}^n \times \mathbb{R}^n) \setminus \{x = y\}$  and let  $T : \mathcal{D} \rightarrow \mathcal{D}'$  be the linear operator associated with the kernel  $K(x, y)$ , that is,

$$\langle T\varphi, \psi \rangle = \iint_{\mathbb{R}^n \times \mathbb{R}^n} K(x, y)\varphi(y)\psi(x) dy dx$$

where  $\varphi, \psi \in \mathcal{D}$  are  $C^\infty$ -test functions on  $\mathbb{R}^n$  with disjoint supports. For convenience, we write

$$\Delta K(x, x'; y, y') = |K(x, y) - K(x', y')| + |K(y, x) - K(y', x')|.$$

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It is customary to assume that  $K(x, y)$  satisfies the following pointwise conditions:

$$(1.1) \quad |K(x, y)| \leq C|x - y|^{-n}, \quad \text{and}$$

$$(1.2) \quad |\Delta K(x, x'; y, y)| \leq C|x' - x|^\gamma |x - y|^{-n-\gamma} \quad \text{for } |x - y| \geq 2|x' - x|,$$

where  $0 < \gamma \leq 1$ .

In their celebrated paper [4], David and Journé characterized the type of kernel  $K(x, y)$  for which  $T$  is a bounded operator on  $L^2$ . This is now called the  $T(1)$  theorem. They proved that under conditions (1.1) and (1.2) on  $K(x, y)$ ,  $T$  extends to a bounded operator on  $L^2$  if and only if both  $T(1)$  and  $T^*(1)$  are BMO functions, and  $T$  has the following *weak boundedness property* (WBP): For  $\varphi, \psi \in \mathcal{D}$  with  $\text{diam}(\text{supp } \varphi), \text{diam}(\text{supp } \psi) \leq t$ ,

$$(1.3) \quad |\langle T\varphi, \psi \rangle| \leq t^n (\|\varphi\|_\infty + t\|\nabla\varphi\|_\infty) (\|\psi\|_\infty + t\|\nabla\psi\|_\infty).$$

Later, Meyer [11] improved the theorem by replacing the pointwise assumption with the following integral assumption on  $K(x, y)$ :

$$(1.1') \quad \sup_{r>0} \int_{r \leq |x-y| < 2r} (|K(x, y)| + |K(y, x)|) dy \leq C, \quad \text{and}$$

$$(1.2') \quad \sum_{k=0}^{\infty} (k+1)B(k) < \infty, \quad \text{with}$$

$$B(k) = \sup_{\substack{r>0 \\ |u|+|v| \leq r}} \left( \int_{2^k r \leq |x-y| < 2^{k+1} r} \Delta K(x+u, x; y+v, y) dy \right).$$

The  $T(1)$  theorem has also been considered by Lemarié on the Besov spaces [10], Frazier *et al* on the Triebel-Lizorkin spaces [7], and Han and Hofmann on both classes of spaces [8]. The definitions of such spaces can be stated as follows (see [13]):

Let  $\mathcal{S}(\mathbb{R}^n)$  be the space of tempered test functions. Let  $\varphi \in \mathcal{S}(\mathbb{R}^n)$  with  $\text{supp } \hat{\varphi} \subset \{\xi \in \mathbb{R}^n : 1/2 \leq |\xi| \leq 2\}$  and  $|\hat{\varphi}(\xi)| \geq c > 0$  for  $3/5 \leq |\xi| \leq 5/3$ ; put  $\varphi_j(x) = 2^{jn}\varphi(2^jx)$  and  $\mathcal{Q}_j(f)(x) = \varphi_j * f(x)$ . For  $\alpha \in \mathbb{R}$  and  $0 < p, q < \infty$ , the Besov spaces  $\dot{B}_p^{\alpha, q}$  is the collection of all  $f \in \mathcal{S}'/\mathcal{P}$  (the tempered distributions modulo polynomials) satisfying

$$\|f\|_{\dot{B}_p^{\alpha, q}} = \left( \sum_j (2^{j\alpha} \|\mathcal{Q}_j f\|_p)^q \right)^{1/q} < \infty.$$

The Triebel-Lizorkin space is defined analogously,  $\dot{F}_p^{\alpha, q}$  being the collection of all  $f \in \mathcal{S}'/\mathcal{P}$  such that

$$\|f\|_{\dot{F}_p^{\alpha, q}} = \left\| \left( \sum_j (2^{j\alpha} |\mathcal{Q}_j f|)^q \right)^{1/q} \right\|_p < \infty.$$

In this paper, we will prove the following two theorems:

**THEOREM 1.1.** *Suppose  $T$  satisfies the WBP (1.3), the kernel  $K(x, y)$  satisfies*

- (i) (1.1') and (1.2'). If  $T(1) = T^*(1) = 0$ , then  $T$  is bounded on  $\dot{B}_p^{0,q}$ ,  $1 \leq p$ ,  $q < \infty$ .
- (ii) (1.1') and  $\sum_{k=0}^{\infty} 2^{k\alpha} B(k) < \infty$ . If  $T(1) = 0$ , then  $T$  is bounded on  $\dot{B}_p^{\alpha,q}$ ,  $0 < \alpha < 1$  and  $1 \leq p, q < \infty$ .

**THEOREM 1.2.** *Suppose  $T$  satisfies the WBP (1.3), the kernel  $K(x, y)$  satisfies*

- (i) (1.1') and  $\sum_{k=0}^{\infty} (k+1)^{2-1/q} B(k) < \infty$ . If  $T(1) = T^*(1) = 0$ , then  $T$  is bounded on  $\dot{F}_p^{0,q}$ ,  $1 \leq p, q < \infty$ .
- (ii) (1.1') and  $\sum_{k=0}^{\infty} 2^{k\alpha} B(k) < \infty$ . If  $T(1) = 0$ , then  $T$  is bounded on  $\dot{F}_p^{\alpha,q}$ ,  $0 < \alpha < 1$  and  $1 \leq p, q < \infty$ .

We remark that the two theorems extend the results of Han and Hofmann [8]; they need to assume that  $B(k) \leq C2^{-k\epsilon}$  for  $0 < \alpha < \epsilon$  in both theorems. For  $-1 < \alpha < 0$ , Theorem 1.1 and Theorem 1.2 also hold by interchanging the role of  $T(1)$  and  $T^*(1)$  because of the duality (the dual of  $\dot{B}_p^{\alpha,q}$  is  $\dot{B}_{p'}^{-\alpha,q'}$  and similarly for  $\dot{F}_p^{\alpha,q}$ ).

Note that  $\dot{F}_p^{0,2}$  is of special interest because it is the Hardy space  $H^1$  when  $p = 1$  and is  $L^p$  when  $p > 1$ . For the Hardy space  $H^1$ , the kernel condition in Theorem 1.2 is  $\sum_k (k+1)^{3/2} B(k) < \infty$ . In [5],  $T$  is proved to be bounded on  $L^2$  under the kernel condition  $\sum_k (k+1)^{1/2} B(k) < \infty$ . By the interpolation theorem, a direct application of the theorem yields the following result, which is stronger than the corresponding case stated in (i).

**COROLLARY 1.3.** *Suppose  $T$  satisfies the WBP (1.3), the kernel  $K(x, y)$  satisfies (1.1') and  $\sum_k (k+1)^{1/2+2|1/p-1/2|} B(k) < \infty$ . If  $T(1) = T^*(1) = 0$ , then  $T$  is bounded on  $L^p$ ,  $1 < p < \infty$ .*

The main tool used in proving the theorems is wavelets, initiated in [2] and [5]. This is quite different from the approaches in [7, 8, 10, 11]. The proof of Theorem 1.1 depends on the Beylkin-Coifman-Rohklin wavelet decomposition of the operator  $T$ . For Theorem 1.2, we first prove the boundedness of  $T$  on  $\dot{F}_1^{\alpha,q}$  using an atomic decomposition on this space. This, together with an interpolation on  $\dot{F}_p^{\alpha,p}$  ( $= \dot{B}_p^{\alpha,p}$ ), yields the boundedness of  $T$  for the other case.

The paper is organized as follows. In Section 2 we will give some preliminaries on wavelets and the BCR decomposition of  $T$ . We also set up the proof in terms of wavelet terminology. The  $T(1)$  theorem on the Besov spaces is proved in Section 3 and on the Triebel-Lizorkin spaces in Section 4.

## 2. Preliminaries

For simplicity, we only consider the one dimensional case. The higher dimensional case is similar.

Let us recall the concept of *multiresolution analysis* in  $L^2(\mathbb{R})$  [12]: it is an increasing sequence of closed linear subspaces  $\{V_j\}_{j \in \mathbb{Z}} \subseteq L^2(\mathbb{R})$  with the following properties:

- (i)  $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$ ,  $\bigcup_{j \in \mathbb{Z}} V_j$  is dense in  $L^2(\mathbb{R})$ ;
- (ii) For every  $j \in \mathbb{Z}$  and  $f \in L^2(\mathbb{R})$ ,  $f \in V_j \iff f(2 \cdot) \in V_{j+1}$ ;
- (iii) There exists a  $\varphi$  in  $V_0$  such that  $\varphi(x - k)$ ,  $k \in \mathbb{Z}$ , is an orthonormal basis for  $V_0$ .

The above  $\varphi$  is called a *scaling function*. Note that by adjusting a normalization constant,  $\sum_k \phi(x - k) = 1$  for all  $x \in \mathbb{R}$  [3]. For each  $j \in \mathbb{Z}$ , we define  $\varphi_{jk}(x) = 2^{j/2} \varphi(2^j x - k)$ ,  $k \in \mathbb{Z}$ . The sequence  $\{\varphi_{jk}\}_{k \in \mathbb{Z}}$  forms an orthonormal basis for  $V_j$ . From  $\varphi$  we can construct a wavelet function  $\psi$ . Then  $\{\psi_{jk}\}_{k \in \mathbb{Z}}$  forms an orthonormal basis for  $W_j$ , the orthogonal complement to  $V_j$  inside  $V_{j+1}$ , that is,  $V_{j+1} = V_j \oplus W_j$ . It follows that  $\{\psi_{jk}\}_{j,k \in \mathbb{Z}}$  is an orthonormal basis for  $L^2(\mathbb{R})$ . In this paper, we assume that the wavelets are compactly supported, say  $\text{supp } \varphi, \text{supp } \psi \subseteq [0, M]$  for some integer  $M$ . Also we assume that they have the desirable degree of smoothness whenever needed.

We need the following characterizations of the Besov and Triebel-Lizorkin spaces [6, 12].

**PROPOSITION 2.1.** *Suppose  $\psi \in C^\gamma$  is a compactly supported wavelet and  $\{\psi_{jk}\}_{j,k \in \mathbb{Z}}$  forms an orthonormal basis of  $L^2(\mathbb{R})$ . Let  $f$  be locally integrable and write  $f(x) = \sum_{j,k} \alpha(j, k) \psi_{jk}(x)$  formally.*

- (i) For  $0 < \alpha < \gamma$ ,  $1 \leq p, q < \infty$ ,  $f \in \dot{B}_p^{\alpha,q}$  if and only if

$$\left( \sum_j \left( \sum_k |2^{(-1/p + \alpha + 1/2)j} \alpha(j, k)|^p \right)^{q/p} \right)^{1/q} < \infty.$$

- (ii) For  $0 < \alpha < \gamma$ ,  $1 \leq p, q < \infty$ ,  $f \in \dot{F}_p^{\alpha,q}$  if and only if

$$A(f)(x) = \left( \sum_{j,k} 2^{(\alpha+1/2)jq} |\alpha(j, k)|^q \chi(2^j x - k) \right)^{1/q} \in L^p(\mathbb{R}),$$

where  $\chi$  denotes the characteristic function of  $[0, 1)$ . In this case,  $\|f\|_{\dot{F}_p^{\alpha,q}} \approx \|A(f)\|_p$ .

Let  $P_j : L^2(\mathbb{R}) \rightarrow V_j$  be the orthonormal projection and  $Q_j = P_{j+1} - P_j$ . Then  $Q_j : L^2(\mathbb{R}) \rightarrow W_j$  is the corresponding orthonormal projection. In [2], Beylkin,

Coifman and Rokhlin give a decomposition of  $T$  in terms of  $P_j$  and  $Q_j$ :

$$(2.1) \quad T = \sum_{-\infty}^{\infty} P_j T Q_j + \sum_{-\infty}^{\infty} Q_j T P_j + \sum_{-\infty}^{\infty} Q_j T Q_j.$$

The corresponding distribution kernel is

$$(2.2) \quad K(x, y) = \sum_{jkl} a(j, k, l) \varphi_{jk}(x) \psi_{jl}(y) + \sum_{jkl} b(j, k, l) \psi_{jk}(x) \varphi_{jl}(y) \\ + \sum_{jkl} c(j, k, l) \psi_{jk}(x) \psi_{jl}(y),$$

where

$$a(j, k, l) = \langle T \psi_{jl}, \varphi_{jk} \rangle = \langle K, \varphi_{jk} \otimes \psi_{jl} \rangle, \\ b(j, k, l) = \langle T \varphi_{jl}, \psi_{jk} \rangle = \langle K, \psi_{jk} \otimes \varphi_{jl} \rangle, \\ c(j, k, l) = \langle T \psi_{jl}, \psi_{jk} \rangle = \langle K, \psi_{jk} \otimes \psi_{jl} \rangle.$$

We call such  $a(j, k, l)$ ,  $b(j, k, l)$ ,  $c(j, k, l)$  the BCR-coefficients.

It is easy to show that

**PROPOSITION 2.2.** *Suppose  $T$  satisfies the conditions in Theorem 1.1 or Theorem 1.2, then  $T^*(1) = 0$  implies that for any  $j, l \in \mathbb{Z}$ ,  $\sum_k a(j, k, l) = 0$ ; similarly  $T(1) = 0$  implies that for any  $j, k \in \mathbb{Z}$ ,  $\sum_l b(j, k, l) = 0$ .*

**PROOF.** Assuming that  $T^*(1) = 0$  and using  $\sum_k \varphi(x - k) = 1$ , we have

$$\sum_k a(j, k, l) = \sum_k \langle T \psi_{jl}, \varphi_{jk} \rangle = \langle T \psi_{jl}, 2^{j/2} \rangle = \langle \psi_{jl}, 2^{j/2} T^*(1) \rangle = 0.$$

The second part can be proved similarly. □

**PROPOSITION 2.3.** *Suppose  $T$  satisfies the conditions in Theorem 1.1 or Theorem 1.2. Let*

$$A(m) = \sup_{j,l} \left\{ \sum_{k: 2^m \leq |k-l| < 2^{m+1}} (|a(j, k, l)| + |a(j, l, k)| + |b(j, k, l)| + |b(j, l, k)|) \right\}.$$

*Then there exists  $C$  such that  $A(m) \leq C B(m)$  for all  $m \geq 0$ .*

*Moreover we have*

$$(2.3) \quad \sup_{j,l} \sum_k (|c(j, k, l)| + |c(j, l, k)|) < \infty.$$

**PROOF.** We first observe that  $A(m) < \infty$  for each  $m \geq 0$ . This comes directly from the WBP in (1.3) and the expressions for  $a(j, k, l)$  and  $b(j, k, l)$ . By using this, it suffices to prove the inequality for  $2^m > M$ .

Let  $y_0 = 2^{-j}l$ , then

$$\begin{aligned} a(j, k, l) &= \iint K(x, y) \varphi_{jk}(x) \psi_{jl}(y) dx dy \\ &= 2^j \iint (K(x, y + y_0) - K(x, y_0)) \varphi(2^j x - k) \psi(2^j y) dx dy \\ &\leq C \sup_{y \in [0, 2^{-j}M]} \int_{[k2^{-j}, (k+M)2^{-j}]} |K(x, y + y_0) - K(x, y_0)| dx. \end{aligned}$$

Hence for  $2^m > M$ ,

$$\sum_{k: 2^m \leq |k-l| < 2^{m+1}} |a(j, k, l)| \leq C \sup_{y \in [0, 2^{-j}M]} \int_E |K(x, y + y_0) - K(x, y_0)| dx,$$

where  $E = \{x \in \mathbb{R} : 2^{m-j} \leq |x - y_0| \leq 2^{m+1-j} + 2^{-j}M\}$ . According to the definition of  $B(m)$ , we have  $\sum_{k: 2^m \leq |k-l| < 2^{m+1}} |a(j, k, l)| \leq CB(m)$  for  $2^m > M$ . The same argument applies to the other terms in  $A(m)$ , which completes the proof for the first assertion.

The proof of (2.3) is essentially the same. We consider

$$\sup_{j, l} \left\{ \sum_{k \in J} (|c(j, k, l)| + |c(j, l, k)|) \right\}.$$

It is bounded if  $J = \{k : |k - l| \leq M\}$ ; and it is  $\leq CB(m)$  if  $J = \{k : 2^m \leq |k - l| < 2^{m+1}\}$  if  $2^{m+1} > M$ . This implies (2.3).  $\square$

### 3. $T(1)$ Theorem on Besov spaces

In view of Proposition 2.2 and Proposition 2.3, we will prove the following theorem in terms of the BCR-coefficients, which implies Theorem 1.1.

**THEOREM 3.1.** *Let  $T : \mathcal{D} \rightarrow \mathcal{D}'$  be a Calderón-Zygmund operator with the wavelet decomposition as in (2.1), (2.2) and satisfying (1.1'), (2.3).*

(i) *If  $\sum_{m=0}^{\infty} (m+1)A(m) < \infty$  and  $\sum_k a(j, k, l) = \sum_l b(j, k, l) = 0$ , then  $T$  is a bounded operator on  $\dot{B}_p^{0,q}$ ,  $1 \leq p, q < \infty$ .*

(ii) *If  $\sum_{m=0}^{\infty} 2^{\alpha m} A(m) < \infty$  and  $\sum_l b(j, k, l) = 0$  for any  $j, k \in \mathbb{Z}$ , then  $T$  is bounded on  $\dot{B}_p^{\alpha,q}$ ,  $0 < \alpha < 1$ ,  $1 \leq p, q < \infty$ .*

Let us rewrite

$$T = \sum_{-\infty}^{\infty} P_j T Q_j + \sum_{-\infty}^{\infty} Q_j T P_j + \sum_{-\infty}^{\infty} Q_j T Q_j = T^{(1)} + T^{(2)} + T^{(3)}.$$

We will first consider the term  $T^{(2)}$ . Its distributional kernel is

$$K^{(2)}(x, y) = \sum_{jkl} b(j, k, l) \psi_{jk}(x) \varphi_{jl}(y).$$

Let  $J_m = \{(k, l) : 2^m \leq |k - l| < 2^{m+1}\}$ , and

$$(3.1) \quad b_m(j, k, l) = \begin{cases} b(j, k, l) & (k, l) \in J_m; \\ -\sum_{n:(k,n) \in J_m} b(j, k, n) & l = k; \\ 0 & \text{otherwise,} \end{cases}$$

where  $m = 0, 1, 2, \dots$ . The definition implies that  $\sum_l b_m(j, k, l) = 0$  for each  $j, k \in \mathbb{Z}$ . Since  $\sum_l b(j, k, l) = 0$  by assumption, we have

$$b(j, k, k) = -\sum_{m=0}^{\infty} \sum_{l:(k,l) \in J_m} b(j, k, l).$$

Hence  $K^{(2)}(x, y)$  can be decomposed as

$$(3.2) \quad \begin{aligned} K^{(2)}(x, y) &= \sum_{jk} b(j, k, k) \psi_{jk}(x) \varphi_{jk}(y) \\ &\quad + \sum_{jk} \sum_{l:l \neq k} b(j, k, l) \psi_{jk}(x) \varphi_{jl}(y) \\ &= \sum_{jk} \left( -\sum_{m=0}^{\infty} \sum_{l:(k,l) \in J_m} b(j, k, l) \right) \psi_{jk}(x) \varphi_{jk}(y) \\ &\quad + \sum_{m=0}^{\infty} \sum_{jk} \sum_{l:(k,l) \in J_m} b(j, k, l) \psi_{jk}(x) \varphi_{jl}(y) \\ &= \sum_{m=0}^{\infty} \sum_{jkl} b_m(j, k, l) \psi_{jk}(x) \varphi_{jl}(y) = \sum_{m=0}^{\infty} K_m^{(2)}(x, y). \end{aligned}$$

Let  $T_m^{(2)}$  denote the operator with distributional kernel  $K_m^{(2)}(x, y)$ . Then we can decompose  $T^{(2)}$  as:  $T^{(2)} = \sum_{m=0}^{\infty} T_m^{(2)}$ . We call each  $T_m^{(2)}$  a *block operator*. The following lemma together with the assumption on  $A(m)$  will imply that  $T^{(2)}$  is a bounded operator on  $\dot{B}_p^{\alpha, q}$ .

**LEMMA 3.2.** *Under the hypothesis of Theorem 3.1, let  $1 \leq p, q < \infty$ ,  $0 \leq \alpha < 1$ . Then  $T_m^{(2)}$  is a bounded operator on  $\dot{B}_p^{\alpha, q}$ , and the operator norm satisfies*

$$\|T_m^{(2)}\| \leq \begin{cases} C(m+1)A(m) & \alpha = 0; \\ C2^{\alpha m}A(m) & 0 < \alpha < 1, \end{cases}$$

where  $C$  is independent of  $m$ .

**PROOF.** Let  $f(y) = \sum_{jk} \alpha(j, k) \psi_{jk}(y)$  be in  $\dot{B}_p^{\alpha, q}$ , and let  $g(x) = \sum_{jk} \beta(j, k) \psi_{jk}(x)$  be in the dual space  $\dot{B}_{p'}^{-\alpha, q'}$ . Noting that  $\langle \varphi_{jl}, \psi_{j'k'} \rangle \neq 0$  implies that  $j > j'$  and  $2^{j-j'}k' \leq l \leq 2^{j-j'}(k' + M) + M$  (recall that  $\varphi, \psi$  have compact supports contained in  $[0, M]$ ), one can write

$$\begin{aligned} \langle T_m^{(2)} f, g \rangle &= \sum_{j'k'} \sum_{jkl} \alpha(j', k') b_m(j, k, l) \beta(j, k) \langle \varphi_{jl}, \psi_{j'k'} \rangle \\ &= \sum_{j, j': 0 < j - j' \leq m} \left( \sum_{kk'l} \alpha(j', k') b_m(j, k, l) \beta(j, k) \langle \varphi_{jl}, \psi_{j'k'} \rangle \right) \\ &\quad + \sum_{j, j': j - j' \geq m+1} \left( \sum_{kk'l} \alpha(j', k') b_m(j, k, l) \beta(j, k) \langle \varphi_{jl}, \psi_{j'k'} \rangle \right) \\ &= \text{I} + \text{II}. \end{aligned}$$

Let  $\Gamma_{kk'l}^{mj's} = b_m(j' + s, k, l) \langle \varphi_{j'+sl}, \psi_{j'k'} \rangle$ . By using Proposition 2.1 and the Hölder inequality, we obtain

$$\begin{aligned} |\text{II}| &\leq \sum_{s=1}^m \sum_{j'} \left| \sum_{kk'l} \alpha(j', k') b_m(j' + s, k, l) \beta(j' + s, k) \langle \varphi_{j'+sl}, \psi_{j'k'} \rangle \right| \\ &\leq \sum_{s=1}^m \left\{ \sum_{j'} \left( \sum_{k'} |\alpha(j', k')|^p \right)^{1/p} \left( \sum_{k'l} \left| \sum \beta(j' + s, k) \Gamma_{kk'l}^{mj's} \right|^{p'} \right)^{1/p'} \right\} \\ &\leq C \sum_{s=1}^m 2^{(1/p' + \alpha - 1/2)s} \|f\|_{\dot{B}_p^{\alpha, q}} \\ &\quad \times \left\{ \sum_{j'} \left( \sum_{k'} \left| 2^{(-1/p' - \alpha + 1/2)(j'+s)} \sum_{kl} \beta(j' + s, k) \Gamma_{kk'l}^{mj's} \right|^{p'} \right)^{q'/p'} \right\}^{1/q'}. \end{aligned}$$



We will make a separate estimation of  $\sum_{k'} \dots$ . Note that

$$\begin{aligned} \sum_{kl} \left| \Gamma_{kk'l}^{mj's} \right| &\leq 2^{-s/2} \sum_l \sum_k |b_m(j' + s, k, l) \langle \varphi(x - l), \psi(2^{-s}x - k') \rangle| \\ &\leq 2^{-s/2} A(m) \sum_l |\langle \varphi(x - l), \psi(2^{-s}x - k') \rangle| \\ &\leq C 2^{s/2} A(m), \end{aligned}$$

(the last inequality holds because  $\varphi$  has compact support and  $\sum_l |\varphi(x - l)|$  is bounded) and

$$\begin{aligned} \sum_{k'l} \left| \Gamma_{kk'l}^{mj's} \right| &\leq 2^{-s/2} \sum_l |b_m(j' + s, k, l)| \left\langle |\varphi(x - l)|, \sum_{k'} |\psi(2^{-s}x - k')| \right\rangle \\ &\leq C 2^{-s/2} A(m). \end{aligned}$$

Hence

$$\begin{aligned} (3.3) \quad \sum_{k'} \left| \sum_{kl} \beta(j' + s, k) \Gamma_{kk'l}^{mj's} \right|^{p'} &\leq \sum_{k'} \left\{ \left( \sum_{kl} |\beta(j' + s, k)|^{p'} |\Gamma_{kk'l}^{mj's}| \right) \left( \sum_{kl} |\Gamma_{kk'l}^{mj's}| \right)^{p'/p} \right\} \\ &\leq C \left( \sum_k |\beta(j' + s, k)|^{p'} \sum_{k'l} |\Gamma_{kk'l}^{mj's}| \right) (2^{s/2} A(m))^{p'/p} \\ &\leq C \left( \sum_k |\beta(j' + s, k)|^{p'} \right) (2^{-s/2} A(m)) (2^{s/2} A(m))^{p'/p} \\ &= C 2^{s(p'/p-1)/2} |A(m)|^{p'} \sum_k |\beta(j' + s, k)|^{p'}. \end{aligned}$$

It follows that

$$\begin{aligned} |\mathbf{I}| &\leq C \sum_{s=1}^m 2^{(1/p'+\alpha-1/2)s} 2^{s(1/p-1/p')/2} \cdot A(m) \|f\|_{\dot{B}_p^{\alpha,q}} \\ &\quad \times \left\{ \sum_{j'} \left( \sum_k |2^{(-1/p'-\alpha+1/2)(j'+s)} \beta(j' + s, k)|^{p'} \right)^{q'/p'} \right\}^{1/q'} \\ &\leq C \sum_{s=1}^m 2^{\alpha m} A(m) \|f\|_{\dot{B}_p^{\alpha,q}} \|g\|_{\dot{B}_{p'}^{-\alpha,q'}} \\ &\leq \begin{cases} C(m+1)A(m) \|f\|_{\dot{B}_p^{\alpha,q}} \|g\|_{\dot{B}_{p'}^{-\alpha,q'}} & \alpha = 0; \\ C 2^{\alpha m} A(m) \|f\|_{\dot{B}_p^{\alpha,q}} \|g\|_{\dot{B}_{p'}^{-\alpha,q'}} & 0 < \alpha < 1. \end{cases} \end{aligned}$$

We now estimate the expression II. For convenience, we use the same notation  $C$  to denote the different constants in the different place. Define

$$g_{jk}(x) = \int_{-\infty}^x \sum_l b_m(j, k, l) \varphi(y - l) dy \quad \text{and} \quad \tilde{\Gamma}_{kk'}^{j's} = \langle g_{j'+sk}(x), \psi'(2^{-s}x - k) \rangle.$$

Then

$$\begin{aligned} |\text{III}| &= \left| \sum_{j, j': j-j' \geq m+1} \sum_{kk'} \alpha(j', k') b_m(j, k, l) \beta(j, k) \langle \varphi_{j,l}, \psi_{j'k'} \rangle \right| \\ &= C \sum_{s=m+1}^{\infty} \sum_{j'=-\infty}^{\infty} \sum_{kk'} 2^{-3s/2} \left| \alpha(j', k') \beta(j' + s, k) \tilde{\Gamma}_{kk'}^{j's} \right| \\ &\leq C \sum_{s=m+1}^{\infty} 2^{-3s/2} \|f\|_{\dot{B}_p^{\alpha, q}} \\ &\quad \times \left\{ \sum_{j'} 2^{(-1/p' - \alpha + 1/2)j'q'} \left( \sum_{k'} \left| \sum_k \beta(j' + s, k) \tilde{\Gamma}_{kk'}^{j's} \right|^{p'} \right)^{q'/p'} \right\}^{1/q'}. \end{aligned}$$

We claim that

$$(3.4) \quad \sum_{k'} \left| \sum_k \beta(j' + s, k) \tilde{\Gamma}_{kk'}^{j's} \right|^{p'} \leq C 2^{(s+m)(p'-1)} 2^m |A(m)|^{p'} \sum_k |\beta(j' + s, k)|^{p'}.$$

In fact, the condition  $\sum_l b_m(j, k, l) = 0$  implies that  $\text{supp } g_{jk} \subseteq [k - 2^{m+1}, k + 2^{m+1}M]$ . Then

$$\begin{aligned} \sum_{k'} \left| \tilde{\Gamma}_{kk'}^{j's} \right| &\leq C \left\langle |g_{j'+sk}(x)|, \sum_{k'} |\psi'(2^{-s}x - k')| \right\rangle \\ &\leq C \langle |g_{j'+sk}(x)|, 1 \rangle \leq C 2^m A(m). \end{aligned}$$

On the other hand, for  $s = j - j'$  since  $\text{supp } \psi'(2^{-s}x - k') \subseteq [2^s k', 2^s(k' + M)]$ , we know that  $|k - 2^s k'| \leq 2^s M + 2^m + M \leq C 2^s$ . It follows that

$$\sum_k \left| \tilde{\Gamma}_{kk'}^{j's} \right| \leq C \sum_k \langle |g_{j'+sk}(x)|, |\psi'(2^{-s}x - k')| \rangle \leq C 2^{s+m} A(m).$$

Combining these two estimates we have

$$\begin{aligned} & \sum_{k'} \left| \sum_k \beta(j' + s, k) \tilde{\Gamma}_{kk'}^{j's} \right|^{p'} \\ & \leq \sum_{k'} \left( \sum_k |\beta(j' + s, k)|^{p'} \left| \tilde{\Gamma}_{kk'}^{j's} \right| \right) \left( \sum_k \left| \tilde{\Gamma}_{kk'}^{j's} \right| \right)^{p'/p} \\ & \leq C 2^{(s+m)(p'-1)} 2^m |A(m)|^{p'} \sum_k |\beta(j' + s, k)|^{p'}. \end{aligned}$$

This proves the claim. We return to the estimate of |II|

$$\begin{aligned} |\text{II}| & \leq C \sum_{s=m+1}^{\infty} 2^{(1/p'+\alpha-2)s} \|f\|_{\dot{B}_p^{\alpha,q}} \left\{ \sum_{j'} \left( 2^{(s+m)(p'-1)} 2^m |A(m)|^{p'} \right. \right. \\ & \quad \left. \left. \times \sum_k \left| 2^{(-1/p'-\alpha+1/2)(j'+s)} \beta(j' + s, k) \right|^{p'} \right)^{q'/p'} \right\}^{1/q'} \\ & \leq C \left( \sum_{s=m+1}^{\infty} 2^{(\alpha-1)s} 2^m A(m) \right) \|f\|_{\dot{B}_p^{\alpha,q}} \|g\|_{\dot{B}_p^{-\alpha,q'}} \\ & \leq C 2^{am} A(m) \|f\|_{\dot{B}_p^{\alpha,q}} \|g\|_{\dot{B}_p^{-\alpha,q'}}. \end{aligned}$$

By the estimates of |I| and |II| we conclude that  $T^{(2)}$  is a bounded operator on  $\dot{B}_p^{\alpha,q}$ ,  $1 \leq p, q < \infty$ ,  $0 \leq \alpha < 1$  with the operator norm as specified.  $\square$

**LEMMA 3.3.** *Under the hypothesis of Theorem 3.1, let  $1 \leq p, q < \infty$ ,  $0 \leq \alpha < 1$ . Then  $T^{(1)}$  is a bounded operator on  $\dot{B}_p^{\alpha,q}$ .*

**PROOF.** We can write

$$\langle T^{(1)} f, g \rangle = \sum_{jj'kk'} a(j, k, l) \beta(j', k') \alpha(j, l) \langle \varphi_{jk}, \psi_{j'k'} \rangle$$

with  $j > j'$  and  $2^{j-j'} k' \leq l \leq 2^{j-j'} (k' + M) + M$  as in Lemma 3.2. For the case  $\alpha = 0$ , we have by assumption that  $\sum_k a(j, k, l) = 0$ , and we can apply the same proof as in Lemma 3.2 (by replacing  $\sum_l b(j, k, l) = 0$ ).

For the case  $0 < \alpha < 1$ , we have not assumed that  $\sum_k a(j, k, l) = 0$  and we need

to modify the proof by separating out the diagonal term (as in (3.2)):

$$\begin{aligned} |\langle T^{(1)} f, g \rangle| \leq & \left| \sum_{j, j': j-j' > 0} \sum_{kk'} \beta(j', k') a(j, k, k) \alpha(j, k) \langle \varphi_{jk}, \psi_{j'k'} \rangle \right| \\ & + \left| \sum_{m=0}^{\infty} \sum_{j, j': j-j' > 0} \sum_{(k, l) \in J_m} \sum_{k'} \beta(j', k') a(j, k, l) \alpha(j, l) \langle \varphi_{jk}, \psi_{j'k'} \rangle \right|. \end{aligned}$$

By using the same argument as in Lemma 3.2, we can show that the first term is bounded by  $C \sum_{s=1}^{\infty} 2^{-\alpha s} \|f\|_{\dot{B}_p^{\alpha, q}} \|g\|_{\dot{B}_p^{-\alpha, q'}}$  and the second term is bounded by  $CA(m) \sum_{s=1}^{\infty} 2^{-\alpha s} \|f\|_{\dot{B}_p^{\alpha, q}} \|g\|_{\dot{B}_p^{-\alpha, q'}}$  (note that in here the term  $\sum_{s=1}^{\infty} 2^{-\alpha s}$  converges and we only need to use the estimation for |I| without recouring to the estimation |II| in the last proof). This proves the lemma.  $\square$

**LEMMA 3.4.** *Under the hypothesis of Theorem 3.1 let  $1 \leq p, q < \infty$ ,  $0 \leq \alpha < 1$ . Then  $T^{(3)}$  is a bounded operator on the Besov spaces  $\dot{B}_p^{\alpha, q}$ .*

**PROOF.** We can write

$$\langle T^{(3)} f, g \rangle = \sum_{jkl} c(j, k, l) \alpha(j, k) \beta(j, l),$$

where  $f, g$  are defined as in Lemma 3.2. By using the duality and condition (2.3), it can be checked as in Lemma 3.2 that  $|\langle T^{(3)} f, g \rangle| \leq C \|f\|_{\dot{B}_p^{\alpha, q}} \|g\|_{\dot{B}_p^{-\alpha, q'}}$ .  $\square$

Theorem 3.1 follows directly from Lemmas 3.2–3.4.  $\square$

#### 4. $T(1)$ Theorem on Triebel-Lizorkin spaces

In view of Proposition 2.2 and Proposition 2.3, we will prove the following theorem in terms of the BCR-coefficients, which implies Theorem 1.2.

**THEOREM 4.1.** *Let  $T : \mathcal{D} \rightarrow \mathcal{D}'$  be a Calderón-Zygmund operator with the wavelet decomposition as in (2.1), (2.2) and satisfying (1.1'), (2.3).*

(i) *If  $\sum_{m=0}^{\infty} (m+1)^{2-1/q} A(m) < \infty$ , and  $\sum_k a(j, k, l) = \sum_l b(j, k, l) = 0$ , then  $T$  is a bounded operator on  $\dot{F}_p^{0, q}$ ,  $1 \leq p, q < \infty$ .*

(ii) *If  $\sum_{m=0}^{\infty} 2^{\alpha m} A(m) < \infty$  and  $\sum_l b(j, k, l) = 0$  for any  $j, k \in \mathbb{Z}$ , then  $T$  is bounded on  $\dot{F}_p^{\alpha, q}$ ,  $0 < \alpha < 1$ ,  $1 \leq p, q < \infty$ .*

We will first prove the theorem for  $\dot{F}_1^{\alpha, q}$ , then apply an interpolation theorem

on  $\dot{F}_1^{\alpha,q}$  and  $\dot{F}_q^{\alpha,q}(= \dot{B}_q^{\alpha,q})$ , and a duality argument to conclude the theorem. We need the following notion of atom which can be found in [9].

**DEFINITION.** Let  $a(x) = \sum_{j,k} \alpha(j,k) \psi_{jk}(x)$  be a locally integrable function. We say that  $a(x)$  is an  $(\alpha, 1, q)$ -atom if there is a dyadic cube  $I \subset \mathbb{R}$  such that

- (i)  $\text{supp } a(x) \subset I$ ;
- (ii)  $\int_I a(x) dx = 0$ ;
- (iii)  $\|A(a)\|_q \leq |I|^{1/q-1}$ ,

where  $A(a)$  is defined by

$$(4.1) \quad A(a)(x) = \left( \sum_{j,k} 2^{(\alpha+1/2)jq} |\alpha(j,k)|^q \chi(2^j x - k) \right)^{1/q}.$$

For  $j, k \in \mathbb{Z}$ , let  $I_{jk}$  denote the interval  $[2^{-j}k, 2^{-j}(k+1)]$ . Let

$$(4.2) \quad a_{jk}(x) = \sum_{j',k': I_{j'k'} \subset I_{jk}} \alpha(j',k') \psi_{j'k'}(x).$$

Note that the sum is actually adding all  $j', k'$  with  $j \leq j', k' \in [2^{j'-j}k, 2^{j'-j}(k+1))$ . It follows that the support of  $a_{jk}(x)$  is contained in  $[2^{-j}k, 2^{-j}(k+M)]$ . By the definition, we know that  $a_{jk}(x)$  is an  $(\alpha, 1, q)$ -atom if

$$\|a_{jk}\|_{(\alpha,1,q)} := \left\{ 2^{-j(q-1)} \sum_{j',k': I_{j'k'} \subset I_{jk}} 2^{(-1/q+\alpha+1/2)j'q} |\alpha(j',k')|^q \right\}^{1/q} < \infty.$$

**LEMMA 4.2.** Let  $a_{jk}(x)$  be the atom as in (4.2), and let  $T_m^{(2)}$  be defined as in Lemma 3.2. Then we have

$$\|T_m^{(2)} a_{jk}\|_{\dot{F}_1^{\alpha,q}} \leq \begin{cases} C(m+1)^{2-1/q} A(m) \|a_{jk}\|_{(\alpha,1,q)} & \alpha = 0; \\ C2^{\alpha m} A(m) \|a_{jk}\|_{(\alpha,1,q)} & 0 < \alpha < 1, \end{cases}$$

where  $C$  is independent of  $m, j, k$ .

**PROOF.** Without loss of generality we consider  $a_{00}(y)$  and denote it by  $a(y)$  for simplicity, that is,  $a(y) = \sum_{0 \leq j} \sum_{0 \leq k < 2^j} \alpha(j,k) \psi_{jk}(y)$ . Noting that  $\langle \varphi_{j1}, \psi_{j'k'} \rangle \neq 0$

implies that  $j > j'$  and  $2^{j-j'}k' \leq l \leq 2^{j-j'}(k' + M) + M$ . One can write

$$\begin{aligned}
(T_m^{(2)}a)(x) &= \sum_{jkl} \sum_{j'k'} \alpha(j', k') b_m(j, k, l) \langle \varphi_{jl}, \psi_{j'k'} \rangle \psi_{jk}(x) \\
&= \sum_{j \leq m} \left( \sum_{j'kk'l} \alpha(j', k') b_m(j, k, l) \langle \varphi_{jl}, \psi_{j'k'} \rangle \psi_{jk}(x) \right) \\
&\quad + \sum_{j > m} \left( \sum_{j': 0 < j-j' \leq m} \sum_{kk'l} \alpha(j', k') b_m(j, k, l) \langle \varphi_{jl}, \psi_{j'k'} \rangle \psi_{jk}(x) \right) \\
&\quad + \sum_{j > m} \left( \sum_{j': j-j' > m} \sum_{kk'l} \alpha(j', k') b_m(j, k, l) \langle \varphi_{jl}, \psi_{j'k'} \rangle \psi_{jk}(x) \right) \\
&= a_1(x) + a_2(x) + a_3(x).
\end{aligned}$$

Since  $\dot{B}_1^{\alpha,1} = \dot{F}_1^{\alpha,1} \subset \dot{F}_1^{\alpha,q}$ ,  $1 \leq q < \infty$ , it follows that there exists a constant  $C$  such that  $\|f\|_{\dot{F}_1^{\alpha,q}} \leq C\|f\|_{\dot{B}_1^{\alpha,1}}$ . Using Proposition 2.1 and the Hölder inequality, we obtain

$$\begin{aligned}
\|a_1\|_{\dot{F}_1^{\alpha,q}} &\leq C \left\| \sum_{j \leq m} \sum_{j'kk'l} \alpha(j', k') b_m(j, k, l) \langle \varphi_{jl}, \psi_{j'k'} \rangle \psi_{jk} \right\|_{\dot{B}_1^{\alpha,1}} \\
&\leq C \sum_{j \leq m} \sum_{j'kk'l} 2^{(\alpha-1/2)j} |\alpha(j', k') b_m(j, k, l) \langle \varphi_{jl}, \psi_{j'k'} \rangle| \\
&\leq CA(m) \sum_{0 \leq j \leq m} \sum_{0 < j' < j} \sum_{0 \leq k' < 2^{j'}} 2^{(j-j')/2} 2^{(\alpha-1/2)j} |\alpha(j', k')| \\
&\leq CA(m) \left\{ \sum_{0 \leq j \leq m} 2^{\alpha j} \left( \sum_{0 < j' < m} 2^{-\alpha j' q'} \right)^{1/q'} \right\} \\
&\quad \times \left( \sum_{j'k'} 2^{(-1/q + \alpha + 1/2)j'q} |\alpha(j', k')|^q \right)^{1/q} \\
&\leq \begin{cases} C(m+1)^{2-1/q} A(m) \|a\|_{(\alpha,1,q)} & \alpha = 0; \\ C2^{\alpha m} A(m) \|a\|_{(\alpha,1,q)} & 0 < \alpha < 1. \end{cases}
\end{aligned}$$

To estimate  $\|a_2\|_{\dot{F}_1^{\alpha,q}}$ , we first observe that the condition  $0 \leq k' < 2^{j'}$  implies that  $0 \leq 2^{-j'}l \leq M$ . By the expression of  $a_2(x)$ , we know that  $k \in E_j = [-2^{m+1}, 2^{m+1} + 2^j + 2^j M]$ , and hence

$$\text{supp } a_2(x) \subseteq \bigcup_{j > m} \bigcup_{k \in E_j} [2^{-j}k, 2^{-j}(k + M)] \subseteq [-1, 2M].$$

Let  $\Gamma_{kk'l}^{mjs} = b_m(j, k, l)\langle\varphi_{jl}, \psi_{j-sk'}\rangle$ . Then

$$\begin{aligned} \|a_2\|_{\dot{F}_1^{\alpha,q}} &\leq C \left\| \sum_{j>m} \sum_{j':0<j-j'\leq m} \sum_{kk'l} \alpha(j', k') b_m(j, k, l) \langle\varphi_{jl}, \psi_{j'k'}\rangle \psi_{jk}(x) \right\|_{\dot{F}_1^{\alpha,q}} \\ &\leq C \sum_{s=1}^m \left\| \sum_{j=m+1}^{\infty} \sum_{kl} \sum_{0\leq k'<2^{j-s}} \alpha(j-s, k') \Gamma_{kk'l}^{mjs} \psi_{jk} \right\|_{\dot{F}_1^{\alpha,q}} \\ &\leq C \sum_{s=1}^m \int_{\mathbf{R}} \left( \sum_{j=m+1}^{\infty} \sum_k 2^{(\alpha+1/2)jq} \left| \sum_{k'l} \alpha(j-s, k') \Gamma_{kk'l}^{mjs} \right|^q \chi(2^j x - k) \right)^{1/q} dx \\ &\leq C \sum_{s=1}^m \left\{ \sum_{j=m+1}^{\infty} \sum_k 2^{-j} 2^{(\alpha+1/2)jq} \left| \sum_{k'l} \alpha(j-s, k') \Gamma_{kk'l}^{mjs} \right|^q \right\}^{1/q}. \end{aligned}$$

Similar to the estimate (3.3) in Lemma 3.2, we have

$$\sum_k \left| \sum_{k'l} \alpha(j-s, k') \Gamma_{kk'l}^{mjs} \right|^q \leq C 2^{s(1-q/q')/2} |A(m)|^q \sum_{k'} |\alpha(j-s, k')|^q.$$

It follows that

$$\begin{aligned} \|a_2\|_{\dot{F}_1^{\alpha,q}} &\leq C A(m) \sum_{s=1}^m \left\{ \sum_{j=m+1}^{\infty} 2^{-j+(\alpha+1/2)jq} 2^{s(1-q/q')/2} \sum_{k'} |\alpha(j-s, k')|^q \right\}^{1/q} \\ &\leq C A(m) \sum_{s=1}^m 2^{\alpha s} \left\{ \sum_{j=m+1}^{\infty} \sum_{k'} 2^{(-1/q+\alpha+1/2)(j-s)q} |\alpha(j-s, k')|^q \right\}^{1/q} \\ &\leq \begin{cases} C(m+1)A(m)\|a\|_{(\alpha,1,q)} & \alpha = 0; \\ C2^{\alpha m} A(m)\|a\|_{(\alpha,1,q)} & 0 < \alpha < 1. \end{cases} \end{aligned}$$

We now turn to estimate the term  $a_3(x)$ . Like  $a_2(x)$ ,  $\text{supp } a_3(x) \subseteq [-1, 2M]$ . Let  $g_{jk}(x)$  be defined as in the estimate of  $|\text{II}|$  of Lemma 3.2 and  $\tilde{\Gamma}_{kk'}^{js} = \langle g_{jk}(x), \psi'(2^{-s}x - k') \rangle$ . Then

$$\begin{aligned} \|a_3(x)\|_{\dot{F}_1^{\alpha,q}} &\leq C \left\| \sum_{j>m} \sum_{j':j-j'>m} \sum_{kk'l} \alpha(j', k') b_m(j, k, l) \langle\varphi_{jl}, \psi_{j'k'}\rangle \psi_{jk}(x) \right\|_{\dot{F}_1^{\alpha,q}} \\ &\leq C \sum_{s=m+1}^{\infty} 2^{-3s/2} \left\| \sum_{j=m+1}^{\infty} \sum_{kk'} \alpha(j-s, k') \tilde{\Gamma}_{kk'}^{js} \psi_{jk}(x) \right\|_{\dot{F}_1^{\alpha,q}} \\ &\leq C \sum_{s=m+1}^{\infty} 2^{-3s/2} \left\{ \sum_{j=m+1}^{\infty} \sum_k 2^{-j+(\alpha+1/2)jq} \left| \sum_{k'} \alpha(j-s, k') \tilde{\Gamma}_{kk'}^{js} \right|^q \right\}^{1/q}. \end{aligned}$$

By the same estimate as (3.4) in Lemma 3.2, we obtain

$$\sum_k \left| \sum_{k'} \alpha(j-s, k') \tilde{\Gamma}_{kk'}^{js} \right|^q \leq C 2^{s+m} 2^{mq/q'} |A(m)|^q \sum_{k'} |\alpha(j-s, k')|^q.$$

Hence

$$\begin{aligned} \|a_3\|_{\dot{F}_1^{\alpha,q}} &\leq C \sum_{s=m+1}^{\infty} 2^{-3s/2} \left\{ \sum_{j=m+1}^{\infty} 2^{-j+(\alpha+1/2)jq} 2^{s+m} 2^{mq/q'} \sum_{k'} |\alpha(j-s, k')|^q \right\}^{1/q} \\ &\leq C A(m) \sum_{s=m+1}^{\infty} 2^m 2^{(\alpha-1)s} \left\{ \sum_{jk'} 2^{(-1/q+\alpha+1/2)(j-s)q} |\alpha(j-s, k')|^q \right\}^{1/q} \\ &\leq C 2^{am} A(m) \|a\|_{(\alpha,1,q)}. \end{aligned}$$

We have hence proved Lemma 4.2 by combining the estimates of  $a_1(x)$ ,  $a_2(x)$  and  $a_3(x)$ .  $\square$

Now we state the following atomic decomposition of  $\dot{F}_1^{\alpha,q}$ , which is given in [1] for  $\dot{F}_1^{0,2}(= H^1)$  and in [6] and [9] for the general case. For completeness we modify their proof and sketch it here.

**LEMMA 4.3.** *Let  $f(x) = \sum_{jk} \alpha(j, k) \psi_{jk}(x)$  be in  $\dot{F}_1^{\alpha,q}$ . Then there exists a sequence  $\{h_{sn}(x)\}_{s,n}$  of  $(\alpha, 1, q)$ -atoms and  $\{\lambda_{sn}\} \in \mathbb{R}$  such that*

$$(4.3) \quad f(x) = \sum_{s,n} \lambda_{sn} h_{sn}(x) \quad \text{and} \quad C_1 \|f\|_{\dot{F}_1^{\alpha,q}} \leq \sum_{s,n} |\lambda_{sn}| \leq C_2 \|f\|_{\dot{F}_1^{\alpha,q}}$$

for some fixed  $C_1, C_2 > 0$  independent of  $f$ .

**PROOF.** By Proposition 2.1,  $f \in \dot{F}_1^{\alpha,q}$  has an equivalent norm given by  $\|f\|_{\dot{F}_1^{\alpha,q}} \approx \|A(f)\|_1$ .

If  $2^s$ ,  $s \in \mathbb{Z}$ , is a given threshold, we define  $\Omega_s = \{x : A(f)(x) > 2^s\}$ . This allows us to write  $\Omega_s = \bigcup_{n \in \mathbb{N}} Q_{sn}$ , where each  $Q_{sn}$  is a maximal dyadic interval in  $\Omega_s$ . The intervals  $Q_{sn}$ , being dyadic and maximal, are either identical or disjoint. For each  $s \in \mathbb{Z}$ ,  $n \in \mathbb{N}$ , consider the family  $\mathcal{F}_{sn}$  of all dyadic intervals  $I_{jk}$  such that  $I_{jk} \subset Q_{sn}$  which are contained in no  $Q_{s+1p}$  for any  $p$ . By the above construction, we can write

$$\bigcup_s \Omega_s = \bigcup_{s,n} \left( \bigcup_{I_{jk} \in \mathcal{F}_{sn}} I_{jk} \right).$$



For each  $s \in \mathbb{Z}$ ,  $n \in \mathbb{N}$ , let

$$(4.4) \quad h_{sn}(x) = |\lambda_{sn}|^{-1} \sum_{I_{jk} \in \mathcal{F}_{sn}} \alpha(j, k) \psi_{jk}(x),$$

where  $\lambda_{sn} = |\mathcal{Q}_{sn}|^{1-1/q} \left\{ \sum_{I_{jk} \in \mathcal{F}_{sn}} 2^{(-1/q + \alpha + 1/2)jq} |\alpha(j, k)|^q \right\}^{1/q}$ . Then,  $h_{sn}(x)$  is an  $(\alpha, 1, q)$ -atom and

$$(4.5) \quad f(x) = \sum_{j,k} \alpha(j, k) \psi_{jk}(x) = \sum_{s,n} \lambda_{sn} h_{sn}(x)$$

gives an atomic decomposition of  $f$ . For the details we refer the reader to [1, 6, 9].  $\square$

**PROOF OF THEOREM 4.1.** By Lemma 4.4 we can write  $f(x) \in \dot{F}_1^{\alpha,q}$  as an atomic decomposition  $f(x) = \sum_{s,n} \lambda_{sn} h_{sn}(x)$ , where each  $h_{sn}(x)$  is an  $(\alpha, 1, q)$ -atom defined in (4.4). For each  $h_{sn}(x)$ , we can rewrite it as the form in (4.2) by assigning  $\alpha(j, k) = 0$  for  $I_{jk} \subset \mathcal{Q}_{sn}$  but  $I_{jk} \notin \mathcal{F}_{sn}$ . Using Lemma 4.2 and Lemma 4.3, we have

$$\|T_m^{(2)} f\|_{\dot{F}_1^{\alpha,q}} \leq \begin{cases} C(m+1)^{2-1/q} A(m) \sum_{s,n} |\lambda_{sn}| & \alpha = 0; \\ C2^{\alpha m} A(m) \sum_{s,n} |\lambda_{sn}| & 0 < \alpha < 1, \end{cases}$$

where  $C$  is independent of  $m, s, n$ . It follows that  $T^{(2)}$  is bounded on  $\dot{F}_1^{\alpha,q}$ . Similarly as in Lemma 3.3, and Lemma 3.4, we can show that both  $T^{(1)}$  and  $T^{(3)}$  are also bounded operators on  $\dot{F}_1^{\alpha,q}$ . Hence, we have proved that  $T$  is bounded on  $\dot{F}_1^{\alpha,q}$ , and

$$\|T\|_{(\dot{F}_1^{\alpha,q}, \dot{F}_1^{\alpha,q})} \leq \begin{cases} C \sum_{m=0}^{\infty} (m+1)^{2-1/q} A(m) & \alpha = 0; \\ C \sum_{m=0}^{\infty} 2^{\alpha m} A(m) & 0 < \alpha < 1, \end{cases}$$

where  $0 \leq \alpha < 1$ ,  $1 \leq q < \infty$ .

Since  $T$  is bounded on  $\dot{B}_q^{\alpha,q} (= \dot{F}_q^{\alpha,q})$ , the interpolation theorem [13] implies that  $T$  is bounded on  $\dot{F}_p^{\alpha,q}$ ,  $0 \leq \alpha < 1$ ,  $1 \leq p \leq q$ . Similarly as in the proof of Theorem 3.1 and Theorem 4.1, we can show, by interchanging the role of  $T(1)$  and  $T^*(1)$ , that  $T$  is bounded on both  $\dot{F}_q^{\alpha,q}$  and  $\dot{F}_1^{\alpha,q}$  with  $-1 < \alpha \leq 0$ ,  $1 \leq q < \infty$ . Again, applying the interpolation theorem and the duality,  $T$  is bounded on  $\dot{F}_p^{\alpha,q}$ ,  $0 \leq \alpha < 1$ ,  $p \geq q$ . This finishes the proof of Theorem 4.1.  $\square$

**PROOF OF COROLLARY 1.3.** With the above notation, we have  $\|T_m^{(2)}\|_{(H^1, H^1)} \leq C(m+1)^{3/2} A(m)$  and  $\|T_m^{(2)}\|_{(L^2, L^2)} \leq C(m+1)^{1/2} A(m)$  for some  $C > 0$  independent of  $m$ . For  $1 < p < 2$ , by the interpolation theorem we have  $\|T_m^{(2)}\|_{(L^p, L^p)} \leq C(m+1)^{1/2+2(1/p-1/2)} A(m)$ . Using the duality argument, for  $2 < p < \infty$  we have  $\|T_m^{(2)}\|_{(L^p, L^p)} \leq C(m+1)^{1/2+2|1/p-1/2|} A(m)$ . It follows from the assumption on  $A(m)$  that  $T^{(2)}$  is bounded on  $L^p$ . Similarly,  $T^{(1)}$  and  $T^{(3)}$  are bounded operators on  $L^p$ ; so is  $T$ . This completes the proof of Corollary 1.3.  $\square$

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## References

- [1] J. Aguirre, M. Escobedo, J. C. Perel and Ph. Tchamitchian, ‘Basis of wavelets and atomic decompositions of  $H^1(\mathbb{R}^n)$  and  $H^1(\mathbb{R}^n \times \mathbb{R}^n)$ ’, *Proc. Amer. Math. Soc.* **111** (1991), 683–693.
- [2] G. Beylkin, R. Coifman and V. Rokhlin, ‘Fast wavelet transforms and numerical algorithms’, *Comm. Pure Appl. Math.* **44** (1991), 141–183.
- [3] I. Daubechies, *Ten lectures on wavelets*, CBMS–NSF Regional Conference Series in Appl. Math. 61 (SIAM, Philadelphia, 1992).
- [4] G. David and J. L. Journé, ‘A boundedness criterion for generalized Calderón-Zygmund operators’, *Ann. of Math.* **120** (1984), 371–397.
- [5] D. G. Deng, L. X. Yan and Q. X. Yang, ‘Blocking analysis and  $T(1)$  theorem’, *Science in China* **41** (1998), 800–808.
- [6] M. Frazier, B. Jawerth and G. Weiss, *Littlewood-Paley theory and the study of functions*, CBMS–Regional Conference Series in Mathematics 79 (AMS, Providence, RI, 1991).
- [7] M. Frazier, R. Torres and G. Weiss, ‘The boundedness of Calderón-Zygmund operator on the spaces  $\dot{F}_p^{\alpha,q}$ ’, *Rev. Mat. Iberoamericana* **4** (1998), 41–72.
- [8] Y. Han and S. Hofmann, ‘ $T(1)$  theorem for Besov and Triebel-Lizorkin spaces’, *Trans. Amer. Math. Soc.* **237** (1993), 839–853.
- [9] Y. Han, M. Paluszynski and G. Weiss, ‘A new atomic decomposition for the Triebel-Lizorkin spaces’, *Contemporary Math.* **189** (1995), 235–249.
- [10] P. G. Lemarié, ‘Continuité sur les espaces de Besov and opérateurs définis par des intégrales singulières’, *Ann. Inst. Fourier (Grenoble)* **35** (1985), 175–187.
- [11] Y. Meyer, *La minimalité de l’espace de Besov  $\dot{B}_1^{0,1}$  et la continuité des opérateurs définis par des intégrales singulières*, Monografias de Matematicas, 4 (Univ. Autonoma de Madrid, 1986).
- [12] ———, *Ondelettes et opérateurs*, Vols I, II (Hermann, Paris, 1990).
- [13] H. Triebel, *Theory of function spaces* (Birkhäuser, Basel, 1983).

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