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TWO-SIDED IDEALS IN GROUP NEAR-RINGS

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Abstract

The two-sided ideals of group near-rings are characterized and studied. Various examples are presented to illustrate the interplay between ideals in the base near-ring R and the corresponding group near-ring $R[G]$. Some results concerning the Jacobson radicals of $R[G]$ are also discussed.

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1. Introduction

In [2] group near-rings have been defined in their most general form. Since then, some work has been done on the ideal theory of group near-rings (see [1, 5]), but only for certain special cases, such as for near-rings which are distributively generated. This paper is meant to be the first step towards laying the groundwork for the ideal theory of general group near-rings. These ideals are characterized and some of their fundamental properties are revealed. Several results from matrix near-ring theory are utilized in order to do so.

Throughout this paper, R denotes a right near-ring with identity 1 and G denotes a (multiplicatively written) group with identity e and with $|G| \geq 2$. For general results on near-rings, the reader is referred to a standard textbook such as [9]. Recall that for any (additively written) group H, the set of all mappings $f : H \to H$ under the operations of pointwise addition and composition, forms a near-ring, denoted by $M(H)$. We need this in the following

DEFINITION 1.1 ([2]). Let R^G denote the direct sum of $|G|$ copies of the group $(R, +)$. The *group near-ring* constructed from R and G, denoted $R[G]$, is the subnear-ring of $M(R^G)$ generated by the set $\{[r, g] \in M(R^G) : r \in R, g \in G\}$, where $[r, g] : R^G \to R^G$ is defined by $([r, g](\mu))(h) = r\mu(hg)$, for all $\mu \in R^G$ and $h \in G$.

It follows that in case of a finite group G, with $|G| = n$, the near-ring $R[G]$ is closely related to the $n \times n$ matrix near-ring over R. Hence we pertinently give the following definition, due to Meldrum and van der Walt [6].

DEFINITION 1.2. Let R^n denote the direct sum of n copies of the group $(R, +)$. The $n \times n$ *matrix near-ring* over R, denoted $M_n(R)$, is the subnear-ring of $M(R^n)$ generated by the set $\{f_{ij}^r \in M(R^n) : r \in R, 1 \le i, j \le n\}$, where $f_{ij}^r : R^n \to R^n$ is defined by $f_{ij}^r(\alpha) = \iota_i(r\pi_j(\alpha))$, for all $\alpha \in R^n$. Here, $\iota_i : R \to R^n$ and $\pi_j : R^n \to R$ denote the i-th and j-th co-ordinate injection and projection functions respectively. For typographical reasons we also sometimes write $[r; i, j]$ for the matrix f_{ij}^r.

The interested reader should consult [1, 2, 5] for basic results on group near-rings and [6, 7] for general results on matrix near-rings. Note that when R happens to be a ring, then both $R[G]$ and $M_n(R)$ revert to the standard situation in ring theory.

Our first result relates $R[G]$ and $M_n(R)$ in case G is a finite group. As usual, S_n denotes the symmetric group on the set $\{1, 2, \ldots, n\}$.

THEOREM 1.3. *If G is a finite group with $|G| = n$, then $R[G]$ is a subnear-ring of $M_n(R)$, sharing the same identity element* $[1, e] = f_{11}^1 + f_{22}^1 + \cdots + f_{nn}^1$.

PROOF. The elements of both $R[G]$ and $M_n(R)$ are mappings of the form $R^n \to R^n$. Hence it is sufficient to show that each mapping in $R[G]$ is also in $M_n(R)$. In fact, it is sufficient to show that each generator $[r, g]$ of $R[G]$ is an $n \times n$ matrix.

To this end, let $[r, g] \in R[G]$, where $G = \{g_1, g_2, \ldots, g_n\}$. We can use the elements of G to index the co-ordinates of any $\alpha \in R^n$, that is, the i-th co-ordinate of α is $\alpha(g_i) = \pi_i(\alpha)$. Now consider an arbitrary $\alpha = \langle s_{g_1}, s_{g_2}, \ldots, s_{g_n}\rangle \in R^n$, where $\alpha(g_i) = s_{g_i}$, $i = 1, 2, \ldots, n$. Then

$$\begin{aligned}[r, g]\langle s_{g_1}, s_{g_2}, \ldots, s_{g_n}\rangle &= \langle rs_{g_1g}, rs_{g_2g}, \ldots, rs_{g_ng}\rangle \\ &= \langle rs_{g_{\rho(1)}}, rs_{g_{\rho(2)}}, \ldots, rs_{g_{\rho(n)}}\rangle \quad \text{for some } \rho \in S_n \\ &= \big([r; 1, \rho(1)] + [r; 2, \rho(2)] + \cdots + [r; n, \rho(n)]\big)\alpha.\end{aligned}$$

Note that ρ depends on g only. It follows that $[r, g] = \sum_{i=1}^{n}[r; i, \rho(i)] \in M_n(R)$. □

2. Basic results on the ideal theory of $R[G]$

An important question now arises: Given an ideal A of R, how do we relate a corresponding ideal in $R[G]$? This problem has been studied for matrix near-rings, and satisfactory results have been obtained (see [3, 4, 7, 8, 11]). Keeping in mind that when R is a ring (with identity), the complete set of (two-sided) ideals of $M_n(R)$ can be obtained by considering $M_n(A)$ for ideals A of R, the natural approach was to define ideals

$$A^+ = \mathrm{Id}\langle f_{11}^a : a \in A\rangle_{M_n(R)}$$

and

$$A^* = (A^n : R^n)_{M_n(R)} = \{U \in M_n(R) : U(R^n) \subseteq A^n\}$$

in $M_n(R)$, for an ideal A of R. We use the notation $\mathrm{Id}\langle X\rangle_R$ to denote the ideal of R generated by the subset $X \subseteq R$. Also note that, since $f_{ij}^a = f_{i1}^1 f_{11}^a f_{1j}^1$, our definition of A^+ agrees with the definition $A^+ = \mathrm{Id}\langle f_{ij}^a : a \in A, 1 \leq i, j \leq n\rangle_{M_n(R)}$ given in [11].

It is easily checked that $A^+ \subseteq A^*$. When R is a ring, $A^+ = A^* = M_n(A)$, but in the general near-ring situation, it can happen that $A^+ \subset A^*$, where '$\subset$' means 'proper inclusion'. Moreover, it turned out that, in certain cases, ideals strictly enveloped between A^+ and A^* for some A, and not equal to B^+ or B^* for any ideal B of R, exist. These ideals were termed *intermediate*, and it was shown in [3] that any ideal of a matrix near-ring must be of the form A^+ or A^* if it is not intermediate.

Following the same strategy for group near-rings leads to similar results, but we do (rather unexpectedly) also get something new. So let A be an ideal of R, and define

$${}^+A = \mathrm{Id}\langle [a, e] : a \in A\rangle_{R[G]}$$

and

$${}^*A = (A^G : R^G)_{R[G]} = \{U \in R[G] : U(R^G) \subseteq A^G\}.$$

Note that we use left superscripts to distinguish between the group near-ring and the matrix near-ring situation. The following result relates all these ideals in case G is finite.

THEOREM 2.1. *Let G be a finite group with $|G| = n$. For any ideal A of R, we have the following inclusions, denoted by arrows*:

$$\begin{array}{ccc} R[G] & \longrightarrow & M_n(R) \\ \uparrow & & \uparrow \\ {}^*A & \longrightarrow & A^* \\ \uparrow & & \uparrow \\ {}^+A & \longrightarrow & A^+ \end{array}$$

PROOF. The fact that each $[a, e]$, $a \in A$, belongs to ${}^*A$ and each f_{11}^a, $a \in A$, belongs to A^* forces ${}^+A \subseteq {}^*A$ and $A^+ \subseteq A^*$. The other inclusions follow from Theorem 1.3 and its proof. □

It is also natural to ask how to construct an ideal in the base near-ring R from a given ideal $\mathcal{A}$ of $M_n(R)$ or $R[G]$. In the matrix near-ring case, the construction is as follows: For an ideal $\mathcal{A}$ of $M_n(R)$, define

$$\mathcal{A}_* = \{\pi_1 U \iota_1(1) : U \in \mathcal{A}\}.$$

Note that this definition is equivalent to the definition

$$\mathcal{A}_* = \{x \in R : x \in \operatorname{Im}(\pi_j U),\ \text{for some } U \in \mathcal{A}, 1 \leq j \leq n\}$$

given in [6]: if $\alpha = \langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \in R^n$, and $U\alpha = \beta = \langle \beta_1, \beta_2, \ldots, \beta_n \rangle$ for some $U \in \mathcal{A}$, then $\pi_j(U\alpha) = \beta_j = \pi_1\big[f_{1j}^1 U(f_{11}^{\alpha_1} + f_{21}^{\alpha_2} + \cdots + f_{n1}^{\alpha_n})\big]\iota_1(1)$. It is clear that $f_{1j}^1 U(f_{11}^{\alpha_1} + f_{21}^{\alpha_2} + \cdots + f_{n1}^{\alpha_n}) \in \mathcal{A}$, since $\mathcal{A}$ is a two-sided ideal.

To make a similar construction in the group near-ring case, we need the analogous in R^G of the element $\iota_1(1)$ in R^n. This is given by $\varepsilon \in R^G$, where $\varepsilon(e) = 1$ and $\varepsilon(g) = 0$ if $g \neq e$. For the remainder of this paper, ε will always denote this particular element of R^G. Now let $\mathcal{A}$ be an ideal of $R[G]$. Then

$${}_*\mathcal{A} = \{(U\varepsilon)(e) : U \in \mathcal{A}\}.$$

It follows that both $\mathcal{A}_*$ and ${}_*\mathcal{A}$ are ideals of R (see [6, Proposition 4.6] and [2, Lemma 4.8]). The following theorem summarizes the basic relationships amongst all these ideals.

THEOREM 2.2. (a) *Let A be an ideal of R and let $\mathcal{A}$ be an ideal of $M_n(R)$. Then*

$$(A^+)_* = A = (A^*)_* \quad \text{and} \quad (\mathcal{A}_*)^+ \subseteq \mathcal{A} \subseteq (\mathcal{A}_*)^*.$$

(b) *Let A be an ideal of R and let $\mathcal{A}$ be an ideal of $R[G]$. Then*

$${}_*({}^+A) = A = {}_*({}^*A) \quad \text{and} \quad \mathcal{A} \subseteq {}^*({}_*\mathcal{A}).$$

(c) *The maps $A \mapsto A^+$, $A \mapsto A^*$, $A \mapsto {}^+A$ and $A \mapsto {}^*A$ (for A an ideal of R) are injective. The maps $\mathcal{A} \mapsto \mathcal{A}_*$ (for $\mathcal{A}$ an ideal of $M_n(R)$) and $\mathcal{A} \mapsto {}_*\mathcal{A}$ (for $\mathcal{A}$ an ideal of $R[G]$) are surjective.*

PROOF. (a) See, for example, [7].
(b) Let $a \in A$. Then $[a, e] \in {}^+A$, so that $a = ([a, e]\varepsilon)(e) \in {}_*({}^+A)$. If $a \in {}_*({}^+A)$, then $a = (U\varepsilon)(e)$ for some $U \in {}^+A \subseteq {}^*A$, which shows that $a \in A$. Hence, $A = {}_*({}^+A)$. The same procedure is followed to show that $A = {}_*({}^*A)$.

Furthermore, it follows from [2, Theorem 4.9] that $\mathcal{A} \subseteq {}^*({}_*\mathcal{A})$.
(c) These properties follow in a straightforward manner from (a), (b), [7, Proposition 1.46] and [2, Theorems 4.4–4.5]. □

Unexpectedly (because of the corresponding result in (a)), the inclusion $^{+}({}_{*}\mathcal{A}) \subseteq \mathcal{A}$ is in general not valid for group near-rings; not even for a commutative group ring, as the next example shows.

EXAMPLE 2.3. Let R be a commutative ring and let G be an Abelian group which contains an element g of order 2. Consider the element $U = [1, e] + [1, g]$ of the commutative group ring $R[G]$, and let $\mathcal{A} = \mathrm{Id}\langle U\rangle_{R[G]}$. Then

$$(U\varepsilon)(e) = (([1, e] + [1, g])\varepsilon)(e) = \varepsilon(e) + \varepsilon(g) = 1.$$

So $1 \in {}_{*}\mathcal{A}$, forcing ${}_{*}\mathcal{A} = R$. But then $^{+}({}_{*}\mathcal{A}) = R[G]$.

We now show that $\mathcal{A}$ is a proper ideal of $R[G]$, from which the desired result $^{+}({}_{*}\mathcal{A}) \not\subseteq \mathcal{A}$ follows. Consider $\zeta \in R^G$, where $\zeta(e) = 1$; $\zeta(g) = -1$ and $\zeta(h) = 0$ if $h \in G \setminus \{e, g\}$. Then $(U\zeta)(h) = \zeta(h) + \zeta(hg) = 0$ for all $h \in G$. Hence $U \in \mathrm{Ann}_{R[G]}(\zeta)$ from which it follows that $\mathcal{A} \subseteq \mathrm{Ann}_{R[G]}(\zeta)$. But since there are elements in $R[G]$ which do not annihilate ζ (such as the identity $[1, e]$), our result follows.

3. Intermediate ideals

As in the case of matrix near-rings, the concept of an intermediate ideal also makes sense for group near-rings. As mentioned before, there is, in general, a gap between ^{+}A and $^{*}A$ for an ideal A of R. One way to measure the 'size' of this gap is to count the number of ideals which occur in this gap.

DEFINITION 3.1. An ideal $\mathcal{A}$ of $R[G]$ such that $^{+}A \subset \mathcal{A} \subset {}^{*}A$ for some ideal A of R, is called an *intermediate ideal* of $R[G]$. (Recall that '$\subset$' denotes proper inclusion.)

Our first task is to show that these ideals do indeed exist.

EXAMPLE 3.2. Consider the zero-symmetric near-ring $\mathbb{Z}_0[x]$ of polynomials over the integers with zero constant term. Addition is the usual addition of polynomials and multiplication is defined to be composition of polynomials. Fix $n \in \mathbb{Z}$, $n \geq 4$, and define R to be the subnear-ring of $\mathbb{Z}_0[x]$ of all polynomials of which the coefficients of $x^2, x^3, \ldots, x^{2n-1}$ are equal to 0, that is,

$$R = \{a_1x + a_{2n}x^{2n} + a_{2n+1}x^{2n+1} + \cdots + a_kx^k : k \geq 2n,\\ a_i \in \mathbb{Z},\ i = 1, 2n, 2n+1, \ldots, k\}.$$

Also, if mR (for a positive integer m) denotes the set of all polynomials in R, the coefficients of which are divisible by m, then one easily checks that mR is an ideal of R.

Let $G = \{e, g\}$. (We could use any finite group here, but the notation becomes more complicated and unnecessarily obscures the clarity of the arguments.) By Theorem 1.3, $R[G]$ is a subnear-ring of $M_2(R)$. Furthermore, by Theorem 2.1, if A is any ideal of R, then ${}^+A \subseteq A^+$. This implies that we can use the results of [8] to prove the following.

RESULT 3.2.1. For any $U \in R[G]$ and for any $\langle p, q\rangle \in R^2$ we have that $U\langle p, q\rangle = \langle \zeta_1(p, q), \zeta_2(p, q)\rangle$, where the ζ_i denote polynomials in two variables over the integers. Moreover, in both $\zeta_1(p, q)$ and $\zeta_2(p, q)$, the coefficients of $p^k q^{2n-k}$ are divisible by $\binom{2n}{k}$, $k = 0, 1, \ldots, 2n$.

RESULT 3.2.2. Let $A = mR$ for some positive integer m and let $U \in {}^+A$. Then, for any $\langle p, q\rangle \in R^2$ we have that $U\langle p, q\rangle = \langle \zeta_1(p, q), \zeta_2(p, q)\rangle$, where the ζ_i denote polynomials in two variables over the integers. Moreover, in both $\zeta_1(p, q)$ and $\zeta_2(p, q)$, the coefficients of $p^k q^{2n-k}$ are divisible by $m\binom{2n}{k}$, $k = 0, 1, \ldots, 2n$.

Now let $m = 2^n$ and consider the ideal $A = mR$. We show that ${}^+A \subset {}^*A$ and that there exists a chain of $n - 2$ ideals $\mathcal{A}_i$, $i = 1, 2, \ldots, n - 2$, such that

$$ {}^+A \subset \mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots \subset \mathcal{A}_{n-2} \subset {}^*A. $$

Consider the elements $W_1, W_2 \in R[G]$ where

$$ W_1 = [x^{2^{n-1}}, e]([x, e] + [x, g]), \quad W_2 = [x^{2^{n-1}}, e]([-x, e] + [x, g]). $$

Define $W = W_1 - W_2$. Then $W\langle p, q\rangle = \langle \zeta(p, q), \zeta(p, q)\rangle$ where

$$ \zeta(p, q) = 2 \sum_{i=1}^{2^{n-2}} \binom{2^{n-1}}{2i - 1} p^{2^{n-1}-2i+1} q^{2i-1}, $$

for all $\langle p, q\rangle \in R^2$.

By using Results 3.2.1 and 3.2.2, the remainder of the proof follows exactly the same lines as the proof of [8, Proposition 3.2], except that we do not have 0 in the second co-ordinates of the elements of R^2, that is, we have here $\langle \zeta(p, q), \zeta(p, q)\rangle$ rather than $\langle \zeta(p, q), 0\rangle$, but this has no effect on what we want to show.

At this point one could raise the question: Although an intermediate ideal $\mathcal{A}$ has the property that ${}^+A \subset \mathcal{A} \subset {}^*A$ for some ideal A of R, isn't it possible that $\mathcal{A} = {}^+B$ or $\mathcal{A} = {}^*B$ for some other ideal B of R? As in the case of matrix near-rings, the answer is no:

THEOREM 3.3. *If $\mathcal{A}$ is an intermediate ideal of $R[G]$, then there is a unique ideal A of R such that ${}^+A \subset \mathcal{A} \subset {}^*A$. Moreover, $\mathcal{A}$ is not equal to ${}^+B$ or ${}^*B$ for any ideal B of R.*

PROOF. By using Theorem 2.2 (b) together with the methods used in [3, Lemmas 2.2–2.3], the results follow. □

For a given intermediate ideal $\mathcal{A}$ of $M_n(R)$, it is known that $\mathcal{A}_*$ is the unique ideal A of R such that $A^+ \subset \mathcal{A} \subset A^*$ (see [3, Corollary 2.5]). It is, however, still an open question whether ${}_*\mathcal{A}$ is always the unique ideal of R enveloping the intermediate ideal $\mathcal{A}$ of $R[G]$.

4. Exceptional ideals

It was shown in [3, Lemma 2.3] that any ideal of $M_n(R)$ that is not intermediate, must be of the form A^+ or A^* for some ideal A of R. This gives a complete characterization of the two-sided ideals of $M_n(R)$.

Surprisingly, the situation is somewhat different for group near-rings. There are, in general, ideals of $R[G]$ that are not intermediate, but also not of the form ${}^+A$ or of the form ${}^*A$, for any ideal A of R.

DEFINITION 4.1. An ideal $\mathcal{A}$ of $R[G]$ that is not intermediate and also not of the form ${}^+A$ or of the form ${}^*A$, for any ideal A of R, is called an *exceptional* ideal of $R[G]$.

Lets continue to study Example 2.3.

EXAMPLE 4.2. In Example 2.3 it was found that ${}^+({}_*\mathcal{A}) \not\subseteq \mathcal{A}$ for the ideal $\mathcal{A} = \mathrm{Ann}_{R[G]}(\zeta)$. We proceed to show that $\mathcal{A}$ is an exceptional ideal of $R[G]$. Suppose that $\mathcal{A} \subseteq {}^*A$ for some ideal A of R. Then, since $(([1,e]+[1,g])\varepsilon)(e) = 1$, it follows that $1 \in A$, implying that $A = R$. This, in turn, implies that ${}^+A = {}^*A = R[G]$. For reference,

$$\mathcal{A} \subseteq {}^*A \quad \text{implies} \quad {}^+A = {}^*A = R[G]. \tag{1}$$

Now suppose that $\mathcal{A}$ is intermediate. Then ${}^+A \subset \mathcal{A} \subset {}^*A$ for an ideal A of R. By (1), ${}^+A = {}^*A$, a contradiction.

Suppose that $\mathcal{A} = {}^+A$ for some ideal A of R. Then, by (1) and Theorem 2.2 (b), $\mathcal{A} = {}^+A = R[G]$, a contradiction, because $\mathcal{A}$ is proper.

Finally, suppose that $\mathcal{A} = {}^*A$ for an ideal A of R. Again, by (1), it follows that $\mathcal{A} = {}^*A = R[G]$, a contradiction.

It is interesting to note that an exceptional ideal could be found in every group near-ring.

THEOREM 4.3. *The augmentation ideal Δ of $R[G]$ is always exceptional.*

PROOF. It was shown in [2, Theorem 4.13] that $\Delta = \mathrm{Id}\langle [1, g] - [1, e] : g \in G\rangle_{R[G]}$. For any $g \neq e$, $[1, g] - [1, e] \in \Delta$, so that $(([1, g] - [1, e])\varepsilon)(e) = -1 \in {}_*\Delta$, forcing ${}_*\Delta = R$. It follows that if $\Delta \subseteq {}^*A$ for an ideal A of R, then ${}^+A = {}^*A = R[G]$, because if $\Delta \subseteq {}^*A$, then ${}_*\Delta \subseteq {}_*({}^*A) = A$, according to Theorem 2.2 (b). Furthermore, because $R[G]/\Delta \cong R$, by [2, Corollary 4.12], and R is assumed to be a nontrivial near-ring, Δ is a proper ideal of $R[G]$. Now follow the same method as in Example 4.2. □

5. Modules over $R[G]$ and the Jacobson radicals

In this last section we would like to present some results regarding the $\mathcal{J}$-radicals of $R[G]$ which means that we need to study some module theory over $R[G]$. Since similar results have been obtained with respect to matrix near-rings, we certainly want to utilize these, henceforth we only focus on the case where G is finite. In particular, we let $G = \{g_1 = e, g_2, \ldots, g_n\}$.

In what follows, the terminology 'ideal', 'R-subgroup', 'simple' and 'R-simple', has the same meaning as in [9, Definitions 1.27 (b), 1.21 (b) and 1.36]. Also note that, because of the way in which $R[G]$ (respectively, $M_n(R)$) is defined, R^n can be viewed in a natural way as a (left) $R[G]$-module (respectively, $M_n(R)$-module). This brings us to

THEOREM 5.1. *If L is an ideal of the module ${}_R R$, that is, L is a left ideal of the near-ring R, then L^n is an ideal of the module ${}_{R[G]}R^n$.*

PROOF. We know that L^n is an ideal of ${}_{M_n(R)}R^n$, by [6, Proposition 4.1]. But since $R[G]$ is a subnear-ring of $M_n(R)$ by Theorem 1.3, the result follows. □

The next step is to show how an arbitrary module over R can be extended to a module over $R[G]$. Since we are only interested in type 0 and type 2 modules, we will assume that all modules are monogenic, that is, if Γ is an R-module then there exists $\gamma \in \Gamma$ such that $R\gamma = \Gamma$. This implies that we can view Γ^n as an $R[G]$-module, as follows: Let $U \in R[G]$ and $\langle \gamma_1, \gamma_2, \ldots, \gamma_n\rangle \in \Gamma^n$. Then there are $r_1, r_2, \ldots, r_n \in R$ such that $r_i\gamma = \gamma_i$, $i = 1, 2, \ldots, n$. Define

$$U\langle \gamma_1, \gamma_2, \ldots, \gamma_n\rangle = (U\langle r_1, r_2, \ldots, r_n\rangle)\gamma,$$

where $\langle s_1, s_2, \ldots, s_n\rangle\gamma = \langle s_1\gamma, s_2\gamma, \ldots, s_n\gamma\rangle$ for every $\langle s_1, s_2, \ldots, s_n\rangle \in R^n$.

Note that this is exactly the way in which Γ^n has been defined as an $M_n(R)$-module (see [10]). Since $R[G]$ is a subnear-ring of $M_n(R)$, this definition makes sense, and ${}_{R[G]}\Gamma^n$ is well-defined.

THEOREM 5.2. *If Γ is a monogenic R-module, then Γ^n is a monogenic $R[G]$-module.*

PROOF. Suppose $R\gamma = \Gamma$ for some $\gamma \in \Gamma$. As before, we can index the coordinates of an $\alpha \in \Gamma^n$ with the elements of G, that is, $\alpha(g_i) = \pi_i(\alpha)$. Consider the element $\eta \in \Gamma^n$, where $\eta(g_1) = \gamma$ and $\eta(g_j) = 0$ for $j \neq 1$. We show that η is a generator for Γ^n over $R[G]$.

Let $1 \leq i \leq n$ and let $r_i \in R$. Then

$$([r_i, g_i]\eta)(h) = \begin{cases} r_i\gamma & \text{if } h = g_i^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

But then $([r_1, g_1] + [r_2, g_2] + \cdots + [r_n, g_n])\eta = \langle r_1\gamma, r_2\gamma, \ldots, r_n\gamma\rangle$. By varying each r_i over the elements of R, we see that $R[G]\eta = \Gamma^n$. □

If Λ is an ideal of the monogenic module ${}_R\Gamma$, we can easily generalize Theorem 5.1 by showing that Λ^n is an ideal of ${}_{R[G]}\Gamma^n$. Also, by Theorem 5.2, since Γ/Λ is a monogenic R-module (via the natural action $r(\gamma + \Lambda) = r\gamma + \Lambda$), we have that $(\Gamma/\Lambda)^n$ is a monogenic $R[G]$-module. Moreover, $(\Gamma/\Lambda)^n \cong_{R[G]} \Gamma^n/\Lambda^n$, a fact which can be proved in a way similar to the proof of [7, Proposition 1.29], where the same result was proved for matrix near-rings.

The following result is needed in the example that follows:

THEOREM 5.3. *Let R be zero-symmetric and let ${}_R\Gamma$ be a monogenic module where $|\Gamma| = 2$. If $G = \{e, g\}$, then the diagonal of Γ^2, $d(\Gamma^2) = \{\langle\gamma, \gamma\rangle : \gamma \in \Gamma\}$, is a non-trivial, proper ideal of the module ${}_{R[G]}\Gamma^2$.*

PROOF. Since $|\Gamma^2| = 4$ and $|d(\Gamma^2)| = 2$, the diagonal is clearly non-trivial and proper. It is also trivially closed under addition. We use induction on the complexity of $U \in R[G]$ (see the discussion following Theorem 2.4 in [2]) to prove that

$$U(\langle\gamma, \gamma\rangle + \langle\alpha, \beta\rangle) - U\langle\alpha, \beta\rangle \in d(\Gamma^2), \tag{2}$$

for all $\langle\gamma, \gamma\rangle \in d(\Gamma^2)$, $\langle\alpha, \beta\rangle \in \Gamma^2$ and $U \in R[G]$. Note that if $R\gamma' = \Gamma$, then each of α and β in (2) vary over the set $\{0, \gamma'\}$.

Let $U \in R[G]$ have complexity 1, that is, $U = [r, e]$ or $U = [r, g]$ for some $r \in R$. Lets say $U = [r, e]$ (the case $U = [r, g]$ being treated similarly). Then

$$\begin{aligned}[r, e](\langle\gamma, \gamma\rangle + \langle\alpha, \beta\rangle) - [r, e]\langle\alpha, \beta\rangle &= \langle r(\gamma + \alpha) - r\alpha, r(\gamma + \beta) - r\beta\rangle \\ &= \langle r\gamma, r\gamma\rangle \in d(\Gamma^2).\end{aligned}$$

Now consider any $U \in R[G]$ with complexity greater than 1, and assume the result to be true for all elements of $R[G]$ which have complexity smaller than that of U. Then either $U = V + W$ or $U = VW$, where the complexity of both V and W are smaller than that of U. On the one hand,

$$\begin{aligned}
&U(\langle\gamma,\gamma\rangle+\langle\alpha,\beta\rangle)-U\langle\alpha,\beta\rangle\\
&\quad=(V+W)(\langle\gamma,\gamma\rangle+\langle\alpha,\beta\rangle)-(V+W)\langle\alpha,\beta\rangle\\
&\quad=V(\langle\gamma,\gamma\rangle+\langle\alpha,\beta\rangle)-V\langle\alpha,\beta\rangle+W(\langle\gamma,\gamma\rangle+\langle\alpha,\beta\rangle)-W\langle\alpha,\beta\rangle\\
&\quad\in d(\Gamma^2)+d(\Gamma^2)=d(\Gamma^2),
\end{aligned}$$

and on the other hand,

$$\begin{aligned}
&U(\langle\gamma,\gamma\rangle+\langle\alpha,\beta\rangle)-U\langle\alpha,\beta\rangle\\
&\quad=(VW)(\langle\gamma,\gamma\rangle+\langle\alpha,\beta\rangle)-(VW)\langle\alpha,\beta\rangle\\
&\quad=V[W(\langle\gamma,\gamma\rangle+\langle\alpha,\beta\rangle)-W\langle\alpha,\beta\rangle+W\langle\alpha,\beta\rangle]-V(W\langle\alpha,\beta\rangle)\\
&\quad=V(\langle\delta,\delta\rangle+W\langle\alpha,\beta\rangle)-V(W\langle\alpha,\beta\rangle)\quad\text{for some }\langle\delta,\delta\rangle\in d(\Gamma^2)\\
&\quad\in d(\Gamma^2),
\end{aligned}$$

and the proof is complete. □

COROLLARY 5.4. *With the same assumptions as in Theorem 5.3, we have that both the $R[G]$-modules $d(\Gamma^2)$ and $\Gamma^2/d(\Gamma^2)$ are of type 2, hence also of type 0.*

PROOF. Both these modules have order 2 and are non-trivial. □

COROLLARY 5.5. *If ${}_R\Gamma$ is simple (R-simple), then ${}_{R[G]}\Gamma^n$ is not necessarily simple ($R[G]$-simple).*

There exists a very natural relationship between the $\mathcal{J}$-radicals of R and the corresponding matrix near-ring $M_n(R)$, namely $\mathcal{J}_\nu(M_n(R)) \subseteq \mathcal{J}_\nu(R)^*$, $\nu \in \{0, 2\}$ [7, Theorem 2.34]. When $\nu = 2$, we even have $\mathcal{J}_2(M_n(R)) = \mathcal{J}_2(R)^*$, which is, of course, a very useful tool.

The key result which enables us to prove these relationships, is the fact that ${}_R\Gamma$ is simple (R-simple) if and only if ${}_{M_n(R)}\Gamma^n$ is simple ($M_n(R)$-simple) [10, Corollary 3.8]. We have just seen in Corollary 5.5 that this flow of simplicity does not necessarily occur between R-modules and $R[G]$-modules. The consequences of this are reflected in the following example, where we construct a finite, Abelian, zero-symmetric near-ring R such that (for $\nu \in \{0, 2\}$) both $\mathcal{J}_\nu(R[G]) \not\subseteq {}^*\mathcal{J}_\nu(R)$ and ${}^*\mathcal{J}_\nu(R) \not\subseteq \mathcal{J}_\nu(R[G])$, where $|G| = 2$. It turns out, though, that ${}^+\mathcal{J}_\nu(R) \subset \mathcal{J}_\nu(R[G])$ for this example. It is still an open question whether ${}^+\mathcal{J}_\nu(R) \subseteq \mathcal{J}_\nu(R[G])$ holds in general.

EXAMPLE 5.6. Consider the (additive) groups

$$M = \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad N = M \oplus \mathbb{Z}_2, \quad H = N \oplus \mathbb{Z}_2.$$

Let M_i, $1 \le i \le 3$, be the two-element subgroups of M and let N_j, $1 \le j \le 4$, be the two-element subgroups of N which are not contained in M. Also, let $m_i \in M_i$, $1 \le i \le 3$, and $n_j \in N_j$, $1 \le j \le 4$, denote the non-zero elements in these groups. Finally, let $h_1, h_2, \dots, h_8$ denote the elements of $H \setminus N$. Define the near-ring R as follows:

$$\begin{aligned} R = \{f \in M_0(H) :\ & f(M_i) \subseteq M_i,\ 1 \le i \le 3;\ f(N_j) \subseteq N_j,\ 1 \le j \le 4; \\ & h, h' \in H \text{ and } h - h' \in M \text{ implies } f(h) - f(h') \in M; \\ & h, h' \in H \text{ and } h - h' \in N \text{ implies } f(h) - f(h') \in N\}, \end{aligned}$$

where $M_0(H)$ is the subnear-ring of $M(H)$ containing the zero-preserving mappings. It turns out that R is a zero-symmetric, Abelian near-ring with identity and R is finite with $|R| = 2^{23}$. We also note that each M_i ($1 \le i \le 3$), each N_j ($1 \le j \le 4$), as well as the group H/N can be viewed as an R-module because of the way that R has been defined. We study the group near-ring $R[G]$ where G is the group $\{e, g\}$.

First, define the following ideals of ${}_R R$:

$$\begin{aligned} K &= \{f \in R : f(h_i) \in M,\ 1 \le i \le 8;\ 0 \text{ otherwise}\}, \\ L &= \{f \in R : f(h_i) \in N,\ 1 \le i \le 8;\ 0 \text{ otherwise}\}. \end{aligned}$$

Our first observation is that

$$\mathcal{J}_0(R) = \mathcal{J}_2(R) = \operatorname{Ann}_R N \cap \operatorname{Ann}_R(H/N) = L. \tag{3}$$

This follows from the fact that all M_i's, all N_j's, as well as H/N, are R-modules of type 0, since they are all of order 2 and non-trivial (hence also of type 2), the fact that

$$\operatorname{Ann}_R N = \left[\bigcap_{i=1}^{3} \operatorname{Ann}_R M_i\right] \cap \left[\bigcap_{j=1}^{4} \operatorname{Ann}_R N_j\right],$$

and also from the fact that L is nilpotent (see [9, Theorem 5.37 (d)]). From now on, we simply write $\mathcal{J}(R)$ for $\mathcal{J}_0(R) = \mathcal{J}_2(R)$.

An easy application of Corollary 5.4 and by arguments similar to the above leads us to

$$\begin{aligned} \mathcal{J}_0(R[G]) &= \mathcal{J}_2(R[G]) \\ &= \left[\bigcap_{i=1}^{3} \operatorname{Ann}_{R[G]}(d((M_i)^2))\right] \cap \left[\bigcap_{i=1}^{3} \operatorname{Ann}_{R[G]}(M_i^2/d(M_i^2))\right] \\ &\quad \cap \left[\bigcap_{j=1}^{4} \operatorname{Ann}_{R[G]}(d((N_j)^2))\right] \cap \left[\bigcap_{j=1}^{4} \operatorname{Ann}_{R[G]}(N_j^2/d((N_j^2))\right] \\ &\quad \cap \operatorname{Ann}_{R[G]}(d((H/N)^2)) \cap \operatorname{Ann}_{R[G]}((H/N)^2/d((H/N)^2)), \end{aligned}$$

which, from now on, will simply be denoted by $\mathcal{J}(R[G])$.

Next, observe that, since $(\mathcal{J}(R))^2 = 0$ (by (3)), we have that $({}^+\mathcal{J}(R))^2 = 0$ in $R[G]$. This follows from [1, Lemma 3.1] and the fact that ${}^+\mathcal{J}(R) \subseteq {}^*\mathcal{J}(R)$ (Theorem 2.1). Consequently, ${}^+\mathcal{J}(R) \subseteq \mathcal{J}(R[G])$.

We now show that there are also elements of nilpotency degree 3 in $\mathcal{J}(R[G])$, implying that

$$(4) \qquad {}^+\mathcal{J}(R) \subset \mathcal{J}(R[G]).$$

To this end, consider the ideal

$$\mathcal{A} = \mathrm{Ann}_{R[G]}\, K^2 \cap \mathrm{Ann}_{R[G]}(L^2/K^2) \cap \mathrm{Ann}_{R[G]}(R^2/L^2).$$

Since $\mathcal{A}^3 = 0$, we have that $\mathcal{A} \subseteq \mathcal{J}(R[G])$.

Also consider the elements $a, b, c, d \in R$, defined as follows:

$$\begin{aligned} a(h_i) &= n_1,\ 1 \le i \le 8;\ 0 \text{ otherwise},\\ b(h_i) &= n_2,\ 1 \le i \le 8;\ 0 \text{ otherwise},\\ c(m_3) &= m_3;\ 0 \text{ otherwise},\\ d(n_j) &= n_j,\ 1 \le j \le 4;\ 0 \text{ otherwise}, \end{aligned}$$

where $n_1 = (0, 1, 1, 0)$, $n_2 = (1, 0, 1, 0)$ and $m_3 = (1, 1, 0, 0)$.

Direct computation shows that $V = [a, e] + [b, g] + [c, e]([d, e] + [d, g]) \in R[G]$ is an element of $\mathcal{A}$, hence an element of $\mathcal{J}(R[G])$. It is, however, not an element of ${}^+\mathcal{J}(R)$, because $V^2 \neq 0$. (Note that $V^2\langle 1, 0\rangle = \langle c(da + db), c(da + db)\rangle \neq \langle 0, 0\rangle$, since $c(da + db)(h_1) \neq 0$.) So (4) is proved.

Our next task is to show that

$$(5) \qquad \mathcal{J}(R[G]) \not\subseteq {}^*\mathcal{J}(R).$$

Consider $U = [1, e] + [1, g]$. Since R (hence $R[G]$) has characteristic 2, the diagonal of any (Abelian) $R[G]$-module Γ^2 is mapped to 0, and all other (non-diagonal) elements are mapped into the diagonal ($U\langle \gamma_1, \gamma_2\rangle = \langle \gamma_1 + \gamma_2, \gamma_1 + \gamma_2\rangle$). It follows that $U \in \mathcal{J}(R[G])$. But since $U\langle 1, 0\rangle = \langle 1, 1\rangle \notin (\mathcal{J}(R))^2$, it is immediate that $U \notin {}^*\mathcal{J}(R)$, thus (5) follows.

We finally show that there are elements in ${}^*\mathcal{J}(R)$ which are not in $\mathcal{J}(R[G])$. One such element is $W = [s, e]([t, e] + [t, g])$, where

$$\begin{aligned} s(m_3) &= m_3;\ 0 \text{ otherwise},\\ t(m_i) &= m_i,\ i = 1, 2;\ 0 \text{ otherwise}. \end{aligned}$$

To see this, let K_0 be the R-subgroup of K generated by $k_1, k_2 \in K$, where

$$k_1(h_1) = m_1;\ 0 \text{ otherwise},$$
$$k_2(h_1) = m_2;\ 0 \text{ otherwise}.$$

In other words, $K_0 = \{f \in R : f(h_1) \in M;\ 0 \text{ otherwise}\}$.

It is easy to see that the ideal generated by any non-zero element of the $R[G]$-module K_0^2, is all of K_0^2, which means that the module is simple. It is also monogenic with generator $\langle k_1, k_2 \rangle$, hence a type 0 module. But this implies that

$$\mathcal{J}(R[G]) \subseteq \text{Ann}_{R[G]} K_0^2. \tag{6}$$

We find that $W\langle r, r' \rangle = \langle s(tr + tr'), s(tr + tr') \rangle$ for any $\langle r, r' \rangle \in R^2$. Furthermore, direct computation shows that $s(tr + tr')(N) = 0$ and $s(tr + tr')(H) \subseteq N$, and it follows that $W \in {}^*\mathcal{J}(R)$, by (3).

However, $W\langle k_1, k_2 \rangle = \langle s(tk_1 + tk_2), s(tk_1 + tk_2) \rangle$ where

$$s(tk_1 + tk_2)(h_1) = s(t(m_1) + t(m_2)) = s(m_1 + m_2) = s(m_3) = m_3 \neq 0.$$

Consequently, $W \notin \text{Ann}_{R[G]} K_0^2$, and, by (6), ${}^*\mathcal{J}(R) \not\subseteq \mathcal{J}(R[G])$ is proved.

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