ON GROUPS WITH ALL SUBGROUPS ALMOST SUBNORMAL

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Abstract

In this paper we consider groups in which every subgroup has finite index in the nth term of its normal closure series, for a fixed integer n. We prove that such a group is the extension of a finite normal subgroup by a nilpotent group, whose class is bounded in terms of n only, provided it is either periodic or torsion-free.

A subgroup H of a group G is said to be *almost subnormal* if it has finite index in some subnormal subgroup of G. This occurs when H has finite index in some term $H^{G,n}$, $n \geq 0$, of its normal closure series in G; recall that $H^{G,0} = G$ and $H^{G,n} = H^{H^{G,n-1}}$.

A finite-by-nilpotent group has every subgroup almost subnormal, and for finitely generated groups the converse holds (see [8, 6.3.3]). Note that, if a group G has a finite normal subgroup N such that G/N is nilpotent of class n, then each subgroup H of G has finite index in $H^{G,n}$. For $n = 1$, the converse is settled by a well-known theorem of Neumann [10]: a group G, in which every subgroup H has finite index in its normal closure H^G, is finite-by-abelian. Later, Lennox [7] considered the case in which n is larger than 1 and there is also a bound on the indices. He proved that there exists a function μ such that if $|H^{G,n} : H| \leq c$ for every subgroup H of a group G, where n and c are fixed integer, then the $\mu(n+c)$-th term $\gamma_{\mu(n+c)}(G)$ of the lower central series of G is finite of order at most $c!$. Recall that a theorem by Roseblade states that a group G in which $H = H^{G,n}$ for every subgroup H, is nilpotent and $\gamma_{\rho(n)+1}(G) = 1$, for a well-defined function ρ. Recently, Casolo and Mainardis in [2, 3] gave a description of the structure of groups with all subgroups almost subnormal, proving, in particular, that such groups are finite-by-soluble.

In this paper we consider the class A_n, $n \geq 1$, of groups G in which $|H^{G,n} : H|$ is finite for every subgroup H of G, but no bound on the indices $|H^{G,n} : H|$ is assumed. In particular, we give a generalization of Neumann's theorem to periodic A_n-groups:

THEOREM 1. *There exists a function δ of n, such that if G is a torsion group with the property that $|H^{G,n} : H| < \infty$ for every subgroup H of G, then $\gamma_{\delta(n)}(G)$ is finite.*

We then consider torsion-free groups. By a result due to Casolo and Mainardis [2], torsion-free A_n-groups have every subgroup subnormal and so they turn out to be nilpotent, by a recent result by Smith [14] (see also Casolo [1]). Here, we give a different proof of their nilpotency and, in particular, a bound on their nilpotency class, thus generalizing Neumann's theorem to torsion-free A_n-groups:

THEOREM 2. *There exists a function η of n such that each torsion-free group G in which $|H^{G,n} : H| < \infty$ for every subgroup H, is nilpotent of class at most $\eta(n)$.*

This also gives a different proof of Roseblade's theorem for torsion-free groups with all subgroups subnormal of bounded defect.

Finally, we observe that Smith in [13] gives examples of A_2-groups which are not finite-by-nilpotent. Thus, Theorem 1 and Theorem 2 are no longer true if we drop the assumptions that G is either periodic or torsion-free. Also, Casolo and Mainardis, in [2], construct a non-hypercentral A_2-group. On the other hand, in Proposition 13 we shall prove that locally nilpotent A_n-groups are hypercentral, partially answering the question posed in[8, page 191]. Recall that Heineken-Mohamed groups [6] are example of groups in which every subgroups is almost subnormal but they do not belong to any of the classes A_n.

1. A_n^+-groups

In order to achieve our result on periodic A_n-groups, we find it convenient to study a larger class of groups. We denote by A_n^+ the class of all groups G in which there exists a finite subgroup F with the property that every subgroup H containing F has finite index in the nth term $H^{G,n}$ of its normal closure series. By abuse of notation, we shall denote the above by $(G, F) \in A_n^+$. Note that $A_n \subseteq A_n^+$ but $A_n \neq A_n^+$. Indeed, the group described in [4, Proposition 4] is a periodic A_2^+-group but it is not finite-by-nilpotent, and so, by Theorem 1, it does not belong to A_n.

Also, we denote by $\mathfrak{U}_n^+$ the class of all groups G in which there exists a finite subgroup F such that every subgroup of G containing F is subnormal of defect at most n in G. Clearly, $\mathfrak{U}_n^+ \subseteq A_n^+$, but $\mathfrak{U}_n^+ \neq A_n^+$, since Smith's groups [13] are locally nilpotent A_2-groups which are not finite-by-nilpotent while, for $\mathfrak{U}_n^+$-groups, the following holds:

THEOREM 3 (Detomi [4]). *There exists a function $\beta(n)$ of n, such that if G belongs to $\mathfrak{U}_n^+$ and it is either a locally nilpotent group or a torsion group with $\pi(G)$ finite, then $\gamma_{\beta(n)}(G)$ is finite. In particular, if G is locally nilpotent, then G is nilpotent and its nilpotency class is bounded by a function depending on n and $|F|$.*

Here $\pi(G)$ denotes the set of primes dividing the orders of the elements of G.

The following are two known result which we include without proofs. If N is a subgroup (normal subgroup) with finite index in G, then we write $N \leq_f G$ ($N \trianglelefteq_f G$).

LEMMA 4. *Let G be a countable residually finite group and let H be a finite subgroup of G. Then $H = \bigcap_{N \trianglelefteq_f G} HN$.*

LEMMA 5. *Let G be a group and let F be a finitely generated subgroup of a subgroup H of G. If $[G,_n V] \leq V$ for every finitely generated subgroup V of H such that $F \leq V$, then $[G,_n H] \leq H$.*

We establish an elementary property of periodic A_n^+-groups:

LEMMA 6. *A periodic A_n^+-group is locally finite and finite-by-soluble.*

PROOF. Let $(G, F) \in A_n^+$. Then $F \leq_f F^{G,n}$ gives that $F^{G,n}$ is finite and that every section $F^{G,i}/F^{G,i+1}$ belongs to A_n. Since, by the already mentioned result by Casolo-Mainardis, every A_n-group is finite-by-soluble, the group G has a finite series in which each factor is finite or soluble.

Let X be a finitely generated subgroup of G. Clearly X has a finite series with finite or soluble factors. Hence, since a finitely generated torsion soluble group is finite and a subgroup with finite index in a finitely generated group is finitely generated, each factor in this series of X is finite, and so X is finite. This proves that G is locally nilpotent.

Now, since G has a finite series with finite or soluble factors, to prove that G is finite-by-soluble, it is sufficient to show that soluble-by-finite periodic A_n^+-groups are finite-by-soluble.

Let $(G, F) \in A_n^+$ be a torsion group and let A be a soluble normal subgroup with finite index in G. We can assume that $A \trianglelefteq G$, since A_G has finite index in G. Let τ be a left transversal to A in G and set $H = \langle \tau, F \rangle$. As H has finite index in $K = H^{G,n}$, K is finitely generated and hence finite, by the local finiteness of G. Note that $G = AK$.

We proceed by induction on the defect d of subnormality of K in G. If K is normal in G, then $G/K \cong A/A \cap K$ is soluble, and we are done. If $d > 1$, then, as K has defect of subnormality bounded by $d-1$ in K^G, we can apply the induction hypothesis to K^G, obtaining that some term of the derived series of K^G is finite (and normal in G). Therefore, as $G/K^G \cong A/A \cap K^G$ is soluble, we get that G is finite-by-soluble, which is the desired conclusion.   □

With the same argument as in [4, Lemma 9], it is easy to see that:

LEMMA 7. *Let $G \in A_n^+$ be a locally finite group. If there exists a subgroup A with finite index in G such that $\gamma_{m+1}(A)$ is finite, then $\gamma_{nm+1}(G)$ is finite.*

Roughly speaking, the next proposition says that periodic A_n^+-groups are near to being $\mathfrak{U}_n^+$-groups.

PROPOSITION 8. *Let G be a countable residually finite torsion group and let $G \in A_n^+$. Then there exists a subgroup A with finite index in G such that $A \in \mathfrak{U}_n^+$.*

PROOF. Assume that the lemma is false and let G be a counterexample. Proceeding recursively we construct

(a) a descending chain $\{K_i \mid i \in \mathbb{N}\}$ of subgroups with finite index in G,
(b) an ascending chain $\{F_i \mid i \in \mathbb{N}\}$ of finitely generated subgroups of $\bigcap_{i=0}^{\infty} K_i$, and
(c) a sequence of elements $\{x_i \in [K_{i-1},_n F_i]\backslash K_i \mid 1 \leq i \in \mathbb{N}\}$.

Set $K_0 = G$ and let F_0 be a finite subgroup of G such that $|H^{G,n} : H| < \infty$ whenever $F_0 \leq H \leq G$.

Suppose we have already defined F_i, K_i, and $x_i \in [K_{i-1},_n F_i]\backslash K_i$. As F_i is a finitely generated subgroup of $K_i \leq_f G$, and as G is a counterexample, there exists a subgroup $F_i \leq H \leq K_i$ which is not subnormal of defect less or equal to n in K_i, that is, $[K_i,_n H] \not\leq H$. So, by Lemma 5, there exists a finitely generated subgroup F_{i+1} of H with $F_i \leq F_{i+1}$ and $[K_i,_n F_{i+1}] \not\leq F_{i+1}$. Let us fix an element $x_{i+1} \in [K_i,_n F_{i+1}]\backslash F_{i+1}$. Since, by Lemma 6, G is locally finite, we can apply Lemma 4 to the finitely generated, hence finite subgroup F_{i+1}, and so we get that $x_{i+1} \notin F_{i+1}N$ for a suitable subgroup $N \trianglelefteq_f K_i$. Then we set $K_{i+1} = F_{i+1}N$, so that $F_{i+1} \leq K_{i+1} \leq_f G$ and $x_{i+1} \in [K_i,_n F_{i+1}]\backslash K_{i+1}$. Note that K_{i+1} contains all the subgroups $F_0, \ldots, F_{i+1}$.

Now we consider the subgroups $K = \bigcap_{i \in \mathbb{N}} K_i$ and $H = \langle F_i \mid i \in \mathbb{N} \rangle$. Since $H \geq F_0$, by assumption we have that H has finite index in $H^{G,n}$. So, the chain $\{H^{G,n} \cap K_i\}_{i \in \mathbb{N}}$, stretching from $H^{G,n}$ to H, is finite and there exists an integer i such that $H^{G,n} \cap K_i = H^{G,n} \cap K_j$ for every $j \geq i$. But, since $[G,_n H] \leq H^{G,n}$ and $F_{i+1} \leq H \cap K_i$, we get that

$$x_{i+1} \in [K_i,_n F_{i+1}] \leq [K_i,_n H \cap K_i] \leq [G,_n H] \cap K_i$$
$$\leq H^{G,n} \cap K_i = H^{G,n} \cap K_{i+1},$$

that is $x_{i+1} \in K_{i+1}$, in contradiction to our construction.   □

THEOREM 9. *There exists a function $\delta(n)$ of n, such that if G is a periodic A_n^+-group and if either G is locally nilpotent or $\pi(G)$ is finite, then $\gamma_{\delta(n)}(G)$ is finite. In particular, if G is locally nilpotent then G is nilpotent.*

PROOF. Set $\delta(1) = 2$ and define recursively $\delta(n) = 2n(\beta(n)-1) + 2\delta(n-1) + 1$, where β is the function defined in Theorem 3.

Assume first that G is countable. We shall proceed by induction on n. Let F be a finite subgroup of G such that every subgroup H containing F has finite index in $H^{G,n}$.

If $n = 1$ then $|F^G : F| < \infty$ and F^G is finite. Since $G/F^G \in A_1$, the quotient $G'F^G/F^G$ is finite by Neumann's theorem. Hence $G' = \gamma_2(G)$ is finite.

Let now $n > 1$ and let X be a finitely generated subgroup of G with $X \geq F$. Because G is locally finite, X is finite. Observe that, for every subgroup H of X^G containing X, we have $H^G = X^G$ and so $|H^{X^G,n-1} : H| < \infty$. Thus X^G belongs to A_{n-1}^+ and by the inductive hypothesis we get that $\gamma_{\delta(n-1)}(X^G)$ is finite. Now, by a theorem of Hall it follows that $\zeta_{2\delta(n-1)-2}(X^G)$ has finite index in X^G. Thus, the index of $C_G(X^G/\zeta_{2\delta(n-1)-2}(X^G))$ in G is finite and, denoting by $R = \bigcap_{N \trianglelefteq_f G} N$ the finite residual of G, we obtain that $[R, X^G] \leq \zeta_{2\delta(n-1)-2}(X^G)$. In particular,

$$[R, _{2\delta(n-1)} X^G] \leq \left[R, X^G, _{2\delta(n-1)-2} X^G\right] = 1.$$

Therefore, if we take $s = 2\delta(n-1)$ elements in G, say $x_1, \ldots, x_s$, and we consider the finitely generated subgroup $X = \langle x_1, \ldots, x_s, F \rangle$, then we get $[R, x_1, \ldots, x_s] \leq [R, _s X^G] = 1$, which implies $R \leq \zeta_s(G)$.

Now, as $G/R \in A_n^+$ is a countable residually finite torsion group, by Proposition 8 it follows that there exists a subgroup A with finite index in G, such that $A/R \in \mathfrak{U}_n^+$. Also, A/R satisfies the assumptions of Theorem 3 and so $\gamma_{\beta(n)}(A/R)$ is finite. By Lemma 7 it follows that $\gamma_{n(\beta(n)-1)+1}(G/R)$ is finite and then Hall's theorem gives that $\zeta_{2n(\beta(n)-1)}(G/R)$ has finite index in G/R. Therefore, as $R \leq \zeta_s(G)$, clearly $\zeta_{2n(\beta(n)-1)+s}(G)$ has finite index in G and, by a theorem of Baer (see [12, 14.5.1]), we conclude that $\gamma_{2n(\beta(n)-1)+s+1}(G)$ is finite. This proves that $\gamma_{\delta(n)}(G)$ is finite, for every countable group G satisfying the assumption of the theorem.

For the general case, we assume, contrary to our claim, that there exists a group G, satisfying the assumption of the theorem, such that $\gamma_{\delta(n)}(G)$ is not finite.

Let T be a countable and not finite subset of $\gamma_{\delta(n)}(G)$. Then we can find a countable set of commutators $x_i = [y_{1,i}, \ldots, y_{\delta(n),i}], i \in \mathbb{N}, y_{j,i} \in G$, such that $T \leq \langle x_i \mid i \in \mathbb{N} \rangle$. Let $Y = \langle F, y_{j,i} \mid j = 1, \ldots, \delta(n), i \in \mathbb{N} \rangle$. As Y is a countable A_n^+-group, by the first part of the proof, $\gamma_{\delta(n)}(Y)$ is finite. Thus $T \subseteq \gamma_{\delta(n)}(Y)$ is finite, against our assumption.

Finally, if G is locally nilpotent, since every finite normal subgroup is contained in some term of the upper central series (by a theorem of Mal'cev and McLain [12, 12.1.6]), it follows that G is nilpotent, and the proof is complete. □

As a consequence, we get the announced result on periodic A_n-groups:

PROOF OF THEOREM 1. Let G be a periodic A_n-group. By a result of Casolo and

Mainardis [3], there exists a finite normal subgroup N of G such that G/N has every subgroup subnormal. In particular, G/N is locally nilpotent. Now Theorem 9 gives that $\gamma_{\delta(n)}(G/N)$ is finite and, as N is finite, the result follows.   □

2. Torsion-free A_n-groups

First we observe some basic properties of isolators in locally nilpotent groups. Recall that the *isolator* of a subgroup H in a group G is defined to be the set $I_G(H) = \{x \in G \mid x^n \in H \text{ for some } 1 \leq n \in \mathbb{N}\}$. If G is a locally nilpotent group then $I_G(H)$ is a subgroup of G and if G is also torsion-free then $\gamma_n(I_G(H)) \leq I_G(\gamma_n(H))$ (see, for example, [5, 9]).

LEMMA 10. *Let G be a locally nilpotent group and let $H \leq G$. Then*
(1) *if $I_G(H)$ is finitely generated, then $|I_G(H) : H| < \infty$;*
(2) *if G is torsion-free and H is cyclic, then $I_G(H)$ is locally cyclic.*

PROOF. (1) As $K = I_G(H)$ is a finitely generated nilpotent group, H is subnormal in K, say $H = H^{K,n}$ for an integer n, and every section $H^{K,i}/H^{K,i+1}$ is finitely generated and nilpotent, for $i = 1, \ldots, n-1$. Furthermore, by definition of $I_G(H)$, each $H^{K,i}/H^{K,i+1}$ is periodic and hence finite. Thus, H has finite index in K.

(2) Let K be a finitely generated subgroup of $I_G(H)$. As H is cyclic, we can assume that $H \leq K$. Since K is torsion-free and nilpotent, it has a central series with infinite cyclic factors (see [12, 5.2.20]). So, if K is not cyclic, there is a cyclic normal subgroup N of K with infinite index in K. Now, since, by (1), H has finite index in K, then $H \cap N \neq 1$. Therefore, as H is cyclic, $|K/N| \leq |NH/N| = |H/H \cap N|$ is finite, a contradiction.   □

We state now a consequence of a well-known argument by Robinson (see [12, 5.2.5]). Recall that the Hirsch length of a polycyclic group G is the number of infinite factors in a series of G with cyclic factors.

LEMMA 11. *Let H be a nilpotent group of class c. If H/H' can be generated by r elements, then the Hirsch length h of H is bounded by a function $g(c, r)$ of c and r.*

The already mentioned theorem of Mal'cev and McLain [12, 12.1.6] states that each principal factor of a locally nilpotent group is central. The following consequence is well known, but we include the easy proof for the convenience of the reader:

LEMMA 12. *Let G be a locally nilpotent group and let N be a finitely generated normal subgroup of G. Then there exists an integer n such that $N \leq \zeta_n(G)$. Moreover, if N is torsion-free with Hirsch length h, then $N \leq \zeta_h(G)$.*

PROOF. The theorem of Mal'cev and McLain implies that if N is finite then it is contained in $\zeta_m(G)$ for an integer m bounded by the composition length of N. Also, when N is torsion-free with Hirsch length h, we get that N/N^p is finite and so $N/N^p \leq \zeta_h(G/N^p)$ for every prime p; therefore $[N,_h G] \leq \bigcap_p N^p = 1$ by a residual property of torsion-free finitely generated nilpotent groups (see for example [11, page 170]). Since the torsion subgroup of a finitely generated normal subgroup of G is finite, the lemma follows. □

PROPOSITION 13. *Let G be a locally nilpotent A_n-group. Then G is hypercentral.*

PROOF. By an already cited result of Casolo and Mainardis, A_n-groups are finite-by-soluble and so G is soluble. It is sufficient to prove that G has a non trivial centre. We proceed by induction on the derived length of G. Let A be the centre of G'; by inductive assumption, $A \neq 1$. Let H be a finitely generated subgroup of G. As $|H^{G,n} : H|$ is finite, $H^{G,n}$ is finitely generated and so nilpotent; in particular, $[A,_n H]$ is finitely generated. Since $A = \zeta(G')$, $[A,_n H]^g = [A,_n H^g] \leq [A,_n H[H, \langle g \rangle]] = [A,_n H]$ for $g \in G$, and so $[A,_n H]$ is normal in G. Thus Proposition 12 gives that $[A,_n H] \leq \zeta_k(G)$ for some $k \geq 1$. So, if $[A,_n H] \neq 1$, then $\zeta(G) \neq 1$. Otherwise, $[A,_n H] = 1$ for any finitely generated subgroup of G; thus $A \leq \zeta_n(G)$ and we again conclude that $\zeta(G) \neq 1$. □

A group G is said n-*Engel* if $[x,_n y] = 1$ for all $x, y \in G$. We recall that a torsion-free soluble n-Engel group G with positive derived length d is nilpotent of class at most n^{d-1} (see [11, 7.36]).

Our interest on Engel groups is motivated by the following fact:

LEMMA 14. *A torsion-free A_n-group is $(n + 1)$-Engel.*

PROOF. Let G be a torsion-free A_n-group and let $1 \neq x \in G$. By the definition of the class A_n, $\langle x \rangle$ has finite index in $\langle x \rangle^{G,n}$, so that $\langle x \rangle^{G,n}$ is a finitely generated subgroup of $I_G(\langle x \rangle)$. By the already mentioned result in [2], every subgroup of G is subnormal, so that G is locally nilpotent. Thus, by Lemma 10, $\langle x \rangle^{G,n}$ is cyclic, so that $\langle x \rangle \operatorname{char} \langle x \rangle^{G,n}$, and hence $\langle x \rangle$ is subnormal of defect at most n in G, that is $[G,_n x] \leq \langle x \rangle$. Therefore, $[G,_{n+1} x] = [G,_n x, x] = 1$, as claimed. □

Now we are in a position to prove the announced result on torsion-free A_n-groups.

PROOF OF THEOREM 2. Let $G \in A_n$ be a torsion-free group. As already noted, by a result in [2], G is locally nilpotent.

Note that, if there exists a function $\eta(n)$ such that $\gamma_{\eta(n)+1}(H) = 1$, for every finitely generated subgroup H of G, then $\gamma_{\eta(n)+1}(G) = 1$. Hence, without loss of generality,

we can assume that G is a finitely generated group. In particular, we get that G is nilpotent and every subgroup of G is finitely generated.

Proceeding by induction on n, we prove that there exists a function $\eta(n)$ such that every torsion-free finitely generated A_n-group has nilpotency class at most $\eta(n)$.

If $n = 1$, then Neumann's theorem gives that G' is finite. Hence, since G is torsion-free, G is abelian, and so we can set $\eta(1) = 1$.

Let now $n > 1$ and let H be a subgroup of G. Set $H^{G,i} = H_i$ for every i, so that, by the definition of the class A_n, we have

$$H \leq_f H_n \trianglelefteq H_{n-1} \trianglelefteq \cdots \trianglelefteq H_2 \trianglelefteq H_1 \trianglelefteq G.$$

Note that, for every subgroup K such that $H \leq K \leq H_1$, we get $K^G = H^G = H_1$ and $K \leq_f K^{K^{G,n-1}}$. Hence $H_1/H_2 \in A_{n-1}$. With the same argument it is easy to see that the factor H_i/H_{i+1}, for $i = 1, \ldots, n - 1$, belongs to A_{n-i}. By the induction hypothesis, the factor $H_i/I_{H_i}(H_{i+1})$, being a finitely generated torsion-free A_{n-i}-group, has nilpotency class at most $\eta(n - i)$; hence,

$$\gamma_{\eta(n-i)+1}(H_i) \leq I_{H_i}(H_{i+1}) \leq I_G(H_{i+1}).$$

Thus,

$$\gamma_{\eta(n-i)+1}(I_G(H_i)) \leq I_G(\gamma_{\eta(n-i)+1}(H_i)) \leq I_G(I_G(H_{i+1})) = I_G(H_{i+1}),$$

for every i, so that

$$\gamma_{\eta(n-1)+1}(I_G(H_1)) \leq I_G(H_2),$$
$$\gamma_{\eta(n-2)+1}\left(\gamma_{\eta(n-1)+1}(I_G(H_1))\right) \leq \gamma_{\eta(n-2)+1}(I_G(H_2)) \leq I_G(H_3),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\gamma_{\eta(1)+1}\left(\gamma_{\eta(2)+1}\left(\cdots\left(\gamma_{\eta(n-1)+1}(I_G(H_1))\right)\cdots\right)\right) \leq \gamma_{\eta(1)+1}(I_G(H_{n-1}))$$
$$\leq I_G(H_n) = I_G(H),$$

where the last equality is due to the fact that $H \leq_f H_n \leq I_G(H)$.

In particular, for $k = k(n) = \sum_{i=1}^{n-1}(\eta(i) + 1)$, the kth term $H_1^{(k)}$ of the derived series of H_1 is a subgroup of $I_G(H)$, so that $I_G(H_1^{(k)}) \leq I_G(H)$. Now, by Lemma 14, $H_1/I_G(H_1^{(k)})$ is a soluble torsion-free $(n + 1)$-Engel group and so $H_1/I_G(H_1^{(k)})$ is nilpotent of class at most $(n + 1)^{k-1}$. Thus, for $c = c(n) = (n + 1)^{k-1} + 1$, we get that $\gamma_c(H_1) \leq I_G(H_1^{(k)}) \leq I_G(H)$. This proves that $\gamma_c(H^G) \leq I_G(H)$, for every subgroup H of G.

Now take c elements of G, say $x_1, \ldots, x_c$, and consider the subgroup $H = \langle x_1, \ldots, x_c \rangle$. Clearly we can write $H_1 = H^G$ as a product of the c normal subgroups $\langle x_i \rangle^G$. Since $\gamma_c(\langle x_i \rangle^G) \leq I_G(\langle x_i \rangle)$ and, by Lemma 10, $I_G(\langle x_i \rangle)$ is a cyclic

group, then $[\gamma_c(\langle x_i\rangle^G), x_i] = 1$. Moreover $[\gamma_c(\langle x_i\rangle^G), x_i^g] = 1$ for every $g \in G$. Thus $\gamma_c(\langle x_i\rangle^G) \le \zeta(\langle x_i\rangle^G)$ and $\langle x_i\rangle^G$ has nilpotency class at most c. Therefore H_1 is generated by c normal nilpotent subgroups of class at most c, and by Fitting's theorem it follows that H_1 is nilpotent with class $\mathrm{cl}(H_1) \le c^2$.

Now, since H is a c-generated torsion-free nilpotent group of class $\mathrm{cl}(H) \le \mathrm{cl}(H_1) \le c^2$, Lemma 11 implies that the Hirsch length $h(H)$ of H is bounded by

$$g_1 = g(c^2, c) = \frac{c^{c^2+1} - 1}{c - 1}.$$

Also, by Lemma 10, $|I_G(H) : H| < \infty$, so that $h(I_G(H)) = h(H) \le g_1$.

Therefore, $\gamma_c(H_1)$ is a finitely generated normal subgroup of G with Hirsch length $h(\gamma_c(H_1)) \le h(I_G(H)) \le g_1$ and so, by Proposition 12, $\gamma_c(H_1) \le \zeta_{g_1}(G)$. In particular, $[x_1, \ldots, x_c, y_1, \ldots, y_{g_1}] = 1$ for every $y_1, \ldots, y_{g_1}$ in G, so that

$$\gamma_{c+g_1}(G) = 1.$$

Finally, since $c = c(n)$ and $g_1 = g_1(n)$ depend only on n, the result follows on defining $\eta(n) = c + g_1 - 1$. $\square$

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