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A DUAL DIFFERENTIATION SPACE WITHOUT AN EQUIVALENT LOCALLY UNIFORMLY ROTUND NORM

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Abstract

A Banach space $(X, \|\cdot\|)$ is said to be a *dual differentiation* space if every continuous convex function defined on a non-empty open convex subset A of X^* that possesses weak* continuous subgradients at the points of a residual subset of A is Fréchet differentiable on a dense subset of A. In this paper we show that if we assume the continuum hypothesis then there exists a dual differentiation space that does not admit an equivalent locally uniformly rotund norm.

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1. Introduction

Given a Banach space $(X, \|\cdot\|)$ the *Bishop-Phelps* set (or *BP*-set for short) is the set

$$\{x^* \in X^* : \|x^*\| = x^*(x) \text{ for some } x \in B_X\},$$

where B_X denotes the closed unit ball in $(X, \|\cdot\|)$. The Bishop-Phelps theorem, [1] says that the *BP*-set is always dense in X^*. In this paper we are interested in the case when the *BP*-set is residual (that is, contains a dense G_δ subset) in X^*. Certainly, it is known that if the dual norm is Fréchet differentiable on a dense subset of X^* then the *BP*-set is residual in X^* (see the discussion in [13]). However, the converse question (that is, if the *BP*-set is residual in X^* must the dual norm necessarily be Fréchet differentiable on a dense subset of X^*?) remains open. One approach to this problem is to consider the following class of Banach spaces. A Banach space $(X, \|\cdot\|)$ is called a *dual differentiation* space (or *DD*-space for short) if every continuous convex function defined on a non-empty open convex subset A of X^* that possesses weak* continuous subgradients at the points of a residual subset of A is Fréchet differentiable on a dense subset of A. It follows then that in a *DD*-space if the *BP*-set is residual in X^* then the dual norm is Fréchet differentiable on a dense subset of X^*. Hence one way to solve our problem would be to show that every Banach space is a *DD*-space. Unfortunately, to date, we have been unable to achieve this.

In the study of *DD*-spaces the authors introduced in [2] a class of Banach spaces defined in terms of the continuity properties of 'quasi-continuous' mappings. Let $f : T \to X$ be a mapping acting from a topological space T into a Banach space $(X, \|\cdot\|)$. Then f is said to be *hyperplane minimal* if for each open half space H of X and open subset U of T with $f(U) \cap H \neq \emptyset$ there exists a non-empty open subset V of U such that $f(V) \subseteq H$ (see [2] for the original definition). Using this definition the authors in [14, page 242] said that a Banach space $(X, \|\cdot\|)$ is a *generic continuity* space (or *GC*-space for short) if every hyperplane minimal mapping acting from a complete metric space M into X is norm continuous at the points of a dense subset of M (see [2, page 414] for the original definition in terms of minimal weak* cuscos). It was shown in [2, Theorem 2.6] that every *GC*-space is in fact a *DD*-space. However, right from its inception, the study of *GC*-spaces has been closely linked to the study of locally uniformly rotund renormings. (Recall that a norm $\|\cdot\|$ is said to be *locally uniformly rotund* if $\lim_{n\to\infty} \|x_n - x\| = 0$ whenever $x, x_n \in B_X$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \|x_n + x\| = 2$.) In the paper [6, Theorem 3.5] it was shown that every Banach space that can be equivalently renormed to have every point of its unit sphere a denting point of its closed unit ball is a *GC*-space while in the paper [3, Theorem 4.5] it was shown that every Banach space that can be equivalently renormed to have every point of its unit sphere a quasi-denting point (originally called α-denting point) of its closed unit ball is a *GC*-space. In both cases it can be shown that the spaces can be equivalently renormed to be locally uniformly rotund (see, [19] and [20] respectively). Following on from this, the authors in [2, Theorem 1.13] showed that every Banach space that can be equivalently renormed to be weakly locally uniformly rotund is a *GC*-space. Nowadays such spaces are known to admit an equivalent locally uniformly rotund norm [11]. However, the story does not end here. In [12] it was shown that every Banach space that can be equivalently renormed so that on the dual sphere the relative weak and weak* topologies agree is a *GC*-space. Then in [7] it was shown that such spaces are sigma-fragmentable. Finally, in [4] it was shown that such spaces admit an equivalent locally uniformly rotund norm (see [18] and [4]). Motivated by these results the authors in [14, Question 1] asked 'Can every *GC*-space be equivalently renormed to be locally uniformly rotund?' Here we show that if we assume the continuum hypothesis then the answer is 'No'. Thus we sever the ties between the study of *GC*-spaces and the study of locally uniformly rotund renormings.

2. A *GC*-space without a Kadeč norm

Our counter-example is modelled on that of Namioka and Pol [15] which in turn, is based upon the following theorem of Kunen (see [16] for a proof).

THEOREM 2.1. *Assume the continuum hypothesis and let X be a subset of* $[0, 1]$. *Then there exists a locally compact, locally countable topology τ on X, stronger than the Euclidean topology, such that, if K is the one-point compactification of (X, τ), then the function space $C(K)$ is hereditarily Lindelöf in the weak topology.*

It is shown in [15, Corollary 3.3] that if X is uncountable then the Banach space $(C(K), \|\cdot\|_\infty)$ is not σ-fragmentable (see, [5] for the definition of σ-fragmentability). In particular, this means that $C(K)$ does not admit a Kadeč norm that is equivalent to the supremum norm on $C(K)$, [5]. (Recall that a norm $\|\cdot\|$ is said to be a *Kadeč* norm if on the unit sphere the relative norm topology coincides with the relative weak topology.) What we shall show is that if X does not contain any uncountable compact subsets (with respect to the Euclidean topology) then $C(K)$ is a *GC*-space. Hence, if X is an uncountable subset of $[0, 1]$ that does not contain any uncountable compact subsets (for example, if X is a Bernstein set, [17, page 23]) then $C(K)$ is a *GC*-space without an equivalent locally uniformly rotund norm. But before we can accomplish this we will need a few more definitions and a few more lemmas. Let $\Phi : T \to 2^X$ be a set-valued mapping acting between topological spaces T and X. We shall say that Φ is *upper semicontinuous* (*lower semicontinuous*) *at a point* $t_0 \in T$ if for each open subset W of X with $\Phi(t_0) \subseteq W$ ($\Phi(t_0) \cap W \neq \emptyset$) there exists a neighbourhood U of t_0 such that $\Phi(t) \subseteq W$ ($\Phi(t) \cap W \neq \emptyset$) for all $t \in U$. Similarly, we shall say that Φ is *quasi upper semicontinuous* (*quasi lower semicontinuous*) *at a point* $t_0 \in T$ if for each open neighbourhood U of t_0 and open subset W of X with $\Phi(t_0) \subseteq W$ ($\Phi(t_0) \cap W \neq \emptyset$) there exists a non-empty open subset V of U such that $\Phi(t) \subseteq W$ ($\Phi(t) \cap W \neq \emptyset$) for all $t \in V$. If Φ is both upper and lower semicontinuous at a point $t_0 \in T$ then we simply say that Φ is *continuous at* t_0.

LEMMA 2.2 ([15, Lemma 6.1]). *If $\mathscr{A}$ is an uncountable family of distinct compact open subsets of a Hausdorff topological space then $\bigcup \mathscr{A}$ is also uncountable.*

This lemma may be used to establish the following fact concerning continuous set-valued mappings.

LEMMA 2.3. *Suppose that τ_1 and τ_2 are Hausdorff topologies on a set X such that every τ_1-compact subset of X is at most countable. If $\Phi : M \to 2^X$ is a set-valued mapping acting from a complete metric space (M, ρ) into τ_1-compact subsets of X such that*: (i) *Φ is τ_1-continuous and* (ii) *for each $m \in M$, $\Phi(m)$ is τ_2-compact and τ_2-open then Φ is constant on some non-empty open subset of M.*

PROOF. In order to obtain a contradiction let us assume that Φ is not constant on any non-empty open subset of M. Let D be the set of all finite sequences of 0's and 1's. We shall inductively (on the length $|d|$ of $d \in D$) define a family $\{V_d : d \in D\}$ of non-empty open subsets of M such that

(i) $\rho\text{-diam}(V_d) < 1/2^{|d|}$;
(ii) $\emptyset = \overline{V_{d0}} \cap \overline{V_{d1}} \subseteq \overline{V_{d0}} \cup \overline{V_{d1}} \subseteq V_d$ for each $d \in D$;
(iii) $\Phi(v) \neq \Phi(v')$ whenever $v \in V_{d0}$ and $v' \in V_{d1}$.

Base Step. Let $V_\emptyset$ be a non-empty open subset of M with $\rho\text{-diam}(V_\emptyset) < 1/2^0$, where $\emptyset$ denotes the empty sequence of length 0.

Assuming that we have already defined the non-empty open sets V_d satisfying the properties (i), (ii) and (iii) for all $d \in D$ with $|d| \leq n$, we proceed to the next step.

Inductive Step. Fix $d \in D$ of length n. Then there are two points v_0 and v_1 in V_d and some point $x \in X$ such that $x \in \Phi(v_0)\backslash\Phi(v_1)$. Since (X, τ_1) is Hausdorff and $\Phi(v_1)$ is τ_1-compact there exist disjoint τ_1-open sets U_0 and U_1 such that $x \in U_0$ and $\Phi(v_1) \subseteq U_1$. From the τ_1-continuity of Φ we can choose open neighbourhoods V_{di} of v_i $(i = 1, 2)$ such that (i) and (ii) are satisfied and $\Phi(v) \cap U_0 \neq \emptyset$ for all $v \in V_{d0}$ and $\Phi(V_{d1}) \subseteq U_1$. In particular, $\Phi(v) \neq \Phi(v')$ whenever $v \in V_{d0}$ and $v' \in V_{d1}$ and so property (iii) is also satisfied. This completes the induction.

For each $n \in \mathbb{N}$, let

$$K_n := \bigcup \left\{\overline{V_d} : d \in D \text{ and } |d| = n\right\}$$

and let $K := \bigcap\{K_n : n \in \mathbb{N}\}$. Then K is an uncountable compact subset of M. Moreover, $\Phi(k) \neq \Phi(k')$ whenever k and k' are distinct elements of K. Therefore, $\mathscr{A} := \{\Phi(k) : k \in K\}$ is an uncountable family of τ_2-compact τ_2-open subsets and so by Lemma 2.2, $\Phi(K) = \bigcup \mathscr{A}$ must be uncountable. On the other hand, since Φ is τ_1-upper semicontinuous and has τ_1-compact images, $\Phi(K)$ is τ_1-compact; which contradicts the hypothesis that X does not contain any uncountable τ_1-compact subsets. Hence Φ must be constant on some non-empty open subset of M. □

Our main result also relies upon the following version of Fort's theorem pioneered by Matejdes, [10].

LEMMA 2.4. *Let $\Phi : T \to 2^M$ be a quasi lower semicontinuous set-valued mapping acting from a Baire space T into compact subsets of a metric space M. Then there exists a residual subset R of T such that Φ is continuous at each point of R.*

PROOF. Let $D := \{t \in T : \Phi(t) \neq \emptyset\}$. Since Φ is quasi lower semicontinuous, $D \subseteq \overline{\text{int}(D)}$. Let $B := \text{int}(D)$. Then by [8, Corollary 2.9] there exists a residual subset R' of B such that $\Phi|_B$ is continuous at each point of R'. Let $R := R' \cup (T\backslash\overline{D})$. Then R is residual in T and Φ is continuous at each point of R. □

LEMMA 2.5 ([14, Theorem 1.1]). *Let $\Phi : T \to X$ be a hyperplane minimal mapping acting from a topological space T into a Banach space $(X, \|\cdot\|)$. Then for each subset D of T, $\Phi(\text{int}\,\overline{D}) \subseteq \overline{\text{co}}\{\Phi(D)\}$. In particular,* $\|\cdot\|$-diam$[\Phi(\text{int}\,\overline{D})] \leq \|\cdot\|$-diam$[\Phi(D)]$.

THEOREM 2.6. *Assume the continuum hypothesis. Then there exists a scattered compact set K such that $(C(K), \|\cdot\|_\infty)$ is a GC-space but $(C(K)$, weak$)$ is not σ-fragmentable. In particular, $C(K)$ does not admit a Kadeč norm equivalent to the supremum norm.*

PROOF. Let X be any uncountable subset of $[0, 1]$ that does not contain any uncountable compact subsets (for example, X is a Bernstein subset of $[0, 1]$) and let K be the one-point compactification, with x_∞ the point at infinity, of the space (X, τ) with Kunen's topology as described in Theorem 2.1. By [15, Corollary 3.3], $(C(K)$, weak$)$ is not σ-fragmentable by the norm. In particular, this means that $C(K)$ does not have a Kadeč norm equivalent to the supremum norm. So it remains to show that $(C(K), \|\cdot\|_\infty)$ is a GC-space. In fact because of the 3-space property given in [14, Theorem 3.7] it is sufficient to show that $(C_0(K), \|\cdot\|_\infty)$ is a GC-space, where $C_0(K) := \{f \in C(K) : f(x_\infty) = 0\}$, that is, the functions that vanish at infinity. To this end, let $f : M \to C_0(K)$ be a hyperplane minimal mapping acting from a complete metric space (M, d) into $C_0(K)$. For each $\varepsilon > 0$, consider the open set

$$O_\varepsilon := \bigcup \{\text{open sets } U : \|\cdot\|_\infty\text{-diam}[f(U)] \leq \varepsilon\}.$$

We claim that for each $\varepsilon > 0$, O_ε is dense in M. We begin the justification of this by considering a non-empty open subset W of M (with the aim of showing that $O_\varepsilon \cap W \neq \emptyset$). By [14, Theorem 2.9] we may assume that $f(M) \subseteq B_{C_0(K)}$, the closed unit ball in $C_0(K)$. Let $\mathscr{F}$ be the countable collection of all finite sets F of rational numbers in $(-1, 1)$ such that the distance of each point in $[-1, 1]$ to F is less than $\varepsilon/2$. For each $F \in \mathscr{F}$ and $n \in \mathbb{N}$, let

$$A_n(F) := \{t \in W : \text{dist}(f(t)(K), F) > 1/n\}.$$

Then the countable family $\{A_n(F) : F \in \mathscr{F} \text{ and } n \in \mathbb{N}\}$ covers W. Since for each $t \in M$, $f(t)(K)$ is a scattered compact subset of $\mathbb{R}$ and hence countable. Therefore there must be some $F \in \mathscr{F}$ with $f(t)(K) \cap F = \emptyset$. It follows then that $t \in A_n(F)$ for some $n \in \mathbb{N}$. Now, since W is a Baire space there is some $F \in \mathscr{F}$ and $n \in \mathbb{N}$ such that $A_n(F)$ is second category in W. Let $F := \{q_1, q_2, \ldots, q_m\}$ where,

$$-1 =: q_0 < q_1 < q_2 < \cdots q_m < q_{m+1} := 1$$

and let $F^* := F \cup \{q_0, q_{m+1}\}$. Then, by possibly making n larger, we may assume that

$$0 < 1/n < 1/2 \min\{|q' - q''| : q', q'' \in F^* \text{ and } q' \neq q''\}.$$

Let i_0 be the integer in $\{2, \ldots, m\}$ such that $0 \in (q_{i_0-1}, q_{i_0})$. For each i with $i_0 \leq i \leq m$, define $N_i : \text{int}[\overline{A_n(F)}] \to 2^K$, $\overline{N_i} : \text{int}[\overline{A_n(F)}] \to 2^K$ and $\widetilde{N}_i : \text{int}[\overline{A_n(F)}] \to 2^K$ by

$$N_i(t) := \{k \in K : f(t)(k) > q_i + 1/n\}, \quad \overline{N_i}(t) := \overline{N_i(t)}^{\tau},$$

and

$$\widetilde{N}_i(t) := \{k \in K : f(t)(k) > q_i - 1/n\}.$$

For each i with $1 \leq i < i_0$, define $N_i : \text{int}[\overline{A_n(F)}] \to 2^K$, $\overline{N_i} : \text{int}[\overline{A_n(F)}] \to 2^K$ and $\widetilde{N}_i : \text{int}[\overline{A_n(F)}] \to 2^K$ by

$$N_i(t) := \{k \in K : f(t)(k) < q_i - 1/n\}, \quad \overline{N_i}(t) := \overline{N_i(t)}^{\tau},$$

and

$$\widetilde{N}_i(t) := \{k \in K : f(t)(k) < q_i + 1/n\}.$$

Now since f is hyperplane minimal, both mappings N_i and $\widetilde{N}_i$, (with $i \in \{1, 2, \ldots, m\}$) are quasi lower semicontinuous on $\text{int}[\overline{A_n(F)}]$ with respect to the discrete topology on K. Therefore, for each $i \in \{1, 2, \ldots, m\}$ the mapping $t \mapsto \overline{N_i}(t)$ has compact (possibly empty) images and is quasi lower semicontinuous with respect to both the τ-topology and the Euclidean topology on K. Hence by Lemma 2.4, there exists a dense G_δ subset G of $\text{int}[\overline{A_n(F)}]$ on which each $\overline{N_i}$, (with $i \in \{1, 2, \ldots, m\}$) is continuous with respect to the Euclidean topology on K. We now show that if $t \in G$ then for each $i \in \{1, 2, \ldots, m\}$, $\overline{N_i}(t) = \widetilde{N}_i(t)$. So consider $t \in G$ and $i \in \{i_0, \ldots, m\}$ (the case $1 \leq i < i_0$ is similar) and suppose, in order to obtain a contradiction, that there is some $k \in \widetilde{N}_i(t) \setminus \overline{N_i}(t)$. Since $\overline{N_i}$ is upper semi continuous with respect to the Euclidean topology at $t \in G$ there exists an open neighbourhood U of t in $\text{int}[\overline{A_n(F)}]$ such that $k \notin \overline{N_i}(U)$. On the other hand, the mapping $\widetilde{N}_i$ is quasi lower semicontinuous with respect to the discrete topology on K and so there is a non-empty open subset V of U such that $k \in \widetilde{N}_i(t')$ for all $t' \in V$. In particular, this would mean that for each $t' \in V$, $f(t')(k) \in (q_i - 1/n, q_i + 1/n]$ and so $\text{dist}(f(t')(K), F) \leq 1/n$. But this is impossible since for each $t' \in V \cap A_n(F) \neq \emptyset$, $\text{dist}(f(t')(K), F) > 1/n$. Hence it must be the case that $\overline{N_i}(t) = \widetilde{N}_i(t)$. Next we successively apply Lemma 2.3 (with τ_1 equal to the Euclidean topology and τ_2 equal to τ) to the mappings $t \mapsto \overline{N_i}(t)$ defined on G—which is completely metrizable—to obtain a decreasing sequence

$$U_m \subseteq U_{m-1} \subseteq \cdots U_2 \subseteq U_1 \subseteq W$$

of non-empty open subsets of $\text{int}[\overline{A_n(F)}]$ such that each $\overline{N_i}$ is constant on $U_i \cap G$. Let $U := U_m$ then each $\overline{N_i} = \widetilde{N}_i$ is constant on $U \cap G$. For each $0 \leq i \leq m$, let

$J_i : U \cap G \to 2^K$ be defined by $J_i(t) := \{k \in K : f(t)(k) \in [q_i, q_{i+1}]\}$. It is easy to verify that each J_i $(0 \le i \le m)$ is constant on $U \cap G$ and that for each $t \in U \cap G$, $\{J_i(t) : 0 \le i \le m\}$ is a partition of K. Indeed, for each $t \in U \cap G$ if

(i) $i = 0$ then $J_i(t) = \widetilde{N}_{i+1}(t)$;
(ii) $0 < i < i_0 - 1$ then $J_i(t) = \widetilde{N}_{i+1}(t) \backslash \widetilde{N}_i(t)$;
(iii) $i = m$ then $J_i(t) = \widetilde{N}_i(t)$;
(iv) $i_0 \le i < m$ then $J_i(t) = \widetilde{N}_i(t) \backslash \widetilde{N}_{i+1}(t)$;
(v) $i = i_0 - 1$ then $J_i(t) = K \backslash \bigcup\{J_i(t) : 0 \le i \le m \text{ and } i \neq (i_0 - 1)\}$.

Therefore, if $t, t' \in U \cap G$ and $k \in K$ then $|f(t')(k) - f(t)(k)| < |q_{j+1} - q_j| < \varepsilon$, where j is the unique element in $\{0, 1, \ldots, m\}$ such that $k \in J_j(t)$ and $k \in J_j(t')$. Thus, $\|\cdot\|_\infty$-diam$[f(U \cap G)] \le \varepsilon$ and so by Lemma 2.5, $\|\cdot\|_\infty$-diam$[f(U)] \le \varepsilon$. Hence $\emptyset \neq U \subseteq O_\varepsilon \cap W$; which shows that O_ε is dense in M. Therefore f is norm continuous at each point of $\bigcap\{O_{1/n} : n \in \mathbb{N}\}$. □

REMARK. The previous theorem raises two natural questions: (i) Is every weakly Lindelöf Banach space a generic continuity space? (ii) Is there an example (in ZFC) of a weakly Lindelöf Banach space that does not admit an equivalent locally uniformly rotund norm?

We end this paper by reiterating the main problem in the area. Namely, is it true that if the Bishop-Phelps set of a Banach space $(X, \|\cdot\|)$ is residual in X^* then the dual norm is Fréchet differentiable on a dense subset of X^*?

One impediment to finding a counter-example to this question is that, in general, it is difficult to identify those linear functions in the dual of a Banach space that attain their norm. There are a few exceptions to this, for example, if X is reflexive or of the form $C(T)$, for some infinite compact T, with the supremum norm. However, in the latter case the Bishop-Phelps set is known to be always of the first Baire category in $C(T)^*$, [9].

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