

ON $f(n)$ MODULO $\Omega(n)$ AND $\omega(n)$ WHEN f IS A POLYNOMIAL

FLORIAN LUCA

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Abstract

In this paper we show that if $f(X) \in \mathbb{Z}[X]$ is a nonzero polynomial, then $\omega(n) | f(n)$ holds only on a set of n of asymptotic density zero, where for a positive integer n the number $\omega(n)$ counts the number of distinct prime factors of n .

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1. Introduction

Let n be a positive integer and let $\omega(n)$, $\Omega(n)$, $\tau(n)$, $\phi(n)$ and $\sigma(n)$ be the classical arithmetic functions of n , that is, $\omega(n)$, $\Omega(n)$, and $\tau(n)$ count the number of distinct prime divisors of n , the total number of prime divisors of n , and the number of divisors of n , respectively, while $\phi(n)$ and $\sigma(n)$ are the Euler function of n and the sum of divisors function of n , respectively. We also let $f(X) \in \mathbb{Z}[X]$ to be any nonzero polynomial with integer coefficients.

In [2], it was shown that the set of positive integers n for which $\omega(n) | n$ is of density zero, and it was asked whether the same is true for the set of integers n for which $\Omega(n) | n$. This question was answered in a greater generality in [4]. In this paper, we investigate the density of the sets of positive integers n on which one of the given ‘small’ arithmetic function of n divide either $f(n)$, or the value of f in some other arithmetic function of n .

We have the following result.

THEOREM 1.1. (1) *The set of positive integers*

$$(1.1) \quad \{n \mid f(n) \equiv 0 \pmod{\omega(n)}\}$$

is of asymptotic density zero. The same is true for the sets obtained if one replaces $\omega(n)$ in (1.1) by either $\Omega(n)$ or $\tau(n)$.

(2) The set of positive integers

$$(1.2) \quad \{n \mid f(\tau(n)) \equiv 0 \pmod{\omega(n)}\}$$

is of asymptotic density zero. The same is true for the set obtained if one replaces $\omega(n)$ in (1.2) by $\Omega(n)$.

(3) The set of positive integers

$$(1.3) \quad \{n \mid \phi(n) \equiv 0 \pmod{\omega(n)}\}$$

is of asymptotic density one. The same is true for the other five sets obtained from (1.3) by independently replacing $\phi(n)$ by $\sigma(n)$, and $\omega(n)$ by either $\Omega(n)$ or $\tau(n)$.

(4) The set of positive integers

$$(1.4) \quad \{n \mid f(\Omega(n)) \equiv 0 \pmod{\omega(n)}\}$$

has an asymptotic density for every polynomial $f(X) \in \mathbb{Z}[X]$. This density is zero unless f has nonnegative integer roots, in which case it is positive. Similarly, the set obtained if one interchanges $\Omega(n)$ by $\omega(n)$ in (1.4) has an asymptotic density, which is zero unless f has integer roots which are either negative or zero, in which case it is positive.

The densities of the sets appearing at part (4) of Theorem 1.1 are computable. Namely, the density of the set (1.4) is

$$\sum_{\substack{k \geq 0, k \in \mathbb{Z} \\ f(k) = 0}} d_k,$$

where $d_k > 0$ is the Rényi's constant (see [11]) given by

$$d_k := \lim_{x \rightarrow \infty} \frac{\#\{1 \leq n \leq x \mid \Omega(n) - \omega(n) = k\}}{x}.$$

Similarly, the density of the set obtained if one interchanges $\Omega(n)$ by $\omega(n)$ in (1.4) is

$$\sum_{\substack{k \leq 0, k \in \mathbb{Z} \\ f(k) = 0}} d_{|k|}.$$

Theorem 1.1 gives information about the asymptotic densities of the sets of positive integers $\{n \mid f(\phi(n)) \equiv 0 \pmod{\omega(n)}\}$ and likewise when $\omega(n)$ is replaced by either $\Omega(n)$ or $\tau(n)$, or when $\phi(n)$ is replaced by $\sigma(n)$. Indeed, from part (3) of Theorem 1.1

we get that the asymptotic density of such sets is zero unless $f(0) = 0$, in which case the asymptotic density is one. We point out that if one replaces the polynomial $f(n)$ by the n th Fibonacci number F_n in (1.1), then the statement asserted at part (1) of Theorem 1.1 still holds. This has been done in [5], and it is likely that a combination of the arguments from the method of proof from there with our present arguments and some results from [7] can yield a similar result when F_n is replaced by any nondegenerate linearly recurrent sequence of integers.

2. Preliminary results

In this section, we point out a ‘large’ set of integers which is suitable for our purposes, and then in the next section we verify that every positive integer n from our large set satisfies all the congruences or the incongruences asserted by the theorem.

We denote by C_1, C_2, \dots positive computable constants which are either absolute or depend only on the polynomial f . For a positive integer k and positive real number x , we denote by $\log_k x$ the recursively defined function given by $\log_1 x := \max\{\log x, 1\}$ and $\log_k x := \max\{\log(\log_{k-1} x), 1\}$, where \log denotes the natural logarithm function. When $k = 1$, we simply write $\log_1 x$ as $\log x$ and we thus understand that it is always ≥ 1 . We also use the Landau symbols O and o and the Vinogradov symbols \gg and \ll with their usual meanings. We write $\pi(x)$ for the number of prime numbers $p \leq x$. For a positive integer n , we use $P(n)$ and $p(n)$ to denote the largest prime factor of n and the smallest prime factor of n , respectively.

We set $\delta(x) := \log_5 x$, and we use the notations p, q , and r to denote prime numbers. For a positive integer n , we write $\omega_1(n)$ and $\omega_3(n)$ for the number of distinct prime factors of n which are congruent to 1 and 3 modulo 4, respectively. Thus, we always have $\omega(n) = \omega_1(n) + \omega_3(n) + \varepsilon$, where $\varepsilon = 0$ or 1 according to whether n is odd or even.

We begin with the following claim.

LEMMA 2.1. *Let x be a large positive real number and let $A(x)$ be the set of all positive integers n in the range $\sqrt{x} < n < x$ and which satisfy the following conditions:*

- (1) $\max\{|\omega(n) - \log_2 x|, |\Omega(n) - \log_2 x|\} < \delta(x) \log_2^{1/2} x.$
- (2) $\min\{\omega_1(n), \omega_3(n)\} > \log_2(x)/4.$
- (3) Write

$$(2.1) \quad n := \prod_{p^a \parallel n} p^{a_p}.$$

Then, $\max_{p|n}\{a_p\} < \log_7 x$ and $a_p = 1$ when $p \geq \log_7 x$.

Then the set $A(x)$ contains all positive integers $n < x$ except for $o(x)$ of them.

REMARK. Note that if $n \in A(x)$ then, by condition (3) of Lemma 2.1, we have that the inequality

$$(2.2) \quad \Omega(n) - \omega(n) = \sum_{a_p > 1} (a_p - 1) < \pi(\log_7 x)(\log_7 x - 1) < \log_7^2 x < \log_6 x$$

holds for large values of x .

PROOF OF LEMMA 2.1. It is obvious that there are $\lfloor \sqrt{x} \rfloor = o(x)$ numbers $n < x$ which are not in the range $\sqrt{x} < n < x$.

(1) Since both estimates

$$(2.3) \quad \sum_{1 \leq n < x} (\omega(n) - \log_2 x)^2 = O(x \log_2 x)$$

and

$$(2.4) \quad \sum_{1 \leq n < x} (\Omega(n) - \log_2 x)^2 = O(x \log_2 x)$$

hold (see [14]), it follows that there are at most $O(x/\delta^2(x)) = o(x)$ positive integers $n < x$ which fail to satisfy the inequality asserted at part (1) of the lemma.

(2) Let E be any set of prime numbers and for $x > 0$ write

$$(2.5) \quad E(x) := \sum_{\substack{p < x \\ p \in E}} \frac{1}{p}.$$

For any positive integer n , write $\omega(E, n)$ for the number of primes dividing n which belong to E , and let $0 < \alpha < 1$ be any fixed positive real number. Then, a result of Norton (see [9, 10]), says that if $E(x) > 0$, then the number of positive integers $n < x$ such that $|\omega(E, n) - E(x)| > \alpha E(x)$ is at most $C(\alpha)x/E(x)^{1/2}$, where $C(\alpha)$ is some computable number depending on α and E . Take $E = E_i := \{p \mid p \equiv i \pmod{4}\}$ with $i = 1$ or 3 , take $\alpha := 1/3$, and assume that $x > 5$. Then $E_i(x) \neq 0$ and the estimate

$$(2.6) \quad E_i(x) = \sum_{p \equiv i \pmod{4}} \frac{1}{p} = \frac{\log_2 x}{2} + O(1)$$

holds for both $i = 1$ and 3 . And so, if we assume that $n < x$ fails condition (2) of the lemma for some $i = 1$ or 3 , then $\omega_i(n) \leq \log_2(x)/4$ holds for such n , and with estimate (2.6) we conclude that $\omega_i(n) < 2E(x)/3$ holds for such $n < x$ and for large values of x . Thus, the inequality $|\omega(E_i, n) - E(x)| > E(x)/3$ holds for such n , and

by the above result from [9, 10], we know that the number of such positive integers $n < x$ is

$$\ll \frac{x}{E(x)^{1/2}} \ll \frac{x}{\log_2^{1/2} x} = o(x).$$

(3) Suppose first that there exists a prime number $p \geq \log_7 x$ such that $p^2 | n$. The totality of such $n < x$ is at most

$$\sum_{p \geq \log_7 x} \frac{x}{p} = O\left(\frac{x}{\log_7 x \log_8 x}\right) = o(x).$$

Assume now that $a_p \geq \log_7 x$ holds for some p . Since we may assume that $p < \log_7 x$, it follows that the number of such positive integers $n < x$ is at most

$$\sum_{2 \leq p < \log_7 x} \frac{x}{p^{\log_7 x}} < \frac{x\pi(\log_7 x)}{2^{\log_7 x}} < \frac{x \log_7 x}{(\log_6 x)^{C_1}} = o(x),$$

where we put $C_1 := \log 2$. □

LEMMA 2.2. *Let x be a large positive real number and let $A(x)$ be the subset appearing in Lemma 2.1. Let $B(x)$ be the subset of $n \in A(x)$ with the following property.*

Write

$$\omega(n) := w_1(n)w_2(n), \quad \text{where } w_1(n) := \prod_{\substack{q^{b_q} || n \\ q \leq \log_3 x \log_5 x}} q^{b_q}.$$

Then, $w_2(n) > \log_2^{1/3} x$ is squarefree, coprime to n , and has $p(w_2(n)) < 2 \log_2^{1/2} x$.

Similarly, if one writes

$$\Omega(n) := W_1(n)W_2(n), \quad \text{where } W_1(n) := \prod_{\substack{r^{c_r} || \Omega(n) \\ r \leq \log_3 x \log_5 x}} r^{c_r},$$

then $W_2(n) > \log_2^{1/3} x$ is squarefree, coprime to n , and has $p(W_2(n)) < 2 \log_2^{1/2} x$.

Then the set $B(x)$ contains all positive integers $n < x$ except for $o(x)$ of them.

PROOF OF LEMMA 2.2. We shall deal only with the statement concerning the function $\omega(n)$ because the statement about $\Omega(n)$ can be dealt with in an entirely similar way.

Assume that $n \in A(x)$ but that $w_2(n) \leq \log_2^{1/3} x$. Then $\omega(n) = w_1(n)w_2(n)$, where $w_1(n) < \log_2 x + \delta(x) \log_2^{1/2} x < 2 \log_2 x$, and $P(w_1(n)) \leq \log_3 x \log_5 x$. We

estimate the number of values $w_1(n)$ can take. Suppose that $y > z > 0$ and put $\Psi(y, z) := \#\{n \leq y \mid P(n) \leq z\}$. We shall show that if $z := 2 \log y \log_3 y$, then $\Psi(y, z) = y^{o(1)}$. To see this, we put

$$Z := \frac{\log y}{\log z} \log \left(1 + \frac{z}{\log y} \right) + \frac{z}{\log z} \log \left(1 + \frac{\log y}{z} \right)$$

and then, by [13, Theorem 2 on page 359], we know that the estimate

$$(2.7) \quad \log \psi(y, z) = Z \left(1 + O \left(1 + \frac{1}{\log z} + \frac{1}{\log_2(2y)} \right) \right)$$

holds uniformly in $y > z > 0$. It is clear that the factor that multiplies Z appearing on the right-hand side of (2.7) is $O(1)$, and with our choice for z we have

$$Z \ll \frac{z}{\log z} + \frac{\log y}{\log z} \ll \frac{\log y \log_3 y}{\log_2 y}.$$

And thus, we have

$$\psi(y, z) = \exp(O(Z)) = \exp \left(O \left(\frac{\log y \log_3 y}{\log_2 y} \right) \right) = y^{o(1)}.$$

Setting $y := 2 \log_2 x$, and noting that $\log_3 x \log_5 x < 2 \log y \log_3 y = z$, we get

$$(2.8) \quad \psi(2 \log_2 x, \log_3 x \log_5 x) = (2 \log_2 x)^{o(1)} = (\log_2 x)^{o(1)}.$$

In particular, the inequality

$$(2.9) \quad \psi(2 \log_2 x, \log_3 x \log_5 x) < \log_2^{1/12} x$$

holds for large values of x . Inequality (2.9) tells us that $w_1(n)$ can take no more than $\log_2^{1/12} x$ values. Thus, the total number of values of $\omega(n) = w_1(n)w_2(n)$ which are smaller than $2 \log_2 x$ and for which $w_2(n) \leq \log_2^{1/3} x$ holds is at most $(\log_2 x)^{1/3+1/12} = \log_2^{5/12} x$. However, from [3, page 303], we know that if j is any fixed positive integer, then the number of positive integers $n < x$ having $\omega(n) = j$ is

$$(2.10) \quad \ll \frac{x}{\log^{1/2} x}.$$

Since our j can take only $\log_2^{5/12} x$ values, we conclude that the number of positive integers $n < x$ for which $w_2(n) < \log^{1/3} x$ is

$$\ll \frac{x}{\log^{1/2} x} \log_2^{5/12} x = \frac{x}{\log_2^{1/12} x} = o(x),$$

which takes care of the first condition from the lemma.

We next show that $w_2(n)$ is squarefree for almost all $n < x$. Assume that $n \in A(x)$ but $w_2(n)$ is not squarefree. Notice that $p(w_2(n)) > \log_3 x \log_5 x$. Pick a prime number $p > \log_3 x \log_5 x$ and assume that $j := \omega(n)$ is a number which is divisible by p^2 . Since $\omega(n) < 2 \log_2 x$, it follows that $p \leq C_2 \log_2^{1/2} x$, where $C_2 := \sqrt{2}$. But j is also a number in the interval

$$(2.11) \quad \mathcal{J} := (\log_2 x - \delta(x) \log_2^{1/2} x, \log_2 x + \delta(x) \log_2^{1/2} x)$$

whose length is $2\delta(x) \log_2^{1/2} x$, and so the number of such numbers j which can be multiples of p^2 is

$$(2.12) \quad \leq \frac{2\delta(x) \log_2^{1/2} x}{p^2} + 1.$$

For every one of these numbers j , the number of positive integers $n < x$ with $\omega(n) = j$ is, by (2.10), $\ll x / \log_2^{1/2} x$. Thus, for a fixed prime number p , the number of positive integers $n \in A(x)$ and for which $p^2 | \omega(n)$ is

$$\ll \frac{x\delta(x)}{p^2} + \frac{x}{\log_2^{1/2} x}.$$

Summing up the above inequalities over all the prime numbers p in the range $\log_3 x \log_5 x < p \leq C_2 \log_2^{1/2} x$, we get that the totality of the positive integers $n \in A(x)$ and for which $w_2(n)$ is not squarefree is

$$(2.13) \quad \ll x\delta(x) \sum_{p > \log_3 x \log_5 x} \frac{1}{p^2} + \frac{x\pi(C_2 \log_2^{1/2} x)}{\log_2^{1/2} x}.$$

Since

$$(2.14) \quad \sum_{p > \log_3 x \log_5 x} \frac{1}{p^2} = O\left(\frac{1}{\log_3 x \log_4 x \log_5 x}\right),$$

and

$$\frac{\pi(C_2 \log_2^{1/2} x)}{\log_2^{1/2} x} = O\left(\frac{1}{\log_3 x}\right),$$

it follows that (2.13) is bounded above by

$$\ll \frac{x\delta(x)}{\log_3 x \log_4 x \log_5 x} + \frac{x}{\log_3 x} = o(x).$$

We now show that n and $w_2(n)$ are coprime for almost all $n < x$. Let $n \in A(x)$ and let p be a prime number dividing both n and $w_2(n)$. We now have $\log_3 x \log_5 x < p < 2 \log_2 x$. Fix such a prime p . By condition (3) from Lemma 2.1, we know that $p \parallel n$, therefore $n = pm$ and $\omega(m) = \omega(n) - 1$. Fix also j such that $\omega(n) = pj$. Then $m < x/p$ and $\omega(m) = pj - 1$ is fixed. The number of such numbers m is, by (2.9),

$$\ll \frac{x}{p} \frac{1}{\log_2^{1/2}(x/p)} \ll \frac{x}{p \log_2^{1/2} x},$$

where the last inequality above follows from the fact that the inequalities $p < 2 \log_2 x < x^{1/2}$ hold for large x . Moreover, since pj is a number in the interval \mathcal{I} shown at (2.11), it follows that j can take at most

$$\frac{2\delta(x) \log_2^{1/2} x}{p} + 1$$

values. Thus, the number of numbers $n \in A(x)$ for which $p \mid \gcd(n, w_2(n))$ with a fixed value of p is

$$\ll \frac{x\delta(x)}{p^2} + \frac{x}{p \log_2^{1/2} x}.$$

Summing up the above inequalities over all the possible values of p , it follows that the number of positive integers $n \in A(x)$ for which $w_2(n)$ and n are not coprime is

$$(2.15) \quad \ll x\delta(x) \sum_{p > \log_3 x \log_5 x} \frac{1}{p^2} + \frac{x}{\log_2^{1/2} x} \sum_{p < 2 \log_2 x} \frac{1}{p}.$$

Since

$$(2.16) \quad \sum_{p < 2 \log_2 x} \frac{1}{p} \ll \log_4 x,$$

we get, with (2.14) and (2.16), that (2.15) is bounded above by

$$\ll \frac{x\delta(x)}{\log_3 x \log_4 x \log_5 x} + \frac{x \log_4 x}{\log_2^{1/2} x} = o(x).$$

Finally, we show that for almost all $n < x$ we have $p(w_2(n)) < 2 \log_2^{1/2} x$. Assume that this is not so for some $n \in A(x)$. In this case, since $\omega(n) < 2 \log_2 x$, it follows that $w_2(n)$ is a prime number $p \geq 2 \log_2^{1/2} x$, and $\omega(n) = pj$, where $j < \log_2^{1/2} x$ has $P(j) \leq \log_3 x \log_5 x$. We now fix the number j and notice that since pj belongs to the interval \mathcal{I} shown at (2.11), then p must be a prime number in the interval

$$(2.17) \quad \mathcal{I}_j := \left(\frac{\log_2 x}{j} - \frac{2\delta(x) \log_2^{1/2} x}{j}, \frac{\log_2 x}{j} + \frac{2\delta(x) \log_2^{1/2} x}{j} \right).$$

Let $\pi_j(x)$ be the number of prime numbers in the interval \mathcal{I}_j shown at (2.17). Then, for j fixed, the number of values of $\omega(n) = pj$ is at most $\pi_j(x)$, and for each one of these values, by (2.10), the number of positive integers $n < x$ with $\omega(n) = pj$ is $\ll x / \log_2^{1/2} x$. So, the number of positive integers $n < x$ for which $\omega(n) = pj$ with j fixed and p prime is $\ll x\pi_j(x) / \log_2^{1/2} x$, and so the totality of the positive integers $n \in A(x)$ for which $p(w_2(n)) \geq 2 \log_2^{1/2} x$ is

$$(2.18) \quad \ll \frac{x}{\log_2^{1/2} x} \sum_{\substack{j < \log_2^{1/2} x \\ P(j) \leq \log_3 x \log_5 x}} \pi_j(x).$$

Let us now notice that the interval \mathcal{I}_j is an interval of length $2\delta(x) \log_2^{1/2} x / j$, and, by a result of Montgomery (see [8, page 34]), any interval of length y can contain no more than $2y / \log y$ prime numbers. Thus, since $\delta(x) > 1$ and $j < \log_2^{1/2} x = o(\delta(x) \log_2^{1/2} x)$, we get the inequality

$$(2.19) \quad \begin{aligned} \pi_j(x) &\ll \frac{\delta(x) \log_2^{1/2} x}{j \log(2\delta(x) \log_2^{1/2} x / j)} \\ &\ll \begin{cases} \frac{\delta(x) \log_2^{1/2} x}{j \log_3 x}, & \text{if } j < \log_2^{1/4} x; \\ \frac{\delta(x) \log_2^{1/2} x}{j}, & \text{if } \log_2^{1/4} x \leq j < \log_2^{1/2} x. \end{cases} \end{aligned}$$

In particular, (2.18) can be bounded from above by

$$(2.20) \quad \ll \frac{x\delta(x)}{\log_3 x} \sum_{P(j) < \log_3^2 x} \frac{1}{j} + \frac{x\delta(x)}{\log_2^{1/4} x} \psi(\log_2^{1/2} x, \log_3 x \log_5 x).$$

Clearly, $\psi(\log_2^{1/2} x, \log_3 x \log_5 x) < \psi(2 \log_2 x, \log_3 x \log_5 x) = (\log_2 x)^{o(1)}$ (see (2.8)), therefore

$$(2.21) \quad \frac{x\delta(x)}{\log_2^{1/4} x} \psi(\log_2^{1/2} x, \log_3 x \log_5 x) = \frac{x\delta(x)}{(\log_2 x)^{1/4+o(1)}} = o(x).$$

Finally, note that

$$\begin{aligned} \sum_{P(j) < \log_3^2 x} \frac{1}{j} &= \prod_{p < \log_3^2 x} \left(1 - \frac{1}{p}\right)^{-1} \\ &= \exp\left(O(1) + \sum_{p < \log_3^2 x} \frac{1}{p}\right) = \exp(\log_5 x + O(1)) \ll \log_4 x, \end{aligned}$$

therefore

$$(2.22) \quad \frac{x \delta(x)}{\log_3 x} \sum_{P(j) < \log_3^2 x} \frac{1}{j} \ll \frac{x \delta(x) \log_4 x}{\log_3 x} = o(x),$$

and now (2.21) and (2.22) imply that the right-hand side of (2.20) is $o(x)$.

Finally, the statement about $\Omega(n)$ follows in an entirely similar way due to the fact that a similar upper bound as (2.11) holds for the number of numbers $n < x$ having a fixed value of $\Omega(n)$. \square

3. The proof of Theorem 1.1

PROOF. We let x be a large positive real number and we shall assume that $n \in B(x)$, where $B(x)$ is the set described in Lemma 2.2. Part (1) of the theorem is the toughest cookie in the jar and so we shall prove it last.

(2) Let $n \in B(x)$, and write n as in (2.1). Then $\tau := \tau(n) = 2^{\omega(n)-k}m$, where $k := \#\{p|n \mid a_p > 1\}$ and $m := \prod_{a_p > 1} (a_p + 1)$. Condition 3 of Lemma 2.1 insures that $k \leq \pi(\log_7 x) < \log_7 x$, and that the inequality

$$m < \exp(\pi(\log_7 x) \log(\log_7 x + 1)) < \exp(2 \log_7 x) = \log_6^2 x$$

holds when x is large. In particular, there are a number $< \log_7 x \log_6^2 x < \log_5 x$ such pairs (k, m) , and they all have $P(m) < \log_7 x + 1$. Let p be a prime number in the interval $\mathcal{J} := (\log_3 x \log_5 x, 2 \log_2^{1/2} x)$. By Lemma 2.2, we know that $\omega(n)$ has such a prime factor for all $n \in B(x)$. Assume now that n is a number in $B(x)$ such that $p|\omega(n)$, write $\omega(n) := pj$, and assume further that n satisfies congruence (1.2). With the fixed value of p , the congruence $f(\tau) \equiv 0 \pmod{p}$ puts τ into at most d residue classes modulo p , where $d := \deg(f)$. Let α be one of these residue classes. Since $\tau = 2^{\omega(n)-k}m \equiv 2^{j-k}m \pmod{p}$, we get that $2^{j-k}m \equiv \alpha \pmod{p}$. Note that both 2 and m are invertible modulo p . Put $t(p)$ for the multiplicative order of 2 modulo p . For fixed values of α, k, m , the congruence $2^{j-k}m \equiv \alpha \pmod{p}$ puts j into a fixed congruence class modulo $t(p)$. In particular, with p fixed, the number $\omega(n) = pj$ belongs to at most $d \log_5 x$ congruence classes modulo $pt(p)$. Since this number is also in the interval \mathcal{J} shown in (2.11), we get that the number of values that pj can assume for a fixed value of p is

$$\ll \left(\frac{\delta(x) \log_2^{1/2} x}{pt(p)} + 1 \right) \log_5 x = \frac{\delta(x) \log_2^{1/2} x \log_5 x}{pt(p)} + \log_5 x.$$

For everyone of these values of pj , by inequality (2.10), there are $\ll x/\log_2^{1/2} x$ numbers numbers n for which $\omega(n) = pj$. Thus, for fixed p , the number of numbers

$n \in B(x)$ satisfying congruence (1.2) and for which $p|\omega(n)$ is

$$\ll \frac{x\delta(x)\log_5 x}{p t(p)} + \frac{x\log_5 x}{\log_2^{1/2} x}.$$

Summing up the above inequalities over all the primes p in the interval \mathcal{J} , and using the obvious fact that $t(p) \gg \log p$, we get that the number of numbers $n \in B(x)$ satisfying congruence (1.2) is

$$(3.1) \quad \ll x\delta(x)\log_5 x \sum_{p > \log_3 x} \frac{1}{p \log p} + \frac{x\log_5 x}{\log_2^{1/2} x} \pi(2\log_2^{1/2} x).$$

Since the estimate $\sum_{p > y} 1/(p \log p) \ll 1/\log y$ holds for all $y > 1$, we get that (3.1) is bounded from above by

$$\ll \frac{x\delta(x)\log_5 x}{\log_4 x} + \frac{x\log_5 x}{\log_3 x} = o(x).$$

The same argument applies when $\omega(n)$ is replaced by $\Omega(n)$.

(3) In [6, Lemma 2], it is shown that there exists an absolute constant C_3 such that if x is large and if we set $g(x) := C_3 \log_2 x / \log_3 x$, then both $\phi(n)$ and $\sigma(n)$ are divisible by the least common multiple of all the prime powers $p^a < g(x)$ for all $n < x$ with $o(x)$ exceptions (in [6, Lemma 2] this is only shown for the function ϕ but the argument from there can be adapted in a straightforward way to yield the corresponding result for the function σ). Let $M(x)$ denote the least common multiple of all the prime powers up to $g(x)$. To get statement (3) of Theorem 1.1 for $\omega(n)$ and $\Omega(n)$, we show that both $\omega(n)$ and $\Omega(n)$ divide $M(x)$. To see this, assume that $p^a \parallel \omega(n)$. If $p \leq \log_3 x \log_5 x$, then, by Lemma 2.2, we have $p^a \leq w_1(n) = \omega(n)/w_2(n) \ll \log_2^{2/3} x = o(g(x))$. Assume now that $p^a \parallel w_2(n)$. In this case, by Lemma 2.2, we have that $a = 1$. If $w_2(n)$ is not prime, then there exists another prime number q (necessarily larger than $\log_3 x \log_5 x$) such that $pq|w_2(n)$. Thus, $p \leq w_2(n)/q \ll \log_2 x / (\log_3 x \log_5 x) = o(g(x))$. Finally, if $w_2(n)$ is prime, then $w_2(n) = p(w_2(n)) < 2\log_2^{1/2} x = o(g(x))$. And thus, we have shown that $\omega(n)$ divides $M(x)$, and therefore both $\phi(n)$ and $\sigma(n)$, and a similar argument applies to $\Omega(n)$.

To see that $\tau(n)$ divides both $\phi(n)$ and $\sigma(n)$, write

$$\tau(n) := \prod_{p^a p \parallel n} (a_p + 1) = \prod_{q^{c_q} \parallel \tau(n)} q^{c_q}.$$

We first assume that q is an odd prime divisor of $\tau(n)$. Then, by condition (3) of Lemma 2.1,

$$q^{c_q} \mid \prod_{\substack{p < \log_7 x \\ p^a p \parallel n}} (a_p + 1),$$

and $a_p < \log_7 x$, therefore we deduce that

$$\log(q^{c_q}) \leq \sum_{\substack{p < \log_7 x \\ p^{a_p} \parallel n}} \log(a_p + 1) \ll \pi(\log_7 x) \log(\log_7 x + 1) \ll \log_7 x < \log_6 x$$

holds for large x , and so $q^{c_q} < \exp(\log_6 x) = \log_5 x = o(g(x))$. Hence, q^{c_q} divides $M(x)$. Assume now that $q = 2$ and for every $p \mid n$ write d_p for the exponent at which 2 divides $a_p + 1$. Then, $c_2 = \sum_{p \mid n} d_p$ and, by condition (3) of Lemma 2.1, we have $d_p = 1$ whenever $p \geq \log_7 x$ and $d_p \leq C_4 \log(\log_7 x + 1) < \log_7 x$ with $C_4 := 1/\log 2$, whenever $p < \log_7 x$. Thus, with conditions (1) and (3) of Lemma 2.1, the inequality

$$(3.2) \quad c_2 \leq \omega(n) + \sum_{\substack{p < \log_7 x \\ p^{a_p} \parallel n}} d_p < \log_2 x + \delta(x) \log_2^{1/2} x + \pi(\log_7 x) \log_7 x < \log_2 x + 2\delta(x) \log_2^{1/2} x$$

holds for large values of x . However, the power at which 2 which divides $\phi(n)$ is, by Lemma 2.1, at least

$$(3.3) \quad \omega(n) + \omega_1(n) - 1 \geq \frac{5 \log_2 x}{4} - \delta(x) \log_2^{1/2} x - 1,$$

and it is clear that the right-hand side of (3.3) is larger than the right-hand side of (3.2) for large x . Thus, $2^{c_2} \mid \phi(n)$. To see the statement for σ , notice that by Lemma 2.1, we have that the power at which 2 divides $\sigma(n)$ is at least

$$(3.4) \quad \omega(n) + \omega_3(n) - 1 - 2\pi(\log_7 x) > \frac{5 \log_2 x}{4} - 1 - \delta(x) \log_2^{1/2} x - \log_7 x,$$

and it is clear that the right-hand side of (3.4) is also larger than the right-hand side of (3.2) for large x . This shows that 2^{c_2} divides $\sigma(n)$ as well. We point out that the fact that the set shown at (1.3) with $\phi(n)$ replaced by $\sigma(n)$ and $\omega(n)$ replaced by $\tau(n)$ is of asymptotic density zero has also been proved in [1].

(4) Write $\Delta(n) := \Omega(n) - \omega(n)$. If n satisfies congruence (1.4), then $f(\Delta(n)) \equiv 0 \pmod{\omega(n)}$. By the remark following Lemma 2.1, we know that $\Delta(n) < \log_6 x$, therefore $|f(\Delta(n))| \ll \log_6^d x$, where we use again d for the degree of the polynomial f . However, by condition (1) of Lemma 2.1, we know that $\omega(n) \gg \log_2 x$, and therefore $|f(\Delta(n))| < \omega(n)$ holds when $n \in B(x)$ and x is large. Thus, except for a set of asymptotic density zero of positive integers n , the congruence (1.4) forces $f(\Delta(n)) = 0$. Since $\Delta(n)$ is a nonnegative integer, this will happen only if f has nonnegative integer roots k , and $\Delta(n) = k$ for such n and with k one of these nonnegative integer roots. Conversely, if f has nonnegative integer roots k , then any number

n with $\Delta(n) = k$ will satisfy congruence (1.4). The corresponding statement about $f(\omega(n))$ being a multiple of $\Omega(n)$ can be dealt with in a similar way.

(1) Let w_2 be a squarefree number belonging to the interval $\mathcal{K} := (\log_2^{1/3} x, 2 \log_2 x)$ having $p(w_2) > \log_3 x \log_5 x$, and assume that n satisfies congruence (1.1) and that $w_2(n) = w_2$. By Lemma 2.2, we know that if $n \in B(x)$, then n has such a factor w_2 which is moreover coprime to n . Since w_2 is squarefree and its smallest prime factor is large, it follows that for large x the congruence $f(n) \equiv 0 \pmod{w_2}$ implies that we may replace f by the product of all its primitive nonassociated factors in $\mathbb{Z}[X]$, that is, we may assume that $f(X)$ is primitive and squarefree as an element of $\mathbb{Z}[X]$. Write again d for the degree of f . The congruence $f(n) \equiv 0 \pmod{w_2}$ puts n into at most $d^{\omega(w_2)}$ congruence classes modulo w_2 . Let $\alpha \pmod{w_2}$ be such a congruence class. Then $n = \alpha + w_2 m$ holds with some nonnegative integer m , and since by Lemma 2.2 we know that w_2 and n are coprime, it follows that α and w_2 are coprime. We also write $\omega(n) = w_2 j$. Fixing j and α , we conclude that $n < x$ is in the arithmetic progression $\alpha \pmod{w_2}$, and has a fixed value of $\omega(n) = w_2 j$. It is known that for every fixed positive integer l , the number of positive integers $n < x$ with $n \equiv \alpha \pmod{w_2}$ having $\omega(n) = l$ is

$$(3.5) \qquad \ll \frac{x}{\phi(w_2) \log_2^{1/2} x}$$

and that estimate (3.5) above is uniform in our range $l < 2 \log_2 x$ and $w_2 < 2 \log_2 x$. Indeed, this can be obtained from the main result in [15] together with inequality (2.10). Now $\omega(n) = w_2 j$ is a number in the interval \mathcal{I} shown at (2.11), and so the number of numbers j when w_2 is fixed is

$$\leq \frac{2\delta(x) \log_2^{1/2} x}{w_2} + 1.$$

Summing up the above inequalities over all the allowable values of j , we conclude that if w_2 and α are fixed, then the number of such numbers $n \in B(x)$ which are congruent to α modulo w_2 is

$$\ll \frac{x\delta(x)}{w_2\phi(w_2)} + \frac{x}{\phi(w_2) \log_2^{1/2} x}.$$

Summing up the above inequalities over all the possible values of α , we conclude that the number of numbers $n \in B(x)$ satisfying congruence (1.1) and for which $w_2(n) = w_2$ is fixed is

$$(3.6) \qquad \ll \frac{x\delta(x)d^{\omega(w_2)}}{w_2\phi(w_2)} + \frac{x d^{\omega(w_2)}}{\phi(w_2) \log_2^{1/2} x}.$$

Finally, summing up inequalities (3.6) over all the allowable values of w_2 , we get that the number of numbers $n \in B(x)$ satisfying congruence (1.1) is

$$(3.7) \quad \ll x\delta(x) \sum_{w_2 \in \mathcal{X}} \frac{d^{\omega(w_2)}}{w_2 \phi(w_2)} + \frac{x}{\log_2^{1/2} x} \sum_{w_2 \in \mathcal{X}} \frac{d^{\omega(w_2)}}{\phi(w_2)}.$$

Notice that $d^{\omega(w_2)} = (2^{\omega(w_2)})^{C_5} < (\tau(w_2))^{C_5}$, with $C_5 := \log d / \log 2$. Since for every $\varepsilon > 0$ the inequality $\tau(t) < t^\varepsilon$ holds for all positive integers $t > t_\varepsilon$, it follows that if we set $\varepsilon := (6C_5)^{-1}$, then the inequality $d^{\omega(w_2)} < w_2^{1/6}$ holds for all $w_2 \in \mathcal{X}$ and for sufficiently large values of x . Since we also know that $\phi(m) \gg m / \log_2 m$ holds for all positive integers m , it follows that the inequality $\phi(w_2) > w_2^{5/6}$ holds for all $w_2 \in \mathcal{X}$ and when x is sufficiently large. Thus, if x is large the expression appearing at (3.7) can be bounded above by

$$\begin{aligned} &\ll x\delta(x) \sum_{w_2 \in \mathcal{X}} \frac{1}{w_2^{5/3}} + \frac{x}{\log_2^{1/2} x} \sum_{w_2 \in \mathcal{X}} \frac{1}{w_2^{2/3}} \\ &\ll x\delta(x) \int_{\log_2^{1/3} x}^{2 \log_2 x} \frac{ds}{s^{5/3}} + \frac{x}{\log_2^{1/2} x} \int_{\log_2^{1/3} x}^{2 \log_2 x} \frac{ds}{s^{2/3}} \\ &\ll \frac{x\delta(x)}{\log_2^{2/9} x} + \frac{x}{\log_2^{1/6} x} = o(x). \end{aligned}$$

The same argument applies when $\omega(n)$ is replaced by $\Omega(n)$.

Finally, to see the statement with $\omega(n)$ replaced by $\tau(n)$ in (1.1), write $h(x) := \lfloor (\log_2 x) / 2d \rfloor$ and let $n \in B(x)$ be a number satisfying congruence (1.1) with $\tau(n)$ instead of $\omega(n)$. By condition (3) of Lemma 2.1, if we set again c_2 to be the exponent at which 2 divides $\tau(n)$, we have that the inequality

$$(3.8) \quad c_2 \geq \omega(n) - \pi(\log_7 x) \geq \log_2 x - \delta(x) \log_2^{1/2} x - \log_7 x > \frac{\log_2 x}{2}$$

holds for large x . We write K for the number of irreducible factors of f , and we write f_1, \dots, f_K for these irreducible factors. The congruence $f(n) \equiv 0 \pmod{2^{c_2}}$ together with the lower bound (3.8) on c_2 , imply that there exists $i = 1, \dots, K$ such that $f_i(n) \equiv 0 \pmod{2^{h(x)}}$ holds. Since all the f_i 's are irreducible, with i fixed the above congruence puts n into at most C_6 arithmetic progressions of ratio $2^{h(x)}$, where one can choose C_6 to be an upper bound for the absolute values of all the discriminants of the polynomials f_i for $i = 1, \dots, K$. Thus, if $n \in B(x)$ satisfies (1.1) with $\omega(n)$ replaced by $\tau(n)$, then n is in at most $C_7 := KC_6$ arithmetic progressions of ratio $2^{h(x)}$. Each one of these arithmetic progressions will contain at most

$$\frac{x}{2^{h(x)}} + 1$$

positive integers $n < x$, thus the total number of numbers $n \in B(x)$ satisfying $f(n) \equiv 0 \pmod{\tau(n)}$ is at most

$$\frac{C_7 x}{2^{h(x)}} + C_7 = o(x),$$

which completes the proof of our theorem. \square

We point out that when $f(n) := n$, the problem of estimating the number of positive integers $n < x$ for which $\tau(n)|n$ was treated by Spiro in [12].

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Mathematical Institute

UNAM

Ap. Postal 61-3 (Xangari)

CP 58 089 Morelia

Michoacán

Mexico

e-mail: fluca@matmor.unam.mx