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AN L^p VERSION OF HARDY'S THEOREM FOR THE DUNKL TRANSFORM

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Abstract

In this paper, we give a generalization of Hardy's theorems for the Dunkl transform $\mathscr{F}_D$ on $\mathbb{R}^d$. More precisely for all $a > 0$, $b > 0$ and $p, q \in [1, +\infty]$, we determine the measurable functions f on $\mathbb{R}^d$ such that $e^{a\|x\|^2} f \in L_k^p(\mathbb{R}^d)$ and $e^{b\|y\|^2} \mathscr{F}_D(f) \in L_k^q(\mathbb{R}^d)$, where $L_k^p(\mathbb{R}^d)$ are the Lebesgue spaces associated with the Dunkl transform.

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1. Introduction

A famous theorem of Hardy [8] asserts that a measurable function f on $\mathbb{R}$ and its Fourier transform $\hat{f}$ cannot be both 'very rapidly decreasing'. More precisely, if $|f(x)| \le Ce^{-ax^2}$ and $|\hat{f}(y)| \le Ce^{-by^2}$ for some constants $C > 0$, $a > 0$ and $b > 0$, then $f = 0$ a.e. if $ab > 1/4$ and there exists nonzero f if $ab \le 1/4$. An L^p version of this result, obtained by Cowling and Price [2] states that for $p, q \in [1, +\infty]$, and at least one of them is finite, if $\|e^{ax^2} f\|_p < +\infty$ and $\|e^{by^2} \hat{f}\|_q < +\infty$ then $f = 0$ a.e. if $ab \ge 1/4$. Generalizations of this result to the Heisenberg group and the motion group have been proved in [6, 15]. In this paper we study an analogue of the theorem of Cowling and Price for the Dunkl transform $\mathscr{F}_D$ on $\mathbb{R}^d$. For $a > 0$, $b > 0$ and $p, q \in [1, +\infty]$, we determine the measurable functions f on $\mathbb{R}^d$ such that $e^{a\|x\|^2} f \in L_k^p(\mathbb{R}^d)$ and $e^{b\|y\|^2} \mathscr{F}_D(f) \in L_k^q(\mathbb{R}^d)$, where $L_k^p(\mathbb{R}^d)$ are the Lebesgue spaces associated with the Dunkl transform. We note that our results, announced in [7], are related to an analogue of the classical Heisenberg-Weyl uncertainty principle for the

Dunkl transform due to Rösler [14]. The Dunkl transform is associated to differential-difference operators corresponding to a finite group of reflections of the Euclidean space $\mathbb{R}^d$. They provide a useful tool in the study of special functions with root systems [5, 9] and they play an important role in the algebraic description of exactly solvable quantum many body systems of Calogero-Moser-Sutherland type (see [10, 11]).

The contents of the paper is as follows: In Section 2 we recall some basic facts from Dunkl's theory, we describe Dunkl operators and we give the main results about Dunkl transform $\mathscr{F}_D$ which generalizes the classical Fourier transform $\mathscr{F}$ on $\mathbb{R}^d$.

We introduce, in the third section, the intertwining Dunkl operator V defined by Dunkl in [5] and studied by de Jeu, Rösler and Trimèche in [3, 13, 16]. We also consider in this section the transposed operator ${}^tV$ of V. These operators V and ${}^tV$ are respectively topological automorphisms of $\mathscr{E}(\mathbb{R}^d)$ (the space of $\mathscr{C}^\infty$-functions on $\mathbb{R}^d$) and $\mathscr{D}(\mathbb{R}^d)$ (the subspace of $f \in \mathscr{E}(\mathbb{R}^d)$ which are compactly supported) and they transmute the Dunkl operators into the partial derivatives. We will give more properties of the operator ${}^tV$ which plays an important role in the proofs of the main results of the paper. In particular, in Theorem 3.1 we prove that it can be extended to the Lebesgue space $L^1_k(\mathbb{R}^d)$ associated with Dunkl theory and satisfies the fundamental relation $\mathscr{F}_D = \mathscr{F} \circ {}^tV$.

In Section 4 we give two lemmas from the complex variable theory which are an L^p version of the Phragmén-Lindelöf theorem and will be used in the sequel. Section 5 is devoted to the L^p version of Hardy's theorem for the Dunkl transform $\mathscr{F}_D$. The proof of this result requires both tools introduced in sections two and three.

In the last section, an analogue of the classical Hardy's theorem is obtained for the Dunkl transform.

2. Dunkl transform

In this section, we recall some basic results from Dunkl's theory which we will use in the sequel.

2.1. Reflection groups and root systems We consider $\mathbb{R}^d$ equipped with the usual scalar product $\langle\cdot,\cdot\rangle$ and the Euclidian norm $\|x\| = \sqrt{\langle x, x\rangle}$.

For $\alpha \in \mathbb{R}^d\backslash\{0\}$, let $H_\alpha \subset \mathbb{R}^d$ be the hyperplane orthogonal to α and

$$\sigma_\alpha(x) = x - (2\langle\alpha, x\rangle\|\alpha\|^{-2})\alpha \quad (x \in \mathbb{R}^d),$$

the reflection with respect to H_α. A finite set $R \subset \mathbb{R}^d\backslash\{0\}$ is called a *root system* if $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$.

For a given root system R, the reflections $\sigma_\alpha, \alpha \in R$, generate a finite group $W \subset O(d)$, called the *reflection group* associated with R and for a given $\beta \in \mathbb{R}^d\backslash\bigcup_{\alpha\in R} H_\alpha$, we fix the positive subsystem $R_+ = \{\alpha \in R; \langle \alpha, \beta \rangle > 0\}$, then for each $\alpha \in R$ either α or $-\alpha$ belong to R_+.

A *multiplicity function* is a function $k : R \to \mathbb{C}$ defined on the root system R which is invariant under the action of the reflection group W.

The *index* γ of the root system is then defined by $\gamma = \sum_{\alpha \in R_+} k(\alpha)$, and the *weight function* is the W-invariant and homogeneous (of degree 2γ) function on $\mathbb{R}^d$ given by: $\omega_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k(\alpha)}$.

For $d = 1$ and $W = \mathbb{Z}_2$, the multiplicity function k is a single parameter denoted $\gamma \geq 0$ and for all $x \in \mathbb{R}$: $\omega_k(x) = |x|^{2\gamma}$.

In the general case, we will need the Mehta-type constant

$$c_k = \left(\int_{\mathbb{R}^d} e^{-\|x\|^2} \omega_k(x)\, dx \right)^{-1}, \tag{1}$$

which is known for all Coxeter groups W (see [3, 5, 9]).

2.2. Dunkl operators and Dunkl kernel The Dunkl operators T_j, $j = 1, \ldots, d$, on $\mathbb{R}^d$, associated with the finite reflection group W and multiplicity function k, are given for a function f of class $\mathscr{C}^1$ on $\mathbb{R}^d$ by

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha \in \mathbb{R}_+} k(\alpha) \alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle}. \tag{2}$$

In the case $k = 0$, the T_j, $j = 1, \ldots, d$, reduce to the corresponding partial derivatives. In this paper we will assume throughout that $k \geq 0$.

For $y \in \mathbb{R}^d$, the system

$$\begin{cases} T_j u(x) = y_j u(x) & j = 1, \ldots, d; \\ u(0) = 1, \end{cases}$$

admits a unique analytic solution on $\mathbb{R}^d$, denoted by $K(x, y)$ and called *Dunkl kernel*. This kernel admits a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$ (see [5]).

For example if $d = 1$ and $W = \mathbb{Z}_2$, the Dunkl operator and Dunkl kernel (see [5]) are given by

$$T_1 f(x) = \frac{df}{dx}(x) + \gamma \frac{f(x) - f(-x)}{x}, \tag{3}$$

and for $z, t \in \mathbb{C}$,

$$K(z, t) = j_{\gamma - 1/2}(izt) + \frac{z}{2\gamma + 1} j_{\gamma + 1/2}(izt), \tag{4}$$

where for $s \geq -1/2$, j_s is the normalized Bessel function defined by

$$j_s(u) = 2^s \Gamma(s + 1) u^{-s} J_s(u)$$

with J_s the Bessel function of the first kind and index s.

The Dunkl kernel possesses the following properties [3, 13]:

(i) For $z, t \in \mathbb{C}^d$, we have $K(z,t) = K(t,z)$; $K(z,0) = 1$ and $K(\lambda z, t) = K(z, \lambda t)$ for all $\lambda \in \mathbb{C}$.

(ii) For all $\nu \in \mathbb{N}^d$, $x \in \mathbb{R}^d$ and $z \in \mathbb{C}^d$ we have

$$|D_z^\nu K(x,z)| \leq \|x\|^{|\nu|} \exp(\|x\| \| \operatorname{Re} z\|), \tag{5}$$

with $D_z^\nu = \partial^{|\nu|}/(\partial z_1^{\nu_1} \cdots \partial z_d^{\nu_d})$ and $|\nu| = \nu_1 + \cdots + \nu_d$. In particular, for all $x, y \in \mathbb{R}^d$, we have $|K(-ix, y)| \leq 1$.

(iii) The function $K(x,z)$ admits for all $x \in \mathbb{R}^d$ and $z \in \mathbb{C}^d$ the following Laplace type integral representation

$$K(x,z) = \int_{\mathbb{R}^d} e^{\langle y,z \rangle}\, d\mu_x(y), \tag{6}$$

where μ_x is a probability measure on $\mathbb{R}^d$ with support in the closed ball $B(0, \|x\|)$ of center 0 and radius $\|x\|$ (see [13]).

When $d = 1$ and $W = \mathbb{Z}_2$, for all $x \in \mathbb{R}\backslash\{0\}$ and $z \in \mathbb{C}$ the representation (6) is of the form (see [4])

$$K(x,z) = \frac{\Gamma(\gamma + 1/2)}{\sqrt{\pi}\,\Gamma(\gamma)} |x|^{-2\gamma} \int_{-|x|}^{|x|} (|x| - y)^{\gamma - 1} (|x| + y)^\gamma e^{yz}\, dy.$$

2.3. Dunkl transform We denote by

– $\mathscr{D}(\mathbb{R}^d)$ the space of $\mathscr{C}^\infty$-functions on $\mathbb{R}^d$ with compact support.

– $\mathscr{S}(\mathbb{R}^d)$ the space of $\mathscr{C}^\infty$-functions on $\mathbb{R}^d$ which are rapidly decreasing together with their derivatives.

– $L_k^p(\mathbb{R}^d)$, $p \in [1, +\infty]$, the space of measurable functions f on $\mathbb{R}^d$ such that

$$\|f\|_{k,p} = \left(\int_{\mathbb{R}^d} |f(x)|^p \omega_k(x)\, dx \right)^{1/p} < +\infty, \quad \text{if } 1 \leq p < +\infty,$$

$$\|f\|_{k,\infty} = \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} |f(x)| < +\infty.$$

The Dunkl transform of a function $f \in \mathscr{D}(\mathbb{R}^d)$ is given by

$$\forall\, y \in \mathbb{R}^d, \mathscr{F}_D(f)(y) = \int_{\mathbb{R}^d} f(x) K(x, -iy) \omega_k(x)\, dx. \tag{7}$$

This transform has the following properties [3, 5]:

(i) For $f \in L_k^1(\mathbb{R}^d)$, we have $\|\mathscr{F}_D(f)\|_{k,\infty} \leq \|f\|_{k,1}$.

(ii) The transform $\mathscr{F}_D$ is a topological isomorphism from $\mathscr{S}(\mathbb{R}^d)$ onto itself. The inverse transform is given by

$$\forall\, x \in \mathbb{R}^d,\ \mathscr{F}_D^{-1}(h)(x) = \frac{c_k^2}{2^{2\gamma+d}} \int_{\mathbb{R}^d} h(y) K(x, iy) \omega_k(y)\, dy. \tag{8}$$

(iii) Let f be in $L^1_k(\mathbb{R}^d)$ such that the function $\mathscr{F}_D(f)$ belongs to $L^1_k(\mathbb{R}^d)$. Then we have the following inversion formula for the Dunkl transform

$$f(x) = \frac{c_k^2}{2^{2\gamma+d}} \int_{\mathbb{R}^d} \mathscr{F}_D(f)(y) K(x, iy) \omega_k(y)\, dy, \quad \text{a.e.} \tag{9}$$

3. The Dunkl dual intertwining operator

In this section we consider the Dunkl intertwining operator V and its dual ${}^tV$ and we give their properties. Next we study the extension of the operator ${}^tV$ to the functions of $L^1_k(\mathbb{R}^d)$.

Let $C(\mathbb{R}^d)$ be the space of continuous functions on $\mathbb{R}^d$. The *Dunkl intertwining operator* V is the operator from $C(\mathbb{R}^d)$ into itself given by

$$V(f)(x) = \int_{\mathbb{R}^d} f(y)\, d\mu_x(y), \quad x \in \mathbb{R}^d, \tag{10}$$

where μ_x is the measure given by the relation (6). In particular, we have

$$(\forall\, x \in \mathbb{R}^d,\ \forall\, z \in \mathbb{C}^d) \quad K(x, z) = V(e^{\langle \cdot, z\rangle})(x). \tag{11}$$

The operator ${}^tV$ defined on $\mathscr{D}(\mathbb{R}^d)$ by the relation

$$\int_{\mathbb{R}^d} {}^tV(f)(y) g(y) dy = \int_{\mathbb{R}^d} V(g)(x) f(x) \omega_k(x) dx, \tag{12}$$

where $f \in \mathscr{D}(\mathbb{R}^d)$ and $g \in C(\mathbb{R}^d)$ is called the *Dunkl dual intertwining operator* (see [16]). This operator has the following integral representation

$${}^tV(f)(y) = \int_{\mathbb{R}^d} f(x) d\nu_y(x) \quad (f \in \mathscr{D}(\mathbb{R}^d)), \tag{13}$$

where for all $y \in \mathbb{R}^d$, ν_y is a positive measure on $\mathbb{R}^d$ whose support is contained in the set $\{x \in \mathbb{R}^d,\ \|x\| \geq \|y\|\}$. Moreover, ${}^tV$ is a topological isomorphism from $\mathscr{D}(\mathbb{R}^d)$ (respectively $\mathscr{S}(\mathbb{R}^d)$) onto itself satisfying the transmutation relations

$$(\forall\ f \in \mathscr{D}(\mathbb{R}^d), \forall\ y \in \mathbb{R}^d) \quad {}^tV(T_j f)(y) = \frac{\partial}{\partial y_j} {}^tV(f)(y), \quad j = 1, \ldots, d,$$

and the following property (see [16])

$$\mathscr{F}_D(f) = \mathscr{F} \circ {}^tV(f), \quad \forall\, f \in \mathscr{S}(\mathbb{R}^d), \tag{14}$$

where $\mathscr{F}$ is the classical Fourier transform on $\mathbb{R}^d$ given by

$$\mathscr{F}(g)(\xi) = \int_{\mathbb{R}^d} g(x)e^{-i\langle x,\xi\rangle}\,dx, \quad \forall\, g \in \mathscr{S}(\mathbb{R}^d). \tag{15}$$

EXAMPLE 1. If $d = 1$ and $W = \mathbb{Z}_2$, the operators V and ${}^tV$ are given for $g \in C(\mathbb{R})$ and $f \in \mathscr{D}(\mathbb{R})$ by (see [4, 16])

$$V(g)(x) = \frac{\Gamma(\gamma + 1/2)}{\sqrt{\pi}\,\Gamma(\gamma)}|x|^{-2\gamma}\int_{-|x|}^{|x|}(|x| - y)^{\gamma-1}(|x| + y)^{\gamma}g(y)\,dy, \quad \forall\, x \in \mathbb{R}\backslash\{0\},$$

and

$${}^tV(f)(y) = \frac{\Gamma(\gamma + 1/2)}{\sqrt{\pi}\,\Gamma(\gamma)}\int_{|x|\geq|y|}(|x| - y)^{\gamma-1}(|x| + y)^{\gamma}f(x)\,dx, \quad \forall\, y \in \mathbb{R}.$$

In the following result we extend the operator ${}^tV$ to the functions in $L^1_k(\mathbb{R}^d)$.

THEOREM 3.1. *Let $(\nu_y)_{y\in\mathbb{R}^d}$, be the family of measures defined in formula (13) and let $f \in L^1_k(\mathbb{R}^d)$. Then for almost all y (with respect to Lebesgue measure on $\mathbb{R}^d$), f is ν_y-integrable, the function*

$$y \to \nu_y(f) = \int_{\mathbb{R}^d} f(x)\,d\nu_y(x),$$

which will also be denoted by ${}^tV(f)$ is defined almost everywhere on $\mathbb{R}^d$ and is Lebesgue integrable. Moreover for all bounded continuous functions g on $\mathbb{R}^d$, we have the formula

$$\int_{\mathbb{R}^d} {}^tV(f)(y)g(y)\,dy = \int_{\mathbb{R}^d} f(x)V(g)(x)\omega_k(x)\,dx. \tag{16}$$

PROOF. We will divide the proof in five steps.

(i) Let us show that the family of measures $(\nu_y)_{y\in\mathbb{R}^d}$ is vaguely continuous. More precisely, we will show that for all $f \in C_c(\mathbb{R}^d)$ (the space of $f \in C(\mathbb{R}^d)$ with compact support), the function

$$y \to {}^tV(f)(y) = \int_{\mathbb{R}^d} f(x)d\nu_y(x) = \nu_y(f),$$

belongs to $C_c(\mathbb{R}^d)$.

Let $f \in C_c(\mathbb{R}^d)$ and $(p_n)_{n>0}$ an approximate identity belonging to $\mathscr{D}(\mathbb{R}^d)$. There is a closed ball $B(0, r)$ of $\mathbb{R}^d$ of center 0 and radius r big enough such that it contains all the supports of the functions f and $p_n * f$, $n > 0$ (where $*$ is the classical convolution on $\mathbb{R}^d$) and a non negative function $\varphi \in \mathscr{D}(\mathbb{R}^d)$ such that $\varphi(x) \geq 1$ for all $x \in B(0, r)$. For all $y \in \mathbb{R}^d$, we then have

$$\begin{aligned}|{}^tV(f)(y) - {}^tV(p_n * f)(y)| &\leq \int_{B(0,r)} |f(x) - p_n * f(x)|\, d\nu_y(x)\\ &\leq \|f - p_n * f\|_\infty \|{}^tV(\varphi)\|_\infty,\end{aligned}$$

where $\|\cdot\|_\infty$ denotes the sup norm, and this implies immediately that ${}^tV(p_n * f)$ converges uniformly to ${}^tV(f)$. This shows that ${}^tV(f)$ is continuous but it is also clearly compactly supported. Thus ${}^tV(f) \in C_c(\mathbb{R}^d)$.

(ii) Let $g \geq 0$ be a continuous and bounded function on $\mathbb{R}^d$. Let us show that the family of measures $(\nu_y)_{y \in \mathbb{R}^d}$ is $g(y)\, dy$ integrable (for the definition see [1, page 17]). Let $(p_n)_{n>0}$ be the approximate identity used in (i). For all $f \in C_c(\mathbb{R}^d)$, by formula (12), we have

$$\int_{\mathbb{R}^d} \nu_y(p_n * f) g(y)\, dy = \int_{\mathbb{R}^d} (p_n * f)(x) V(g)(x) \omega_k(x)\, dx.$$

But the functions $p_n * f$, $n > 0$, have their supports in a fixed closed ball $B(0, r)$ and as in step (i), there is a fixed nonnegative $\varphi \in D(\mathbb{R}^d)$ such that for all $n > 0$ we have $|p_n * f| \leq \varphi$ and $|{}^tV(p_n * f)| \leq {}^tV(\varphi)$ on $\mathbb{R}^d$. Then letting $n \to +\infty$ and using the dominated convergence theorem we obtain immediately

$$\int_{\mathbb{R}^d} \nu_y(f) g(y)\, dy = \int_{\mathbb{R}^d} f(x) V(g)(x) \omega_k(x)\, dx. \tag{17}$$

(iii) We consider $g \geq 0$ a continuous and bounded function on $\mathbb{R}^d$. If f is an integrable function on $\mathbb{R}^d$ with respect to the measure $V(g)(x)\omega_k(x)dx$, the points (i), (ii) and Bourbaki's integration of measures theorem [1, page 17] shows that the function $y \to \nu_y(f)$ exists for almost all $y \in \mathbb{R}^d$ with respect to the measure $g(y)dy$, is integrable with respect to this measure and the relation (17) remains valid for this function f.

(iv) In the particular case where $g \equiv 1$ on $\mathbb{R}^d$, the point (iii) shows that if $f \in L^1_k(\mathbb{R}^d)$, the function $y \to \nu_y(f) = {}^tV(f)(y)$ exists almost everywhere, is Lebesgue integrable on $\mathbb{R}^d$ and we have

$$\int_{\mathbb{R}^d} {}^tV(f)(y)\, dy = \int_{\mathbb{R}^d} f(x) \omega_k(x)\, dx. \tag{18}$$

(v) We deduce easily from the points (iii) and (iv) that for all $f \in L^1_k(\mathbb{R}^d)$ and all bounded $g \in C(\mathbb{R}^d)$ formula (16) is true, which completes the proof of the theorem. $\square$

COROLLARY 3.2. *For all* $f \in L^1_k(\mathbb{R}^d)$, *we have*:

$$\mathscr{F}_D(f)(y) = \mathscr{F} \circ {}^tV(f)(y), \quad y \in \mathbb{R}^d. \tag{19}$$

PROOF. We obtain the result by applying (16) to the function $g(x) = e^{-i\langle x,y\rangle}$ and using the relation (11). □

4. An L^p version of the Phragmén-Lindelöf theorems

The proofs of some theorems in this paper depend on the two complex-variable lemmas which will be presented in the following section.

LEMMA 4.1. *Let h be an entire function on* $\mathbb{C}^d$ *such that*

$$\forall\, z \in \mathbb{C}^d, |h(z)| \le C \prod_{j=1}^{d} e^{a(\operatorname{Re} z_j)^2}, \quad \text{and} \quad \forall\, x \in \mathbb{R}^d, |h(x)| \le C, \tag{20}$$

for some $a > 0$ *and* $C > 0$. *Then h is constant on* $\mathbb{C}^d$.

PROOF. We fix $x_2, \dots, x_d \in \mathbb{R}$. The entire function $z_1 \to h(z_1, x_2, \dots, x_d)$ is $O(e^{a(\operatorname{Re} z_1)^2})$ in the quadrant $\Delta = \{z_1 = x_1 + iy_1; x_1 \ge 0, y_1 \ge 0\}$ and is bounded on the sides of Δ, then by a slight modification of the method used in [2, page 445] it is bounded on Δ. Applying the same method to the functions $h(-z_1, x_2, \dots, x_d)$, $\bar{h}(\bar{z}_1, x_2, \dots, x_d)$ and $\bar{h}(-\bar{z}_1, x_2, \dots, x_d)$ we deduce that $h(z_1, x_2, \dots, x_d)$ is bounded on $\mathbb{C}$, therefore by the Liouville theorem we have

$$h(z_1, x_2, \dots, x_d) = h(0, x_2, \dots, x_d), \quad \forall\, z_1 \in \mathbb{C}.$$

Now by analytic extension we deduce that

$$h(z_1, z_2, \dots, z_d) = h(0, z_2, \dots, z_d), \quad \forall\, z_1, \dots, z_d \in \mathbb{C},$$

and by induction, h is a constant function. □

LEMMA 4.2. *Let* $p \in [1, +\infty[$ *and h an entire function on* $\mathbb{C}^d$. *We suppose*

(i) *there exists* $j \in \{1, \dots, d\}$ *such that*

$$|h(z)| \le M(z_1, \dots, z_{j-1}, z_{j+1}, \dots, z_d) e^{a(\operatorname{Re} z_j)^2}, \quad \forall\, z \in \mathbb{C}^d, \tag{21}$$

for some $a > 0$ *and M a positive function on* $\mathbb{C}^d$.

(ii)

$$\|h_{|\mathbb{R}^d}\|_{k,p} < +\infty, \tag{22}$$

Then $h \equiv 0$.

Proof. From (22) the Fubini's theorem yields that there is a set $E \subset \mathbb{R}^{d-1}$, with E^c of Lebesgue measure zero such that for all $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d) \in E$ we have

$$\int_{\mathbb{R}} |h(x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_d)|^p \omega_k(x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_d)\, dx < +\infty.$$

Let us write for $x \in \mathbb{R}$,

$$h(x) = h(x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_d) \quad \text{and}$$
$$\widetilde{\omega}_k(x) = \omega_k(x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_d).$$

Clearly $\widetilde{\omega}_k(x)$ is of the form $\widetilde{\omega}_k(x) = \prod_{\alpha \in R_+} |a_\alpha + \alpha_j x|^{2k(\alpha)}$, where $a_\alpha = \prod_{i \neq j} \alpha_i x_i$ and there are three cases

(i) $\widetilde{\omega}_k(x)$ is identically zero on $\mathbb{R}$. This case occurs if and only if $a_\alpha = 0$ and $\alpha_j = 0$ for some $\alpha \in R_+$ and can be disregarded because points $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d)$ such that $a_\alpha = 0$ for some $\alpha \in R_+$ are in a set of Lebesgue measure zero in $\mathbb{R}^{d-1}$ and then they can be supposed to belong to E^c.

(ii) $\widetilde{\omega}_k(x)$ is a constant if for all $\alpha \in R_+$, $\alpha_j = 0$.

(iii) $\widetilde{\omega}_k(x)$ vanishes only on a finite number of points, precisely for $x = -a_\alpha/\alpha_j$, $\alpha \in R_+$ and $\alpha_j \neq 0$. In this case the set $\{\widetilde{\omega}_k \leq 1\} = \{x \in \mathbb{R};\ \widetilde{\omega}_k(x) \leq 1\}$ is compact. Now we have

$$\begin{aligned}\int_{\mathbb{R}} |h(x)|^p\, dx &= \int_{\{\widetilde{\omega}_k \leq 1\}} |h(x)|^p\, dx + \int_{\{\widetilde{\omega}_k > 1\}} |h(x)|^p\, dx \\ &\leq \int_{\{\widetilde{\omega}_k \leq 1\}} |h(x)|^p\, dx + \int_{\{\widetilde{\omega}_k > 1\}} |h(x)|^p \widetilde{\omega}_k(x)\, dx < +\infty.\end{aligned}$$

Indeed, the first integral in the right hand side of the above inequality is finite because $x \to |h(x)|^p$ is continuous on the compact set $\{\widetilde{\omega}_k \leq 1\}$ and the second integral is also finite by the initial hypothesis. Therefore we have proved that

$$\int_{\mathbb{R}} |h(x_1, \ldots, x_{j-1}, x, x_{j+1}, \ldots, x_d)|^p\, dx < +\infty,$$

for almost all $(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d) \in \mathbb{R}^{d-1}$. Clearly in case (ii) this is also true.

Now using (21) and applying the same method as in [2] to the function

$$z_j \to h(x_1, \dots, x_{j-1}, z_j, x_{j+1}, \dots, x_d)$$

we see that it is zero on $\mathbb{C}$ for almost all $(x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_d) \in \mathbb{R}^{d-1}$. The continuity of h and analytic extension imply that h is zero on $\mathbb{C}^d$. $\square$

5. An L^p version of Hardy's theorem

THEOREM 5.1. *Let f be a measurable function on $\mathbb{R}^d$ such that*

$$\|e^{a\|x\|^2} f\|_{k,p} < +\infty \quad \text{and} \quad \|e^{b\|y\|^2} \mathscr{F}_D(f)\|_{k,q} < +\infty, \tag{23}$$

for some constants $a > 0$, $b > 0$, $1 \leq p, q \leq +\infty$ and at least one of p and q is finite. Then

- *if $ab \geq 1/4$, we have $f = 0$ almost everywhere.*
- *if $ab < 1/4$, for all $\delta \in]a, 1/4b[$,*

the functions of the form $f(x) = P(x)e^{-\delta\|x\|^2}$, where P is an arbitrary polynomial on $\mathbb{R}^d$, satisfy (23).

To prove this result we need the following three lemmas.

LEMMA 5.2. *Let $a > 0$. For all $y \in \mathbb{R}^d$, we have*

$${}^tV(e^{-a\|x\|^2})(y) = Ce^{-a\|y\|^2}, \tag{24}$$

where $C = 2^{2\gamma+d} a^{d/2} c_k^{-1} \pi^{-d/2}$ with c_k the constant given by (1).

PROOF. As the function $x \to e^{-a\|x\|^2}$ belongs to $\mathscr{S}(\mathbb{R}^d)$, the relation (14) shows that ${}^tV(e^{-a\|x\|^2})(y) = \mathscr{F}^{-1} o \mathscr{F}_D(e^{-a\|x\|^2})(y)$.

But from [12, page 535], we have

$$\mathscr{F}_D(e^{-a\|x\|^2})(\xi) = a^{-\gamma} \pi^{-d/2} c_k^{-1} e^{-\|\xi\|^2/4a}, \tag{25}$$

and we obtain the result of the lemma by applying the classical inverse Fourier transform to relation (25). $\square$

LEMMA 5.3. *Let $p \in [1, +\infty]$ and f a measurable function on $\mathbb{R}^d$ such that $\left\|e^{a\|x\|^2} f\right\|_{k,p} < +\infty$, for some $a > 0$. Then: $\left\|e^{a\|x\|^2}\, {}^tV(f)\right\|_p < +\infty$, where $\|\cdot\|_p$ is the norm of the usual Lebesgue space $L^p(\mathbb{R}^d)$.*

PROOF. From the hypothesis it follows that $f \in L^1_k(\mathbb{R}^d)$. Then by Theorem 3.1, the function ${}^tV(f)$ is defined almost everywhere on $\mathbb{R}^d$. Now we consider two cases.

(i) If $p \in [1, +\infty[$, we have

$$\left\| e^{a\|x\|^2} {}^tV(f) \right\|_p^p \leq \int_{\mathbb{R}^d} e^{ap\|x\|^2} \left(\int_{\mathbb{R}^d} e^{a\|y\|^2} |f(y)| e^{-a\|y\|^2} d\nu_x(y) \right)^p dx.$$

Applying Hölder's inequality in the middle integral we obtain

$$\begin{aligned} \left\| e^{a\|x\|^2} {}^tV(f) \right\|_p^p \leq \int_{\mathbb{R}^d} e^{ap\|x\|^2} & \left(\int_{\mathbb{R}^d} e^{ap\|y\|^2} |f(y)|^p \, d\nu_x(y) \right) \\ & \times \left(\int_{\mathbb{R}^d} e^{-ap'\|y\|^2} \, d\nu_x(y) \right)^{p/p'} dx, \end{aligned}$$

where p' is the conjugate exponent of p. By Lemma 5.2 we deduce that the right hand side of the precedent inequality is equal to $(C(p')^{d/2})^{p/p'} \int_{\mathbb{R}^d} {}^tV((e^{a\|y\|^2}|f|)^p)(x)\, dx$, where C is the constant in formula (24). Using the relation (18), we have

$$\left\| e^{a\|x\|^2} {}^tV(f) \right\|_p \leq \left(C(p')^{d/2} \right)^{1/p'} \left\| e^{a\|y\|^2} f \right\|_{k,p} < +\infty.$$

(ii) If $p = +\infty$, we have

$$|{}^tV(f)(x)| \leq \int_{\mathbb{R}^d} e^{a\|y\|^2} |f(y)| e^{-a\|y\|^2} d\nu_x(y), \leq \left\| e^{a\|y\|^2} f \right\|_{k,\infty} {}^tV\left(e^{-a\|y\|^2}\right)(x),$$

and from Lemma 5.2, we obtain $e^{a\|x\|^2} |{}^tV(f)(x)| \leq C \left\| e^{a\|y\|^2} f \right\|_{k,\infty} < +\infty$, where C is the constant of (24). This completes the proof. □

LEMMA 5.4. *Let $p \in [1, +\infty]$ and f a measurable function on $\mathbb{R}^d$ such that $\left\| e^{a\|x\|^2} f \right\|_{k,p} < +\infty$ for some $a > 0$. Then the function defined on $\mathbb{C}^d$ by*

$$\mathscr{F}_D(f)(z) = \int_{\mathbb{R}^d} f(x) K(x, -iz) \omega_k(x) dx, \tag{26}$$

is well defined and entire on $\mathbb{C}^d$. Moreover there exists a positive constant C such that for all $\xi, \eta \in \mathbb{R}^d$, we have

$$|\mathscr{F}_D(f)(\xi + i\eta)| \leq C e^{\|\eta\|^2/4a}. \tag{27}$$

PROOF. The first assertion follows from the hypothesis on the function f and Hölder's inequality using (5) and the derivation theorem under the integral sign. We will now prove (27).

As the function $f \in L^1_k(\mathbb{R}^d)$, we deduce from (19) that for all $\xi, \eta \in \mathbb{R}^d$, we have

$$\mathscr{F}_D(f)(\xi + i\eta) = \int_{\mathbb{R}^d} {}^tV(f)(x) e^{-i\langle x, \xi + i\eta\rangle} dx.$$

Thus

$$\begin{aligned} |\mathscr{F}_D(f)(\xi + i\eta)| &\leq e^{\|\eta\|^2/4a} \int_{\mathbb{R}^d} e^{a\|x\|^2} \left|{}^tV(f)(x)\right| e^{-a\|x\|^2 + \langle x,\eta\rangle - (\|\eta\|^2/4a)}\, dx, \\ &\leq e^{\|\eta\|^2/4a} \int_{\mathbb{R}^d} e^{a\|x\|^2} \left|{}^tV(f)(x)\right| e^{-a\|x-(\eta/2a)\|^2}\, dx, \end{aligned}$$

and using Hölder's inequality and Lemma 5.3, we obtain

$$|\mathscr{F}_D(f)(\xi + i\eta)| \leq e^{\|\eta\|^2/4a} \left\| e^{a\|x\|^2\,t} V(f) \right\|_p \left(\int_{\mathbb{R}^d} e^{-ap'\|x-(\eta/2a)\|^2}\, dx \right)^{1/p'},$$

where p' is the conjugate exponent of p. Then (27) clearly follows. □

Proof of Theorem 5.1. We will divide the proof in several steps.
Step 1. $ab > 1/4$.
Consider the function h defined on $\mathbb{C}^d$ by

$$h(z) = \left(\prod_{j=1}^d e^{z_j^2/4a} \right) \mathscr{F}_D(f)(z). \tag{28}$$

This function is entire on $\mathbb{C}^d$ and using (27) we obtain

$$|h(\xi + i\eta)| \leq C e^{\|\xi\|^2/4a}, \tag{29}$$

for all $\xi \in \mathbb{R}^d$ and $\eta \in \mathbb{R}^d$. In the following we consider two cases.

(i) If $q < +\infty$, we have

$$\begin{aligned} \|h_{|\mathbb{R}^d}\|^q_{k,q} &= \int_{\mathbb{R}^d} \left| e^{\|y\|^2/4a} \mathscr{F}_D(f)(y) \right|^q \omega_k(y)\, dy, \\ &= \int_{\mathbb{R}^d} \left| e^{b\|y\|^2} \mathscr{F}_D(f)(y) \right|^q e^{q((1/4a)-b)\|y\|^2} \omega_k(y)\, dy. \end{aligned}$$

Using the fact that $ab > 1/4$ and the hypothesis (23), we obtain

$$\|h_{|\mathbb{R}^d}\|_{k,q} \leq \|e^{b\|y\|^2} \mathscr{F}_D(f)\|_{k,q} < +\infty. \tag{30}$$

From relations (29) and (30), it follows from Lemma 4.2 that $h(z) = 0$ for all $z \in \mathbb{C}^d$. Thus $\mathscr{F}_D(f)(y) = 0$ for all $y \in \mathbb{R}^d$. The injectivity of $\mathscr{F}_D$ then implies the result of the theorem in this case.

(ii) Assume $q = +\infty$. As $ab > 1/4$, then from (23) we obtain

$$\|h_{|\mathbb{R}^d}\|_{k,\infty} \leq \|e^{b\|y\|^2}\mathscr{F}_D(f)\|_{k,\infty} < +\infty. \tag{31}$$

From (29), (31) and Lemma 4.1, it follows that there exists a positive constant C such that for all $y \in \mathbb{R}^d$, $h(y) = C$. On the other hand, from (28) we have

$$\mathscr{F}_D(f)(y) = Ce^{-\|y\|^2/4a}, \quad \forall\, y \in \mathbb{R}^d. \tag{32}$$

But the assumption on $\mathscr{F}_D(f)$ is expressed as

$$|\mathscr{F}_D(f)(y)| \leq Me^{-b\|y\|^2} \quad \text{a.e.,} \tag{33}$$

for some constant $M > 0$. The continuity of $\mathscr{F}_D(f)$ on $\mathbb{R}^d$ shows that inequality (33) holds everywhere. Then we must have $Ce^{(b-(1/4a))\|y\|^2} \leq M$ everywhere by (32) and (33). This is impossible since $ab > 1/4$, unless $C = 0$. Thus $\mathscr{F}_D(f)(y) = 0$ everywhere and then $f = 0$ a.e. on $\mathbb{R}^d$.

Step 2. $ab = 1/4$.

(i) If $1 \leq p \leq +\infty$ and $1 \leq q < +\infty$, with the same proof as for the point (i) of the first step, we obtain $f = 0$ a.e. on $\mathbb{R}^d$.

(ii) If $1 \leq p < +\infty$ and $q = +\infty$, we deduce from Lemma 5.3, Corollary 3.2 and (23) that the function ${}^tV(f)$ satisfies

$$\left\|e^{a\|x\|^2}{}^tV(f)\right\|_p < +\infty \quad \text{and} \quad \left\|e^{b\|y\|^2}\mathscr{F}({}^tV(f))\right\|_\infty < +\infty.$$

Then using [6, page 66], we see that ${}^tV(f)(x) = 0$ a.e. on $\mathbb{R}^d$. Thus $\mathscr{F}_D(f)(y) = 0$ for all $y \in \mathbb{R}^d$, which implies that $f = 0$ a.e. and the proof is complete.

Step 3. $ab < 1/4$.

Let $\mathscr{P}$ be the algebra of polynomial functions on $\mathbb{R}^d$. By considering the generalized Hermite polynomials on $\mathbb{R}^d$ studied by Rösler in [12] we deduce that the Dunkl transform of a function $f(x) = P(x)e^{-\delta\|x\|^2}$, where $P \in \mathscr{P}$, is of the form $\mathscr{F}_D(f)(y) = Q(y)e^{-\|y\|^2/4\delta}$ for some $Q \in \mathscr{P}$. These functions clearly satisfy the conditions (23). The proof of Theorem 5.1 is complete. □

6. An analogue of Hardy's theorem

In this section we determine the functions f satisfying (23) in the special case $p = q = +\infty$. The result we obtain, is an analogue for the Dunkl transform of the classical Hardy's theorem.

THEOREM 6.1. *Let f be a measurable function on $\mathbb{R}^d$ such that*

$$(34)\qquad |f(x)| \le Me^{-a\|x\|^2} \quad \text{and} \quad |\mathscr{F}_D(f)(y)| \le Me^{-b\|y\|^2},$$

almost everywhere for $x, y \in \mathbb{R}^d$ and for some constants $a > 0$, $b > 0$ and $M > 0$. Then

(i) *If $ab > 1/4$, we have $f = 0$ a.e.*

(ii) *If $ab = 1/4$, the function f is of the form $f(x) = C_0 e^{-a\|x\|^2}$, for some real constant C_0.*

(iii) *If $ab < 1/4$, there are infinitely many nonzero functions f satisfying (34).*

PROOF. (i) If $ab > 1/4$, the point (ii) of the first step of the proof of Theorem 5.1 gives also the result.

(ii) From (34), Lemma 5.2 and Corollary 3.2, the function ${}^tV(f)$ satisfies

$$|{}^tV(f)(x)| \le CMe^{-a\|x\|^2} \quad \text{and} \quad |\mathscr{F}({}^tV(f))(y)| \le Me^{-a\|y\|^2},$$

for almost all $x, y \in \mathbb{R}^d$, where C is the constant in formula (24). Using Hardy's theorem for the classical Fourier transform (see [15, page 137]) we obtain: ${}^tV(f)(x) = C_1 e^{-a\|x\|^2}$, where C_1 is a real constant. We deduce from (19) that there exists $C_2 \in \mathbb{R}$ such that: $\mathscr{F}_D(f)(y) = C_2 e^{-\|y\|^2/4a}$. Thus by using (25) we have $f(x) = C_0 e^{-a\|x\|^2}$, with C_0 a real constant and the result of point (ii) is proved.

(iii) If $ab < 1/4$, the functions defined in the third step of the proof of Theorem 5.1 clearly satisfy also the conditions (34). This completes the proof of Theorem 6.1. □

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References

[1] N. Bourbaki, *Éléments de Mathématiques. Fascicule XXI. Intégration des mesures. Chapitre 5* (Hermann, Paris, 1967).

[2] M. G. Cowling and J. F. Price, *Generalizations of Heisenberg inequality*, Lecture Notes in Math. 992 (Springer, Berlin, 1983) pp. 443–449.

[3] M. F. E. de Jeu, 'The Dunkl transform', *Invent.Math.* **113** (1993), 147–162.

[4] C. F. Dunkl, 'Integral kernels with reflection group invariance', *Canad. J. Math.* **43** (1991), 1213–1227.

[5] ———, 'Hankel transforms associated to finite reflection groups', *Contemp. Math.* **138** (1992), 123–138.

[6] M. Eguchi, S. Koizumi and K. Kumahara, 'An L^p version of the Hardy theorem for the motion group', *J. Austral. Math. Soc. (Series A)* **68** (2000), 55–67.

[7] L. Gallardo and K. Trimèche, 'Un analogue d'un théorème de Hardy pour la transformation de Dunkl', *C. R. Acad. Sci. Paris, Ser. I* **334** (2002), 849–854.

[8] G. H. Hardy, 'A theorem concerning Fourier transform', *J. London Math. Soc.* **8** (1933), 227–231.

[9] G. J. Heckman, 'An elementary approach to the hypergeometric shift operators of Opdam', *Invent. Math.* **103** (1991), 341–350.

[10] K. Hikami, 'Dunkl operators formalism for quantum many-body problems associated with classical root systems', *J. Phys. Soc. Japan* **65** (1996), 394–401.

[11] L. Lapointe and L. Vinet, 'Exact operator solution of the Calogero-Sutherland model', *Comm. Math. Phys.* **178** (1996), 425–452.

[12] M. Rösler, 'Generalized Hermite polynomials and the heat equation for Dunkl operators', *Comm. Math. Phys.* **192** (1998), 519–542.

[13] ———, 'Positivity of Dunkl's intertwining operator', *Duke Math. J.* **98** (1999), 445–463.

[14] ———, 'An uncertainty principle for the Dunkl transform', *Bull. Austral. Math. Soc.* **59** (1999), 353–360.

[15] A. Sitaram, M. Sundari and S. Thangavalu, 'Uncertainty principles on certain Lie groups', *Proc. Indian Acad. Sci. (Math.Sci.)* **105** (1995), 135–151.

[16] K. Trimèche, 'The Dunkl intertwining operator on spaces of functions and distributions and integral representation of its dual', *Integral Transform. Spec. Funct.* **12** (2001), 349–374.

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