ON QUASINILPOTENT OPERATORS, II

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Abstract

In this paper we continue to modify and expand a technique due to Enflo for producing nontrivial hyperinvariant subspaces for quasinilpotent operators, and thereby obtain such subspaces for some additional quasinilpotent operators on Hilbert space. We also obtain a structure theorem for a certain class of operators.

1. Introduction

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and denote by $\mathcal{L}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. In [1], Per Enflo introduced a new technique involving some 'extremal vectors' for producing hyperinvariant subspaces for certain quasinilpotent operators in $\mathcal{L}(\mathcal{H})$. This technique was then modified and expanded by the present authors in [4] to produce better hyperinvariant subspaces for some additional quasinilpotent operators. This paper is a sequel to [4] in which we continue to explore the limits of Enflo's technique for producing hyperinvariant subspaces. In [4] we recovered, via this new technique, the theorem of Lomonosov [6] that if Q is quasinilpotent in $\mathcal{L}(\mathcal{H})$ and the commutant

$$\{Q\}' = \{C \in \mathcal{L}(\mathcal{H}) : CQ = QC\}$$

of Q has the 'Pearcy-Salinas property' (see [6] and [7] for example), then Q has a nontrivial hyperinvariant subspace. Herein we explore further the basic construction from [1]. This allows us to obtain the existence of hyperinvariant subspaces for

some more quasinilpotent operators, and to discuss certain structural properties of quasinilpotent operators.

2. The basic lemmas

For the reader's convenience, we state the basic lemmas from [4], which are themselves only minor variations of the corresponding lemmas in [1]. For proofs the reader may consult either [1] or [4]. As usual $\mathbb{N}$ will denote the set of positive integers and $\mathbb{K}$ the ideal of compact operators in $\mathscr{L}(\mathscr{H})$.

LEMMA 2.1. *Suppose that u and v are nonzero vectors in $\mathscr{H}$ such that for every $w \in \mathscr{H}$, $\mathrm{Re}(u, w) < 0$ implies that $\mathrm{Re}(v, w) \geq 0$. Then there exists a negative number γ_0 such that $v = \gamma_0 u$.*

LEMMA 2.2. *Suppose $T \in \mathscr{L}(\mathscr{H})$, x_0 is a nonzero vector in the closure of the range of T, and ε satisfies $0 < \varepsilon < \|x_0\|$. Then there exists a unique nonzero vector $y_0 = y_0(x_0, \varepsilon)$ in the set $\{y : \|Ty - x_0\| \leq \varepsilon\}$ satisfying*

(a) $\|y_0\| = \inf\{\|y\| : \|Ty - x_0\| \leq \varepsilon\}$, *and this vector y_0 also satisfies*
(b) $\|Ty_0 - x_0\| = \varepsilon$.

LEMMA 2.3. *Suppose T, x_0, y_0 and ε are as in Lemma 2.2. Then there exists a negative number γ_0 such that $T^*(Ty_0 - x_0) = \gamma_0 y_0$.*

LEMMA 2.4. *Suppose $T \in \mathscr{L}(\mathscr{H})$ is quasinilpotent with dense range, x_0 is a nonzero vector in $\mathscr{H}$, and ε satisfies $0 < \varepsilon < \|x_0\|$. For each $n \in \mathbb{N}$, let $y_n = y_n(x_0, \varepsilon)$ be the unique (nonzero) vector in the set $\{y : \|T^n y - x_0\| \leq \varepsilon\}$ satisfying (via Lemma 2.2)*

(a′) $\|y_n\| = \inf\{\|y\| : \|T^n y - x_0\| \leq \varepsilon\}$, *and*
(b′) $\|T^n y_n - x_0\| = \varepsilon$.
Then there exists a subsequence $\{y_{n_k}\}_{k \in \mathbb{N}}$ of the sequence $\{y_n\}_{n \in \mathbb{N}}$ satisfying

$$(1) \qquad \lim_k \|y_{n_k}\| / \|y_{n_k+1}\| = 0.$$

3. Some results

In this section we use the basic lemmas of Section 2 to produce some new results about quasinilpotent operators. The first might be called the basic construction that our modified version of Enflo's technique produces, and was unknown to us at the time [4] was written.

THEOREM 3.1. *We suppose the following*:

(A) *Q is an arbitrary quasinilpotent operator in $\mathcal{L}(\mathcal{H})$ with dense range;*

(B) *C is an arbitrary nonzero contraction in $\mathcal{L}(\mathcal{H})$;*

(C) *x_0 is an arbitrary vector in $\mathcal{H}$ such that $Cx_0 \neq 0$;*

(D) *ε satisfies $0 < \varepsilon < \|Cx_0\| \ (\leq \|x_0\|)$;*

(E) *$\{y_n\}_{n \in \mathbb{N}}$ is the sequence of vectors in $\mathcal{H}$ associated with Q, x_0, and ε provided by Lemma 2.4, and*

(F) *$\{y_{n_k}\}_{k \in \mathbb{N}}$ is a subsequence of the sequence $\{y_n\}$ satisfying (1) and (without loss of generality by virtue of Lemma 2.4 (b')) also has the property that the sequences $\{Q^{n_k} y_{n_k}\}_{k \in \mathbb{N}}$ and $\{z_k := Q^{n_k+1} y_{n_k+1}\}_{k \in \mathbb{N}}$ are weakly convergent to vectors s_0 and z_0, respectively.*

Then,

   (I) *$\|s_0 - x_0\| \leq \varepsilon$, which implies that $s_0 \neq 0 \neq C s_0$ by virtue of (B) and (D),*

   (II) *$0 < \|z_0 - x_0\| \leq \varepsilon$, which implies that $x_0 \neq z_0 \neq 0$ and that $Q^*(z_0 - x_0) \neq 0$,*

and

   (III) *$\|y_{n_k}\| \|(Q^{n_k+1})^*(z_k - x_0)\| \leq (\|y_{n_k}\| / \|y_{n_k+1}\|)(\varepsilon^2 + \varepsilon\|x_0\|)$,*

which implies, in particular, via (1) that

$$(2) \qquad \lim_k \|y_{n_k}\| \|(Q^{n_k+1})^*(z_k - x_0)\| = 0.$$

PROOF. We know from Lemma 2.4 (b') that $\|Q^n y_n - x_0\| = \varepsilon$, $n \in \mathbb{N}$, and since closed balls in $\mathcal{H}$ are weakly closed, (I) is established as well as the fact that $\|z_0 - x_0\| \leq \varepsilon$, which shows that $z_0 \neq 0$. We next show that $z_0 - x_0 \neq 0$, which will also show that $Q^*(z_0 - x_0) \neq 0$ (since Q has dense range). By the definition of the sequence $\{y_{n_k}\}_{k \in \mathbb{N}}$ and Lemma 2.3, we know that there exists a sequence $\{r_k\}_{k \in \mathbb{N}}$ of negative numbers such that

$$(3) \qquad (Q^{n_k+1})^*(z_k - x_0) = r_k y_{n_k+1}, \quad k \in \mathbb{N}.$$

Therefore,

$$(4) \qquad \begin{aligned} \varepsilon^2 &= \|Q^{n_k+1} y_{n_k+1} - x_0\|^2 \\ &= (y_{n_k+1}, (Q^{n_k+1})^*(z_k - x_0)) - (x_0, z_k - x_0) \\ &= r_k \|y_{n_k+1}\|^2 - (x_0, z_k - x_0), \quad k \in \mathbb{N}. \end{aligned}$$

Thus

$$(5) \qquad r_k \|y_{n_k+1}\|^2 = \varepsilon^2 + (x_0, z_k - x_0)$$

and

$$(6) \qquad (x_0, z_k - x_0) < -\varepsilon^2, \quad k \in \mathbb{N}.$$

Taking the weak limit in (6) as $k \to \infty$, we get $(x_0, z_0 - x_0) \leq -\varepsilon^2$, so $z_0 - x_0 \neq 0$, which establishes (II). Furthermore, from (3) and (5) we see that

$$
\begin{aligned}
(7) \qquad \|y_{n_k+1}\| \|(Q^{n_k+1})^*(z_k - x_0)\| &= |r_k| \|y_{n_k+1}\|^2 \\
&= |\varepsilon^2 + (x_0, z_k - x_0)| \\
&\leq \varepsilon^2 + \varepsilon \|x_0\|, \quad k \in \mathbb{N},
\end{aligned}
$$

which, together with trivial arithmetic, establishes (III). $\qquad\qquad\qquad\square$

REMARK 3.2. With the notation as in Theorem 3.1, note that the quantity

$$
\|y_{n_k}\| \|(Q^{n_k+1})^*(z_k - x_0)\|
$$

in (2) is exactly the trace-norm ($\| \ \|_1$) of the rank-one operator $y_{n_k} \otimes (Q^{n_k+1})^*(z_k - x_0)$, which, of course, belongs to the trace class $\mathscr{C}_1(\mathscr{H})$. Since $\mathscr{C}_1(\mathscr{H})^* = \mathscr{L}(\mathscr{H})$, any rank-one operator $u \otimes v$ may be regarded as the weak*-continuous linear functional on $\mathscr{L}(\mathscr{H})$ defined by $(u \otimes v)(T) = (Tu, v)$, $T \in \mathscr{L}(\mathscr{H})$, and the norm of this linear functional is $\|u \otimes v\|_1 = \|u\| \cdot \|v\|$. This is, of course, the basic idea behind the theory of dual algebras (see for example [2]), and together with (2) gives immediately the following.

COROLLARY 3.3. *With the notation as in Theorem 3.1, let $\{A_{k,m}\}_{k,m \in \mathbb{N}}$ and $\{B_{k,m}\}_{k,m \in \mathbb{N}}$ be any bounded doubly indexed sequences of operators from $\mathscr{L}(\mathscr{H})$. Then*

$$
(8) \qquad \lim_k (Q^{n_k} A_{k,m} y_{n_k}, Q^*(z_k - x_0)) = 0, \quad m \in \mathbb{N},
$$

and

$$
(9) \qquad \lim_k ((Q^{n_k}/\|Q^{n_k}\|) B_{k,m} Q^{n_k} y_{n_k}, Q^*(z_k - x_0)) = 0, \quad m \in \mathbb{N}.
$$

To establish immediately the utility of this corollary, we deduce quickly a modest improvement, due to Pearcy-Shields [8], of the famous result of Lomonosov [5].

COROLLARY 3.4. *Suppose that $Q \neq 0$ is a quasinilpotent operator in $\mathscr{L}(\mathscr{H})$ and there exists a bounded sequence $\{J_n\}_{n=0}^{\infty}$ of nonzero operators in $\mathscr{L}(\mathscr{H})$ such that J_0 is compact and either*

(a) *$J_{n-1}Q = Q J_n$, $n \in \mathbb{N}$ or*
(b) *$Q J_{n-1} = J_n Q$, $n \in \mathbb{N}$*

(which is obviously the case if there is a nonzero compact operator in $\{Q\}'$). Then Q has a nontrivial hyperinvariant subspace.

PROOF. Without loss of generality we may suppose that Q is a quasiaffinity and that each J_{n-1}, $n \in \mathbb{N}$, is a contraction. Moreover, as is well known, to show that Q has a nontrivial hyperinvariant subspace, it suffices to show that Q^* does, so in case (b) holds we take adjoints and obtain instead that (a) holds. Apply Corollary 3.3 (see, for example, (8)) with $C := J_0$ to obtain the sequences $\{n_k\}$, $\{y_{n_k}\}$, and $\{z_k\}$ with the properties set forth there. Let S be an arbitrary nonzero contraction in $\{Q\}'$, and define $A_{k,m} = S J_{n_k}$, $k, m \in \mathbb{N}$. Then, after defining

$$\beta_k = \|y_{n_k}\| \|(Q^{n_k+1})^*(z_k - x_0)\|, \qquad s_k = Q^{n_k} y_{n_k},$$
$$t_k = Q^*(z_k - x_0), \quad k \in \mathbb{N}, \quad \text{and} \quad t_0 = Q^*(z_0 - x_0) \neq 0,$$

we obtain from (8) that

$$|(S J_0 s_k, t_k)| = |(S J_0 Q^{n_k} y_{n_k}, t_k)| = |(S Q^{n_k} J_{n_k} y_{n_k}, t_k)|$$
$$= |(S J_{n_k} y_{n_k}, (Q^{n_k})^* t_k)| \leq \beta_k, \quad k \in \mathbb{N}.$$

Since $\|S J_0 s_k - S J_0 s_0\| \to 0$ (this uses the compactness of J_0), and $\{t_k\}_{k \in \mathbb{N}}$ converges weakly to $t_0 \neq 0$, we obtain $(S J_0 s_0, t_0) = 0$. Since $J_0 s_0 \neq 0$ from above, and S is an arbitrary nonzero contraction in $\{Q\}'$, this shows that $(\{Q\}' J_0 s_0)^-$ is a nontrivial hyperinvariant subspace for Q. $\square$

Along similar lines we obtain the following improvement (in the quasinilpotent case) of Lomonosov's theorem [6] about operators whose commutant has the 'Pearcy-Salinas property'.

COROLLARY 3.5. *Suppose that $Q \neq 0$ is a quasinilpotent operator in $\mathcal{L}(\mathcal{H})$ and there exist bounded sequences $\{J_m\}_{m \in \mathbb{N}}$, $\{L_m\}_{m \in \mathbb{N}}$, and $\{K_n\}_{n \in \mathbb{N}}$ in $\mathcal{L}(\mathcal{H})$ such that $\{K_n\} \subset \mathbb{K}$, $\|L_m - K_m\| \to 0$, $Q^n J_m = L_m Q^n$ for all $m, n \in \mathbb{N}$, and $\{L_m\}$ converges in the weak operator topology to an operator $R \neq 0$. Then Q has a nontrivial hyperinvariant subspace.*

PROOF. We may suppose, without loss of generality, that Q is a quasiaffinity and that the sequences $\{J_m\}$, $\{K_m\}$, and $\{L_m\}$ are all contractions. We apply Theorem 3.1 and Corollary 3.3 (especially (8)) to Q with $C := R$ to obtain the sequences $\{s_k\}$, $\{t_k\}$, and $\{\beta_k\}$ defined in Corollary 3.4. Let now S be an arbitrary nonzero contraction in $\{Q\}'$. We will show that $(S R s_0, t_0) = 0$, and therefore that $(\{Q\}' R s_0)^-$ is the desired nontrivial hyperinvariant subspace for Q. (Note that $R s_0$ and t_0 (= w-$\lim t_k$) are nonzero by Theorem 3.1.) We define the doubly indexed sequence $\{A_{k,m}\}$ of (8) of Corollary 3.3 by $A_{k,m} = S J_m$, $k, m \in \mathbb{N}$, and thus obtain

(10) $$|(S L_m s_k, t_k)| = |(Q^{n_k} S J_m y_{n_k}, t_k)| \leq \beta_k, \quad k, m \in \mathbb{N}.$$

Let $\eta > 0$ be given and note that (since $\{K_m\}$ tends to R in the weak operator topology) it suffices to find $M_\eta \in \mathbb{N}$ such that

$$(11) \qquad |(SK_m s_0, t_0)| \leq \eta, \quad m \geq M_\eta.$$

Choose $M_\beta \in \mathbb{N}$ such that for $k \geq M_\beta$, $\beta_k < \eta/2$, and choose $M_\eta \in \mathbb{N}$ such that

$$(12) \qquad \|L_m - K_m\| < \frac{\eta}{2\|S\|(\sup_k \|s_k\|\|t_k\|)}, \quad m \geq M_\eta.$$

Then, via (10) and (12), we have, for $m \geq M_\eta$ and $k \geq M_\beta$,

$$(13) \qquad |(SK_m s_k, t_k)| \leq |(SL_m s_k, t_k)| + |(S(K_m - L_m)s_k, t_k)| < \eta.$$

Fix an arbitrary integer $m_0 \geq M_\eta$ and note that since $\{s_k\}$ tends weakly to s_0 and SK_{m_0} is compact,

$$(14) \qquad \lim_k \|SK_{m_0} s_k - SK_{m_0} s_0\| = 0.$$

From (13), (14), and the fact that $\{t_k\}$ tends weakly to t_0, we obtain

$$|(SK_{m_0} s_0, t_0)| = \lim_k |(SK_{m_0} s_k, t_k)| \leq \eta, \quad m_0 \geq M_\eta,$$

which establishes (11) and completes the proof. □

The following lemma is elementary and needs no proof.

LEMMA 3.6. *Suppose $\{p_n\}_{n=1}^\infty$ is a sequence of positive numbers such that*

$$\lim_n (p_n)^{1/n} = 0.$$

Then there exists a subsequence $\{p_{n_k}\}_{k=1}^\infty$ of $\{p_n\}$ such that $\lim_k (p_{n_k+1}/p_{n_k}) = 0$.

The next result is a structure theorem for quasinilpotent operators, and has some interesting consequences. Its discovery was motivated by the desire to use (9) of Corollary 3.3 to good advantage. Our first proof of it used Corollary 3.3, and only later did we find the elementary proof below.

THEOREM 3.7. *Suppose Q is a nonnilpotent quasinilpotent operator in $\mathscr{L}(\mathscr{H})$ and there exists some bounded sequence $\{X_n\}_{n\in\mathbb{N}}$ in $\mathscr{L}(\mathscr{H})$ such that either the sequence $\{(Q^n/\|Q^n\|)X_n\}$ or the sequence $\{X_n Q^n/\|Q^n\|\}$ converges in the weak operator topology to a nonzero operator R. Then Q is not a quasiaffinity and thus has a nontrivial hyperinvariant subspace.*

PROOF. We suppose that the sequence $\{(Q^n/\|Q^n\|)X_n\}$ converges to $R \neq 0$ in the weak operator topology and show that $QR = 0$. The other assertion is proved by taking adjoints. Consider the sequence $(Q^{n+1}/\|Q^n\|)X_n$, which obviously converges to QR in the weak operator topology. By Lemma 3.6 there exists a subsequence $\{Q^{n_k}\}_{k=1}^{\infty}$ of $\{Q^n\}$ such that the sequence $\{\|Q^{n_k+1}\|/\|Q^{n_k}\|\}$ converges to zero. Thus,

$$(15) \qquad (Q^{n_k+1}/\|Q^{n_k}\|)X_{n_k} = (\|Q^{n_k+1}\|/\|Q^{n_k}\|)(Q^{n_k+1}/\|Q^{n_k+1}\|)X_{n_k}, \quad k \in \mathbb{N},$$

and since the first term in the product on the right side of (15) tends to zero and the second term is bounded, we get from (15) that $QR = 0$, as desired. $\qquad \square$

This yields immediately the following results.

COROLLARY 3.8. *Suppose that Q is a nonnilpotent quasinilpotent operator in $\mathscr{L}(\mathscr{H})$ and there exists a subspace $\mathscr{M} \neq (0)$ such that the sequence of restrictions $\{[Q^n(Q^*)^n/\|Q^n\|^2]|_{\mathscr{M}}\}$ (mapping $\mathscr{M}$ into $\mathscr{H}$) converges in the weak operator topology to a nonzero operator. Then Q has a nontrivial hyperinvariant subspace.*

PROOF. Let E be the projection in $\mathscr{L}(\mathscr{H})$ whose range is $\mathscr{M}$. The hypotheses yield immediately that the sequence $\{(Q^n/\|Q^n\|)((Q^*)^n/\|Q^n\|)E\}$ converges (weakly) to a nonzero operator, and the result follows from Theorem 3.7. $\qquad \square$

COROLLARY 3.9. *Suppose that Q is a nonnilpotent quasinilpotent operator in $\mathscr{L}(\mathscr{H})$ such that the sequence $\{Q^n(Q^*)^n/\|Q^n\|^2\}$ of positive operators has a common eigenvector e_0 with corresponding (nonnegative) eigenvalues $\{\lambda_n\}$ (≤ 1) such that $\lim_n \lambda_n$ exists and is nonzero. Then Q has a nontrivial hyperinvariant subspace.*

4. Some problems

We close this note with what seem to be some interesting problems about quasinilpotent operators and their invariant subspaces.

(1) Generalize Corollary 3.9 by showing that if Q is a quasinilpotent operator in $\mathscr{L}(\mathscr{H})$ such that the operators in the sequence $\{Q^n(Q^*)^n\}_{n\in\mathbb{N}}$ have a common eigenvector, then Q has a nontrivial hyperinvariant subspace.

(2) Show that if Q is a quasinilpotent operator in $\mathscr{L}(\mathscr{H})$ such that the sequence $\{Q^n(Q^*)^n\}_{n\in\mathbb{N}}$ consists of mutually commuting operators, then Q has a nontrivial hyperinvariant subspace.

(3) It is known that if Q is an arbitrary quasinilpotent operator in $\mathscr{L}(\mathscr{H})$, then there exist compact operators K_1 and K_2 and quasiaffinities X and Y in $\mathscr{L}(\mathscr{H})$ such that $QX = XK_1$ and $YQ = K_2Y$ [3]. (It is also known that not every quasinilpotent

operator in $\mathscr{L}(\mathscr{H})$ is quasisimilar to a compact operator [3].) Use these facts together with Corollary 3.3 to show that every quasinilpotent operator in $\mathscr{L}(\mathscr{H})$ has a nontrivial hyperinvariant subspace.

Acknowledgement

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