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MODULAR LIE REPRESENTATIONS OF FINITE GROUPS

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Abstract

Let K be a field of prime characteristic p and let G be a finite group with a Sylow p-subgroup of order p. For any finite-dimensional KG-module V and any positive integer n, let $L^n(V)$ denote the nth homogeneous component of the free Lie K-algebra generated by (a basis of) V. Then $L^n(V)$ can be considered as a KG-module, called the nth Lie power of V. The main result of the paper is a formula which describes the module structure of $L^n(V)$ up to isomorphism.

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1. Introduction

Let G be a group and K a field. For any finite-dimensional KG-module V, let $L(V)$ be the free Lie algebra over K freely generated by any K-basis of V. Then $L(V)$ may be regarded as a KG-module on which each element of G acts as a Lie algebra automorphism. Furthermore, each homogeneous component $L^n(V)$ is a finite-dimensional submodule, called the nth Lie power of V.

In this paper we consider the case where K has prime characteristic p and G is a finite group with a Sylow p-subgroup of order p. We give a formula which describes $L^n(V)$ up to isomorphism for every finite-dimensional KG-module V. The formula has a strong resemblance to Brandt's character formula in characteristic zero [4], but the proof is much deeper.

In [6] a similar (but slightly simpler) formula was obtained for the case where G is cyclic of order p. The present paper builds on [6] and earlier papers by the author, Kovács and Stöhr: particularly [9]. The results cover the symmetric group of degree r

with $p \leq r < 2p$ and the general linear group $\mathrm{GL}(2, p)$. These cases were studied in [8, 17, 10], but closed formulae could not be given there except in special cases. We shall examine some of the connections between these papers and the present paper in Section 7 below.

For any group G and any field K, we consider the Green ring (representation ring) R_{KG}. This is the ring formed from isomorphism classes of finite-dimensional KG-modules, with addition and multiplication coming from direct sums and tensor products, respectively. For any finite-dimensional KG-module V we also write V for the corresponding element of R_{KG}. Thus V^n corresponds to the nth tensor power of V, and $L^n(V)$ may also be regarded as an element of R_{KG}.

In [5] it is shown that there exist $\mathbb{Z}$-linear functions $\Phi^1_{KG}, \Phi^2_{KG}, \ldots$ on R_{KG} such that, for every finite-dimensional KG-module V and every positive integer n,

$$L^n(V) = \frac{1}{n} \sum_{d|n} \Phi^d_{KG}(V^{n/d}). \tag{1.1}$$

(The sum on the right-hand side is divisible by n in R_{KG}.) The functions Φ^n_{KG} are called the *Lie resolvents* for G over K. As shown in [5],

$$\Phi^n_{KG}(V) = \sum_{d|n} \mu(n/d)\, d\, L^d(V^{n/d}), \tag{1.2}$$

where μ denotes the Möbius function. Furthermore,

$$\Phi^n_{KG} = \mu(n)\psi^n_S \quad \text{when } \mathrm{char}(K) \nmid n; \tag{1.3}$$

here ψ^n_S denotes the nth Adams operation on R_{KG} formed by means of symmetric powers (see Section 2 below). In particular, Φ^1_{KG} is the identity function.

Let G be any group and let K be a field of prime characteristic p. Define $\mathbb{Z}$-linear functions $\zeta^n_{KG} : R_{KG} \to R_{KG}$ as follows. For n not divisible by p define $\zeta^n_{KG} = \mu(n)\psi^n_S$. In particular, ζ^1_{KG} is the identity function. Define $\zeta^p_{KG} = \Phi^p_{KG}$, that is, $\zeta^p_{KG}(V) = pL^p(V) - V^p$ for every finite-dimensional KG-module V. For $k > 1$, with k even, define

$$\zeta^{p^k}_{KG} = -p^{k-2}\left(\psi^{p^k}_S + \zeta^p_{KG} \circ \psi^{p^{k-1}}_S\right).$$

(Note that functions are written on the left and $\circ$ denotes composition of functions.) For $k > 1$, with k odd, define

$$\zeta^{p^k}_{KG} = -p^{k-3}\left(\psi^{p^k}_S + \zeta^p_{KG} \circ \psi^{p^{k-1}}_S + \zeta^{p^2}_{KG} \circ \psi^{p^{k-2}}_S\right).$$

Finally, for $n = p^k m$, where $p \nmid m$, define $\zeta^n_{KG} = \zeta^{p^k}_{KG} \circ \zeta^m_{KG}$. Thus the functions ζ^n_{KG} are defined in terms of pth Lie powers and Adams operations.

THEOREM 1.1. *Let K be a field of prime characteristic p and let G be a finite group with a Sylow p-subgroup of order at most p. Then, for every finite-dimensional KG-module V,*

$$L^n(V) = \frac{1}{n}\sum_{d|n} \zeta^d_{KG}(V^{n/d}).$$

In other words, the Lie resolvents are given by $\Phi^n_{KG} = \zeta^n_{KG}$ for all n. More can be said in the cases where G is a p'-group and where the Sylow p-subgroup is normal: see the beginning of Section 7 and the last part of Section 6, respectively.

COROLLARY 1.2. *Let K, p, G and V be as in the theorem. Let n be a positive integer, and write $n = p^k m$ where $p \nmid m$. Then $\Phi^n_{KG} = \Phi^{p^k}_{KG} \circ \Phi^m_{KG}$ and*

$$L^n(V) = \frac{1}{p^k}\sum_{i=0}^{k} \Phi^{p^i}_{KG}(L^m(V^{p^{k-i}})).$$

The first statement comes from the fact that $\zeta^n_{KG} = \zeta^{p^k}_{KG} \circ \zeta^m_{KG}$, by definition of ζ^n_{KG}. The second statement then follows by (1.1): we write each divisor d of n as $d = p^i q$, where $0 \le i \le k$ and $q \mid m$, and use the facts that $\Phi^d_{KG} = \Phi^{p^i}_{KG} \circ \Phi^q_{KG}$ and each $\Phi^{p^i}_{KG}$ is linear. Hence the structure of arbitrary Lie powers is determined by the functions $\Phi^{p^k}_{KG}$ and mth Lie powers for integers m not divisible by p. It would be interesting to know if the corollary is true for all groups.

If we wish to use Theorem 1.1 for a particular group G we need to be able to calculate the functions ζ^n_{KG}. Thus we need to be able to find ζ^p_{KG} (or, equivalently, pth Lie powers) and the Adams operations ψ^n_S. In Sections 6 and 7 we discuss how this might be done provided that enough information is available about the group G. The calculation of the ψ^n_S is simplified a little by the fact that these functions are periodic in n, as shown in Section 7. It is clear, however, that there will be significant difficulties in practice except in small special cases such as where the Sylow p-subgroup of G is normal and self-centralizing.

2. Preliminaries

Throughout this section K is any field. We start by considering an arbitrary group G, but in the second half of the section G will be finite.

We have already mentioned the Green ring R_{KG}. This is a free $\mathbb{Z}$-module with a basis consisting of the (isomorphism classes of) finite-dimensional indecomposable KG-modules. We write Γ_{KG} for the Green algebra, defined by $\Gamma_{KG} = \mathbb{C} \otimes_{\mathbb{Z}} R_{KG}$.

Thus Γ_{KG} is a commutative $\mathbb{C}$-algebra. The identity element of Γ_{KG}, denoted of course by 1, is the isomorphism class of the trivial one-dimensional KG-module.

For any extension field $\widehat{K}$ of K there is a ring homomorphism $\iota : R_{KG} \to R_{\widehat{K}G}$ determined by $V \mapsto \widehat{K} \otimes_K V$ for every finite-dimensional KG-module V. It follows from the Noether-Deuring Theorem (see [11, (29.7)]) that ι is an embedding.

If $\theta : A \to B$ is a homomorphism of groups, then every KB-module V can be made into a KA-module by taking the action of each element g of A on V to be the same as the action of $\theta(g)$. Thus θ determines a ring homomorphism $\theta^* : R_{KB} \to R_{KA}$. If θ is surjective then θ^* is an embedding. If A is a subgroup of B and θ is the inclusion map then θ^* is called restriction from B to A and, for $V \in R_{KB}$, we sometimes write $V{\downarrow}_A$ instead of $\theta^*(V)$.

If V is a finite-dimensional KG-module then, for every positive integer n, $L^n(V)$ denotes the nth Lie power of V, as already defined. Similarly, $\bigwedge^n(V)$ denotes the nth exterior power of V, and $S^n(V)$ the nth symmetric power of V. All of these are finite-dimensional KG-modules and may be regarded as elements of R_{KG}. The exterior and symmetric powers may be encoded by their Hilbert series $\bigwedge(V,t)$ and $S(V,t)$. These are the power series in an indeterminate t with coefficients in R_{KG} defined by

$$\bigwedge(V,t) = 1 + \bigwedge^1(V)t + \bigwedge^2(V)t^2 + \cdots,$$
$$S(V,t) = 1 + S^1(V)t + S^2(V)t^2 + \cdots.$$

We shall need to use the two types of Adams operations on R_{KG} defined by means of exterior powers and symmetric powers. Following [5] and [6] we denote these by $\psi^n_\wedge$ and ψ^n_S, respectively. We summarise the basic facts and refer to [5] for further details. In the ring of all symmetric functions in variables $x_1, x_2, \ldots$, the nth power sum may be written as a polynomial in the elementary symmetric functions and as a polynomial in the complete symmetric functions:

$$x_1^n + x_2^n + \cdots = \rho_n(e_1, \ldots, e_n) = \sigma_n(h_1, \ldots, h_n). \tag{2.1}$$

For each positive integer n, $\psi^n_\wedge$ and ψ^n_S are $\mathbb{Z}$-linear functions on R_{KG} such that, for every finite-dimensional KG-module V,

$$\psi^n_\wedge(V) = \rho_n(\textstyle\bigwedge^1(V), \ldots, \bigwedge^n(V)), \qquad \psi^n_S(V) = \sigma_n(S^1(V), \ldots, S^n(V)), \tag{2.2}$$

$$\psi^1_\wedge(V) - \psi^2_\wedge(V)t + \psi^3_\wedge(V)t^2 - \cdots = \frac{d}{dt}\log \textstyle\bigwedge(V,t), \tag{2.3}$$

$$\psi^1_S(V) + \psi^2_S(V)t + \psi^3_S(V)t^2 + \cdots = \frac{d}{dt}\log S(V,t). \tag{2.4}$$

Also, $\psi^n_\wedge = \psi^n_S$ when $\mathrm{char}(K) \nmid n$. Furthermore, the following result was established in [5, Theorem 5.4].

LEMMA 2.1. *Let q and n be positive integers such that q is not divisible by* $\mathrm{char}(K)$. *Then* $\psi^q_\wedge \circ \psi^n_\wedge = \psi^{qn}_\wedge$ *and* $\psi^q_S \circ \psi^n_S = \psi^{qn}_S$.

In Section 1 we described the basic properties of the Lie resolvents Φ^n_{KG}. Like the Adams operations, these are $\mathbb{Z}$-linear functions on R_{KG}. Also, in Section 1, we defined $\mathbb{Z}$-linear functions ζ^n_{KG} on R_{KG} in the case where K has prime characteristic p. We shall establish some elementary properties of these various functions on R_{KG}. Whenever we discuss ζ^n_{KG} we assume implicitly that K has prime characteristic p.

LEMMA 2.2. *Let $\theta : A \to B$ be a homomorphism of groups, yielding the ring homomorphism $\theta^* : R_{KB} \to R_{KA}$. Then, for every positive integer n and every finite-dimensional KB-module V,*

$$L^n(\theta^*(V)) = \theta^*(L^n(V)), \quad \bigwedge^n(\theta^*(V)) = \theta^*(\bigwedge^n(V)), \quad S^n(\theta^*(V)) = \theta^*(S^n(V)).$$

PROOF. This is straightforward. □

LEMMA 2.3. *Let $\theta : A \to B$ be a homomorphism of groups, yielding the ring homomorphism $\theta^* : R_{KB} \to R_{KA}$. Then, for every positive integer n,*

$$\psi^n_\wedge \circ \theta^* = \theta^* \circ \psi^n_\wedge, \qquad \psi^n_S \circ \theta^* = \theta^* \circ \psi^n_S,$$
$$\Phi^n_{KA} \circ \theta^* = \theta^* \circ \Phi^n_{KB}, \qquad \zeta^n_{KA} \circ \theta^* = \theta^* \circ \zeta^n_{KB}.$$

PROOF. The results for $\psi^n_\wedge$, ψ^n_S and Φ^n_{KG} follow from (2.2), (1.2) and Lemma 2.2. The result for ζ^n_{KG} follows from its definition. □

LEMMA 2.4. *Let $\iota : R_{KG} \to R_{\widehat{K}G}$ be the ring embedding associated with an extension field $\widehat{K}$ of K. Then, for every positive integer n and every finite-dimensional KG-module V,*

$$L^n(\iota(V)) = \iota(L^n(V)), \quad \bigwedge^n(\iota(V)) = \iota(\bigwedge^n(V)), \quad S^n(\iota(V)) = \iota(S^n(V)),$$
$$\psi^n_\wedge \circ \iota = \iota \circ \psi^n_\wedge, \quad \psi^n_S \circ \iota = \iota \circ \psi^n_S, \quad \Phi^n_{\widehat{K}G} \circ \iota = \iota \circ \Phi^n_{KG}, \quad \zeta^n_{\widehat{K}G} \circ \iota = \iota \circ \zeta^n_{KG}.$$

PROOF. This is similar to the proof of Lemmas 2.2 and 2.3. □

LEMMA 2.5. *Let V be a finite-dimensional KG-module, and I a one-dimensional KG-module. Then, for every positive integer n,*

$$L^n(IV) = I^n L^n(V), \quad \bigwedge^n(IV) = I^n \bigwedge^n(V), \quad S^n(IV) = I^n S^n(V),$$
$$\psi^n_\wedge(IV) = I^n \psi^n_\wedge(V), \quad \psi^n_S(IV) = I^n \psi^n_S(V), \quad \Phi^n_{KG}(IV) = I^n \Phi^n_{KG}(V),$$
$$\zeta^n_{KG}(IV) = I^n \zeta^n_{KG}(V), \quad \psi^n_\wedge(I) = \psi^n_S(I) = I^n, \quad \Phi^n_{KG}(I) = \zeta^n_{KG}(I) = \mu(n) I^n.$$

PROOF. This is mostly straightforward. For the statement about $\Phi^n_{KG}(I)$, note that $L^d(I^{n/d}) = 0$ for divisors d of n such that $d > 1$. The statement about $\zeta^n_{KG}(I)$ comes easily from its definition, using the results for $\psi^n_S(I)$ and $\Phi^p_{KG}(I)$. □

From now on in this section, assume that G is finite, and write $p = \mathrm{char}(K)$. (We are particularly interested in the case where $p \neq 0$.) Let $\widehat{K}$ be the algebraic closure of K and let $G_{p'}$ be the set of all elements of G of order not divisible by p. Let Δ be the $\mathbb{C}$-algebra consisting of all class functions from $G_{p'}$ to $\mathbb{C}$, that is, functions δ such that $\delta(g) = \delta(g')$ whenever g and g' are elements of $G_{p'}$ which are conjugate in G. Let c be the least common multiple of the orders of the elements of $G_{p'}$, and choose and fix primitive cth roots of unity ξ in $\widehat{K}$ and ω in $\mathbb{C}$. Then, for every finite-dimensional KG-module V we may define the Brauer character of V to be the element $\mathrm{Br}(V)$ of Δ such that if $g \in G_{p'}$ has eigenvalues $\xi^{k_1}, \dots, \xi^{k_r}$ in its action on V then $\mathrm{Br}(V)(g) = \omega^{k_1} + \cdots + \omega^{k_r}$. (See [3, Section 5.3].) Furthermore, we may extend the definition linearly so that $\mathrm{Br}(V)$ is defined for an arbitrary element V of Γ_{KG}. Then $\mathrm{Br} : \Gamma_{KG} \to \Delta$ is a $\mathbb{C}$-algebra homomorphism.

For each positive integer n, define a function $\psi^n_0 : \Delta \to \Delta$ by $\psi^n_0(\delta)(g) = \delta(g^n)$ for all $\delta \in \Delta$ and $g \in G_{p'}$. Clearly ψ^n_0 is an algebra endomorphism of Δ and

$$\psi^m_0 \circ \psi^n_0 = \psi^{mn}_0, \tag{2.5}$$

for all positive integers m and n.

LEMMA 2.6. *Let V be a finite-dimensional KG-module. Then, for all n,*

$$\mathrm{Br}(\psi^n_\wedge(V)) = \psi^n_0(\mathrm{Br}(V)) = \mathrm{Br}(\psi^n_S(V)).$$

PROOF. This is well known: however, for the reader's convenience we sketch a proof. If $g \in G_{p'}$ has eigenvalues $\xi^{k_1}, \dots, \xi^{k_r}$ on V, then, for $i = 1, \dots, n$,

$$\mathrm{Br}(\textstyle\bigwedge^i(V))(g) = e_i(\omega^{k_1}, \dots, \omega^{k_r}), \quad \mathrm{Br}(S^i(V))(g) = h_i(\omega^{k_1}, \dots, \omega^{k_r}).$$

Thus, by (2.2) and (2.1),

$$\begin{aligned}\mathrm{Br}(\psi^n_\wedge(V))(g) &= \rho_n\left(e_1(\omega^{k_1}, \dots, \omega^{k_r}), \dots, e_n(\omega^{k_1}, \dots, \omega^{k_r})\right) \\ &= \omega^{k_1 n} + \cdots + \omega^{k_r n} = \mathrm{Br}(V)(g^n) = \psi^n_0(\mathrm{Br}(V))(g).\end{aligned}$$

This gives the result for $\psi^n_\wedge$. The result for ψ^n_S is similar. □

The following result is Brandt's character formula [4], as generalised to Brauer characters (see, for example, [7, (5.4)] or [17, (2.11)]).

LEMMA 2.7. *Let V be a finite-dimensional KG-module. Then, for all n,*

$$\mathrm{Br}(L^n(V)) = \frac{1}{n}\sum_{d|n} \mu(d)\psi_0^d(\mathrm{Br}(V^{n/d})).$$

We can now calculate the Brauer characters associated with Φ^n_{KG} and ζ^n_{KG}.

LEMMA 2.8. *Let V be a finite-dimensional KG-module. Then, for all n,*

$$\mathrm{Br}(\Phi^n_{KG}(V)) = \mu(n)\psi_0^n(\mathrm{Br}(V)) = \mathrm{Br}(\zeta^n_{KG}(V)).$$

PROOF. By (1.1), $\mathrm{Br}(L^n(V)) = \frac{1}{n}\sum_{d|n}\mathrm{Br}(\Phi^d_{KG}(V^{n/d}))$. Hence, by Lemma 2.7 and induction on n, we have $\mathrm{Br}(\Phi^n_{KG}(V)) = \mu(n)\psi_0^n(\mathrm{Br}(V))$. It remains to prove that $\mathrm{Br}(\zeta^n_{KG}(V)) = \mu(n)\psi_0^n(\mathrm{Br}(V))$ for all n.

If $p \nmid n$ then $\zeta^n_{KG}(V) = \mu(n)\psi_S^n(V)$ and the result follows by Lemma 2.6. Also, $\zeta^p_{KG} = \Phi^p_{KG}$, so the result for ζ^p_{KG} follows from the first part. This implies that $\mathrm{Br}(\zeta^p_{KG}(U)) = -\psi_0^p(\mathrm{Br}(U))$ for all $U \in R_{KG}$.

Suppose that $k > 1$ and k is even. Then, by the definition of $\zeta^{p^k}_{KG}$,

$$\mathrm{Br}(\zeta^{p^k}_{KG}(V)) = -p^{k-2}\,\mathrm{Br}(\psi_S^{p^k}(V)) - p^{k-2}\,\mathrm{Br}(\zeta^p_{KG}(\psi_S^{p^{k-1}}(V))).$$

Hence, by Lemma 2.6 and the result for ζ^p_{KG},

$$\mathrm{Br}(\zeta^{p^k}_{KG}(V)) = -p^{k-2}\psi_0^{p^k}(\mathrm{Br}(V)) + p^{k-2}\psi_0^p(\psi_0^{p^{k-1}}(\mathrm{Br}(V))).$$

Therefore, by (2.5), $\mathrm{Br}(\zeta^{p^k}_{KG}(V)) = 0 = \mu(p^k)\psi_0^{p^k}(\mathrm{Br}(V))$. Thus the result holds for $\zeta^{p^k}_{KG}$. The result for $\zeta^{p^k}_{KG}$ when $k > 1$ and k is odd is proved in a similar way using the results for ζ^p_{KG} and $\zeta^{p^2}_{KG}$.

Now suppose that $n = p^k m$, where $p \nmid m$. Then, by the definition of ζ^n_{KG},

$$\begin{aligned}\mathrm{Br}(\zeta^n_{KG}(V)) &= \mathrm{Br}(\zeta^{p^k}_{KG}(\zeta^m_{KG}(V))) = \mu(p^k)\psi_0^{p^k}(\mathrm{Br}(\zeta^m_{KG}(V)))\\ &= \mu(p^k)\psi_0^{p^k}(\mu(m)\psi_0^m(\mathrm{Br}(V))) = \mu(n)\psi_0^n(\mathrm{Br}(V)).\end{aligned}$$

This is the required result. □

Recall that R_{KG} has a $\mathbb{Z}$-basis consisting of the finite-dimensional indecomposable KG-modules. Let $(R_{KG})_{\mathrm{proj}}$ and $(R_{KG})_{\mathrm{nonp}}$ be the $\mathbb{Z}$-submodules spanned, respectively, by the projective and the non-projective indecomposables. Then, for $V \in R_{KG}$, we can write $V = V_{\mathrm{proj}} + V_{\mathrm{nonp}}$, uniquely, where $V_{\mathrm{proj}} \in (R_{KG})_{\mathrm{proj}}$ and $V_{\mathrm{nonp}} \in (R_{KG})_{\mathrm{nonp}}$.

LEMMA 2.9. *Let $U, V \in R_{KG}$. If $U_{\mathrm{nonp}} = V_{\mathrm{nonp}}$ and $\mathrm{Br}(U) = \mathrm{Br}(V)$ then $U = V$. In particular, if G is a p'-group and $\mathrm{Br}(U) = \mathrm{Br}(V)$ then $U = V$.*

PROOF. The hypotheses yield $\mathrm{Br}(U_{\mathrm{proj}}) = \mathrm{Br}(V_{\mathrm{proj}})$. However, if W and W' are finite-dimensional projective KG-modules such that $\mathrm{Br}(W) = \mathrm{Br}(W')$ then $W \cong W'$ (see [3, Corollary 5.3.6]). Thus $U_{\mathrm{proj}} = V_{\mathrm{proj}}$, and so $U = V$. □

3. Exterior and symmetric powers

Throughout this section, let K be a field of prime characteristic p and let G be a finite group with a normal Sylow p-subgroup of order p. As we shall see, there are certain basic indecomposable KG-modules $J_1, J_2, \ldots, J_p$. The main purpose of this section is to give formulae for the power series $\bigwedge(J_r, t)$ and $S(J_r, t)$. The formula for $\bigwedge(J_r, t)$ is due to Kouwenhoven [15] and was also proved by Hughes and Kemper [14]. The formula for $S(J_r, t)$ is a corollary of a result in [14].

Kouwenhoven's results are primarily concerned with $\mathrm{GL}(2, p)$ and go beyond what is required here. In order to keep the treatment as simple as possible we have therefore chosen to follow [14]. However, we use slightly different notation and we consider right KG-modules instead of left KG-modules. If V is a left KG-module then V becomes a right KG-module by defining $vg = g^{-1}v$ for all $v \in V$, $g \in G$. This gives a one-one correspondence between left and right KG-modules. We shall use this correspondence in order to interpret the results of [14] as results about right KG-modules, noting that the correspondence commutes with taking direct sums, tensor products, exterior powers and symmetric powers.

Let P be the (normal) Sylow p-subgroup of G. Thus P has a complement in G, and G is a semidirect product, $G = HP$, where H is a p'-group. Let $P = \{1, a, \ldots, a^{p-1}\}$. There is a right action of P on the group algebra KP given by multiplication and a right action of H given by $a^i \mapsto h^{-1}a^ih$ for all $h \in H$ and $i = 0, \ldots, p-1$. In this way KP becomes a right KG-module. For $r = 1, \ldots, p$, the rth power of the augmentation ideal is $KP(a-1)^r$, and this is invariant under the action of G. Thus, for $r = 1, \ldots, p$, we obtain a right KG-module J_r defined by $J_r = KP/KP(a-1)^r$. It is easily verified that J_r has dimension r and corresponds to the left module V_r of [14]. (Also, the isomorphism class of J_r does not depend on the choice of complement H.) Furthermore, $J_1 = 1$ in the Green ring R_{KG}.

For each $h \in H$, let $m(h)$ be the element of $\{1, \ldots, p-1\}$ determined by $h^{-1}ah = a^{m(h)}$, and let $m(h)$ also denote the corresponding element of the prime subfield of K. There is then a homomorphism $\alpha : H \to K \setminus \{0\}$ given by $\alpha(h) = m(h)$ for all h. This yields a one-dimensional right KH-module, which we also denote by α. Furthermore, we regard α as a right KG-module, by means of the projection $G \to H$. It is easily verified that this module corresponds to the left KG-module denoted by V_α or α in [14]. In R_{KG}, as in R_{KH}, we have $\alpha^{p-1} = 1$. Indeed, α has multiplicative order q where $q = |H/C_H(P)|$.

As shown by the pullback construction described in [14], there exists a finite p'-group $\widetilde{H}$ and an extension field $\widehat{K}$ of K with homomorphisms $\theta : \widetilde{H} \to H$ and $\beta : \widetilde{H} \to \widehat{K} \setminus \{0\}$ such that θ is surjective and $\beta(h)^2 = \alpha(\theta(h))$ for all $h \in \widetilde{H}$. Let $\widetilde{G}$ be the semidirect product $\widetilde{H}P$ with P normal such that, for all $h \in \widetilde{H}$, the action of h on P by conjugation is given by the action of $\theta(h)$. Thus θ extends to a surjective homomorphism $\theta : \widetilde{G} \to G$ which is the identity on P.

We regard the ring R_{KG} as a subring of $R_{\widehat{K}G}$ by means of the embedding $\iota : R_{KG} \to R_{\widehat{K}G}$ described at the beginning of Section 2. Also, we regard $R_{\widehat{K}G}$ as a subring of $R_{\widehat{K}\widetilde{G}}$ by means of the embedding θ^* obtained from $\theta : \widetilde{G} \to G$, as described in Section 2. Thus R_{KG} is a subring of $R_{\widehat{K}\widetilde{G}}$. It is easily verified that the images under $\theta^* \circ \iota$ of the KG-modules J_r and α are isomorphic to the $\widehat{K}\widetilde{G}$-modules defined in the same way for $\widetilde{G}$ over $\widehat{K}$. Thus there is no conflict of notation. By Lemmas 2.2 and 2.4, the exterior and symmetric powers of J_r in R_{KG} are the same as the exterior and symmetric powers of J_r in $R_{\widehat{K}\widetilde{G}}$. Thus we may use $R_{\widehat{K}\widetilde{G}}$ in order to find expressions for $\bigwedge(J_r, t)$ and $S(J_r, t)$.

We regard $R_{\widehat{K}\widetilde{H}}$ as a subring of $R_{\widehat{K}\widetilde{G}}$ by means of the embedding given by the projection $\widetilde{G} \to \widetilde{H}$. Clearly $\alpha \in R_{\widehat{K}\widetilde{H}}$. The homomorphism $\beta : \widetilde{H} \to \widehat{K} \setminus \{0\}$ yields an element of $R_{\widehat{K}\widetilde{H}}$ which we also denote by β. From the properties of β we see that $\beta^2 = \alpha$. Hence $\beta^{2p-2} = 1$ and β^{-1} exists. Note that if $p = 2$ we have $\alpha = 1$ and $\mathrm{char}(\widehat{K}) = 2$: thus the definition of β gives $\beta = 1$ in this case.

As in [14], but using λ instead of μ to avoid the notation for the Möbius function, we extend $R_{\widehat{K}\widetilde{G}}$ by an element λ satisfying $\lambda^2 - \beta^{-1} J_2 \lambda + 1 = 0$ to form a commutative ring $R_{\widehat{K}\widetilde{G}}[\lambda]$. Note that this is a free $R_{\widehat{K}\widetilde{G}}$-module: $R_{\widehat{K}\widetilde{G}}[\lambda] = R_{\widehat{K}\widetilde{G}} \oplus R_{\widehat{K}\widetilde{G}}\lambda$. Also, λ is invertible in $R_{\widehat{K}\widetilde{G}}[\lambda]$. We shall find expressions for $\bigwedge(J_r, t)$ and $S(J_r, t)$ as elements of the power series ring $R_{\widehat{K}\widetilde{G}}[\lambda][[t]]$.

By [14, Lemma 1.3],

$$J_r = \beta^{r-1} \sum_{j=0}^{r-1} \lambda^{r-1-2j}, \tag{3.1}$$

for $r = 1, \ldots, p$. Also, by [14, Theorem 1.4], $R_{\widehat{K}\widetilde{G}}[\lambda]$ is generated by $R_{\widehat{K}\widetilde{H}}$ and λ, that is, $R_{\widehat{K}\widetilde{G}}[\lambda] = R_{\widehat{K}\widetilde{H}}[\lambda]$. Tensoring with $\mathbb{C}$ we obtain $\Gamma_{\widehat{K}\widetilde{G}}[\lambda] = \Gamma_{\widehat{K}\widetilde{H}}[\lambda]$, where $\Gamma_{\widehat{K}\widetilde{G}} = \mathbb{C} \otimes R_{\widehat{K}\widetilde{G}}$ and $\Gamma_{\widehat{K}\widetilde{H}} = \mathbb{C} \otimes R_{\widehat{K}\widetilde{H}}$.

By [12, (81.90)], the algebra $\Gamma_{\widehat{K}\widetilde{G}}$ is semisimple. Thus it is isomorphic to the direct sum of m copies of $\mathbb{C}$, where m is the number of indecomposable $\widehat{K}\widetilde{G}$-modules. Thus there are exactly m non-zero algebra homomorphisms $\Gamma_{\widehat{K}\widetilde{G}} \to \mathbb{C}$. The restrictions to $R_{\widehat{K}\widetilde{G}}$ of these homomorphisms are called the 'species' of $R_{\widehat{K}\widetilde{G}}$. Note that if $U, V \in R_{\widehat{K}\widetilde{G}}$ and $\phi(U) = \phi(V)$ for every species ϕ then $U = V$.

Let M_{2p}^* denote the subset of $\mathbb{C}$ consisting of all $2p$th roots of unity except for 1 and -1. Thus $\gamma^{2p-2} + \gamma^{2p-4} + \cdots + \gamma^2 + 1 = 0$ for all $\gamma \in M_{2p}^*$. By the proof of [14, Theorem 1.6], for each $\gamma \in \{\beta, \beta^{-1}\} \cup M_{2p}^*$ there is a $\mathbb{C}$-algebra homomorphism $\phi_\gamma : \Gamma_{\widehat{K}\widetilde{G}}[\lambda] \to \Gamma_{\widehat{K}\widetilde{H}}$ given by $\phi_\gamma(\chi) = \chi$ for all $\chi \in \Gamma_{\widehat{K}\widetilde{H}}$ and $\phi_\gamma(\lambda) = \gamma$. Also, for each $h \in \widetilde{H}$ there is a $\mathbb{C}$-algebra homomorphism $\varepsilon_h : \Gamma_{\widehat{K}\widetilde{H}} \to \mathbb{C}$ such that, for all $\chi \in \Gamma_{\widehat{K}\widetilde{H}}$, $\varepsilon_h(\chi)$ is the value at h of the Brauer character of χ, that is, $\varepsilon_h(\chi) = \mathrm{Br}(\chi)(h)$. For $\gamma \in \{\beta, \beta^{-1}\} \cup M_{2p}^*$ and $h \in \widetilde{H}$, let $\phi_{h,\gamma} = \varepsilon_h \circ \phi_\gamma$. Thus

$\phi_{h,\gamma}$ is a $\mathbb{C}$-algebra homomorphism $\phi_{h,\gamma} : \Gamma_{\widehat{K}\widetilde{G}}[\lambda] \to \mathbb{C}$. The following result is [14, Theorem 1.6], apart from minor notational differences.

LEMMA 3.1. *For each $\gamma \in \{\beta, \beta^{-1}\} \cup M_{2p}^*$ and each $h \in \widetilde{H}$, the restriction of $\phi_{h,\gamma}$ to $R_{\widehat{K}\widetilde{G}}$ is a species of $R_{\widehat{K}\widetilde{G}}$. The homomorphisms $\phi_{h,\gamma}$ and $\phi_{h',\gamma'}$ restrict to the same species if and only if h and h' are conjugate in $\widetilde{H}$ and $\gamma' \in \{\gamma, \gamma^{-1}\}$. Every species of $R_{\widehat{K}\widetilde{G}}$ arises as the restriction of some $\phi_{h,\gamma}$.*

In particular, $\phi_{h,\beta}$ gives the same species as $\phi_{h,\beta^{-1}}$. Since elements of $R_{\widehat{K}\widetilde{G}}$ are determined by their images under the species, we obtain the following result.

COROLLARY 3.2. *Let $U, V \in R_{\widehat{K}\widetilde{G}}$. If $\phi_{h,\gamma}(U) = \phi_{h,\gamma}(V)$ for all $\gamma \in \{\beta\} \cup M_{2p}^*$ and all $h \in \widetilde{H}$, or if $\phi_\gamma(U) = \phi_\gamma(V)$ for all $\gamma \in \{\beta\} \cup M_{2p}^*$, then $U = V$.*

The description of $\bigwedge(J_r, t)$ is as follows.

THEOREM 3.3 ([15, Lemma, page 1709]; [14, Theorem 1.10]). *For $r = 1, \ldots, p$,*

$$\bigwedge(J_r, t) = \prod_{j=0}^{r-1}(1 + \beta^{r-1}\lambda^{r-1-2j}t).$$

We write $W = J_p - \alpha J_{p-1}$ and $\bar{\alpha} = 1 + \alpha + \cdots + \alpha^{p-2}$, recalling that $\alpha^{p-1} = 1$. By direct calculation from (3.1) we get the following result.

LEMMA 3.4. *For the homomorphisms ϕ_β and ϕ_γ, where $\gamma \in M_{2p}^*$, we have*

$$\begin{aligned} &\phi_\beta(J_p) = 1 + \bar{\alpha}, && \phi_\beta(J_{p-1}) = \bar{\alpha}, && \phi_\beta(W) = 1, \\ &\phi_\gamma(J_p) = 0, && \phi_\gamma(J_{p-1}) = -\gamma^p\beta^{p-2}, && \phi_\gamma(W) = \gamma^p\beta^p. \end{aligned}$$

For $r = 1, \ldots, p$, write

$$X_r = (1 - W^{r-1}t^p)(1 - t^p)^{-1}(1 - {\textstyle\bigwedge}^1(J_r)t + {\textstyle\bigwedge}^2(J_r)t^2 - \cdots)^{-1}.$$

Thus, by Theorem 3.3,

$$X_r = (1 - W^{r-1}t^p)(1 - t^p)^{-1}\prod_{j=0}^{r-1}(1 - \beta^{r-1}\lambda^{r-1-2j}t)^{-1}.$$

Let the homomorphisms ϕ_β and ϕ_γ act on $\Gamma_{\widehat{K}\widetilde{G}}[\lambda][[t]]$ by action on coefficients. Then it is easily verified that $\phi_\beta(X_r) = \prod_{j=0}^{r-1}(1 - \alpha^j t)^{-1}$ and, for $\gamma \in M_{2p}^*$,

$$\phi_\gamma(X_r) = (1 - \beta^{p(r-1)}\gamma^{p(r-1)}t^p)(1 - t^p)^{-1}\prod_{j=0}^{r-1}(1 - \beta^{r-1}\gamma^{r-1-2j}t)^{-1}.$$

Replacing α by $\mathrm{Br}(\alpha)(h)$ and β by $\mathrm{Br}(\beta)(h)$, for $h \in \widetilde{H}$, we obtain expressions for $\phi_{h,\beta}(X_r)$ and $\phi_{h,\gamma}(X_r)$. Comparison with [14, Proposition 1.13] shows that $\phi_{h,\beta}(X_r) = \phi_{h,\beta}(S(J_r,t))$ and $\phi_{h,\gamma}(X_r) = \phi_{h,\gamma}(S(J_r,t))$. Therefore, by Corollary 3.2, $X_r = S(J_r,t)$. Thus we have the following result.

THEOREM 3.5 (based on [14, Proposition 1.13]). *For* $r = 1, \ldots, p$,

$$S(J_r,t) = (1-(J_p - \alpha J_{p-1})^{r-1}t^p)(1-t^p)^{-1}\bigwedge(J_r,-t)^{-1}$$
$$= (1-(J_p - \alpha J_{p-1})^{r-1}t^p)(1-t^p)^{-1}\prod_{j=0}^{r-1}(1-\beta^{r-1}\lambda^{r-1-2j}t)^{-1}.$$

4. Adams operations

We continue to use all the notation of Section 3. In particular, G is a finite group with a normal Sylow p-subgroup of order p. We shall find expressions for the elements $\psi^n_\wedge(J_r)$ and $\psi^n_S(J_r)$ of R_{KG}. By Lemmas 2.3 and 2.4, it suffices to find such expressions within $R_{\widehat{K}\widetilde{G}}$. Recall that $\alpha^{p-1} = 1$ and $\beta^2 = \alpha$, so that $\beta^{2p-2} = 1$. For $r \in \{1, \ldots, p\}$, we write $\alpha_r = 1 + \alpha + \cdots + \alpha^{r-1}$. Of particular importance is α_{p-1}, which we also denote by $\bar{\alpha}$, as in Lemma 3.4 above. For each non-negative integer i, we have $\alpha^i\bar{\alpha} = \bar{\alpha}$. Thus $\alpha_r\bar{\alpha} = r\bar{\alpha}$. The identity element of $R_{\widehat{K}\widetilde{G}}[\lambda]$ is denoted by 1 or J_1, as convenient. As in Section 3, let $W = J_p - \alpha J_{p-1}$.

LEMMA 4.1. *For every non-negative integer* n,

$$W^n = \begin{cases} -\beta^{n+1}J_{p-1} + J_p & \text{if } n \text{ is odd;} \\ \beta^n J_1 + (1-\beta^n)J_p & \text{if } n \text{ is even.} \end{cases}$$

PROOF. We use the homomorphisms ϕ_β and ϕ_γ, for $\gamma \in M^*_{2p}$, as defined in Section 3. Note that these homomorphisms fix α and β. Suppose that n is odd. Then, by Lemma 3.4, we find $\phi_\beta(W^n) = 1 = \phi_\beta(-\beta^{n+1}J_{p-1} + J_p)$ and

$$\phi_\gamma(W^n) = \gamma^p\beta^{n+p-1} = \phi_\gamma(-\beta^{n+1}J_{p-1} + J_p).$$

Thus, by Corollary 3.2, $W^n = -\beta^{n+1}J_{p-1} + J_p$. The proof for even n is similar. □

By Theorem 3.3 and (2.3),

$$\psi^1_\wedge(J_r) - \psi^2_\wedge(J_r)t + \psi^3_\wedge(J_r)t^2 - \cdots = \sum_{j=0}^{r-1}\beta^{r-1}\lambda^{r-1-2j}(1+\beta^{r-1}\lambda^{r-1-2j}t)^{-1}.$$

Hence, as stated in [15, page 1720],

$$\psi_{\wedge}^{n}(J_r) = \beta^{(r-1)n} \sum_{j=0}^{r-1} \lambda^{(r-1-2j)n} \quad \text{for all } r \text{ and } n. \tag{4.1}$$

THEOREM 4.2. *Let k be a positive integer and let $r \in \{1, \ldots, p\}$. If r is odd,*

$$\psi_{\wedge}^{p^k}(J_r) = r\beta^{r-1}(J_1 - J_p) + \alpha_r J_p.$$

If $p = 2$,

$$\psi_{\wedge}^{2^k}(J_2) = \begin{cases} 2(J_2 - J_1) & \text{if } k = 1; \\ 2J_1 & \text{if } k \geq 2. \end{cases}$$

If p is odd and r is even,

$$\psi_{\wedge}^{p^k}(J_r) = \begin{cases} -r\beta^{r} J_{p-1} + \alpha_r J_p & \text{if } k \text{ is odd}; \\ -r\beta^{r+p-1} J_{p-1} + \alpha_r J_p & \text{if } k \text{ is even}. \end{cases}$$

PROOF. We assume that p is odd, noting that the proof for $p = 2$ is similar but much easier. Suppose first that r is odd. By (4.1),

$$\phi_\beta(\psi_{\wedge}^{p^k}(J_r)) = \beta^{(r-1)p^k} \sum_{j=0}^{r-1} \beta^{(r-1-2j)p^k} = \sum_{j=0}^{r-1} \alpha^{(r-1-j)p^k} = \sum_{j=0}^{r-1} \alpha^{r-1-j} = \alpha_r.$$

Also, by Lemma 3.4,

$$\phi_\beta(r\beta^{r-1}(J_1 - J_p) + \alpha_r J_p) = -r\beta^{r-1}\bar{\alpha} + \alpha_r(1 + \bar{\alpha}) = -r\bar{\alpha} + \alpha_r + r\bar{\alpha} = \alpha_r.$$

For $\gamma \in M_{2p}^*$, (4.1) gives

$$\phi_\gamma(\psi_{\wedge}^{p^k}(J_r)) = \beta^{(r-1)p^k} \sum_{j=0}^{r-1} \gamma^{(r-1-2j)p^k} = r\beta^{(r-1)p^k} = r\beta^{r-1}.$$

Also, by Lemma 3.4, $\phi_\gamma(r\beta^{r-1}(J_1 - J_p) + \alpha_r J_p) = r\beta^{r-1}$. Thus, for r odd, the result follows by Corollary 3.2.

Now suppose that r is even. Note that $r + p - p^k \equiv r \pmod{2p-2}$ if k is odd, and $r + p - p^k \equiv r + p - 1 \pmod{2p-2}$ if k is even. Thus it suffices to show that

$$\psi_{\wedge}^{p^k}(J_r) = -r\beta^{r+p-p^k} J_{p-1} + \alpha_r J_p.$$

By (4.1), $\phi_\beta(\psi_{\wedge}^{p^k}(J_r)) = \alpha_r$, just as for r odd. Also, by Lemma 3.4,

$$\phi_\beta(-r\beta^{r+p-p^k} J_{p-1} + \alpha_r J_p) = -r\bar{\alpha} + \alpha_r(1 + \bar{\alpha}) = -r\bar{\alpha} + \alpha_r + r\bar{\alpha} = \alpha_r.$$

For $\gamma \in M_{2p}^*$, (4.1) gives

$$\phi_\gamma(\psi_\wedge^{p^k}(J_r)) = \beta^{(r-1)p^k} \sum_{j=0}^{r-1} \gamma^{(r-1-2j)p^k} = r\beta^{(r-1)p^k}\gamma^{p^k} = r\beta^{r-p^k}\gamma^p.$$

Also, by Lemma 3.4,

$$\phi_\gamma(-r\beta^{r+p-p^k} J_{p-1} + \alpha_r J_p) = r\beta^{r+p-p^k}\gamma^p\beta^{p-2} = r\beta^{r-p^k}\gamma^p.$$

Thus the result again follows by Corollary 3.2. □

LEMMA 4.3. *Let n be a positive integer and $r \in \{1, \ldots, p\}$. Then*

$$\psi_S^n(J_r) - \psi_\wedge^n(J_r) = \begin{cases} 0 & \text{if } n \not\equiv 0 \pmod p; \\ p(J_1 - W^{(r-1)n/p}) & \text{if } n \equiv 0 \pmod p. \end{cases}$$

PROOF. By (2.4) and Theorem 3.5,

$$\begin{aligned} &\psi_S^1(J_r) + \psi_S^2(J_r)t + \cdots \\ &\quad = \frac{d}{dt}\log(1 - W^{r-1}t^p) - \frac{d}{dt}\log(1 - t^p) - \frac{d}{dt}\log \bigwedge(J_r, -t). \end{aligned}$$

Hence, by (2.3) and multiplication by t,

$$\begin{aligned} &(\psi_S^1(J_r) - \psi_\wedge^1(J_r))t + (\psi_S^2(J_r) - \psi_\wedge^2(J_r))t^2 + \cdots \\ &\quad = -pW^{r-1}t^p(1 - W^{r-1}t^p)^{-1} + pt^p(1 - t^p)^{-1}. \end{aligned}$$

The result follows by comparing coefficients. □

THEOREM 4.4. *Let k be a positive integer and let $r \in \{1, \ldots, p\}$. If r is odd,*

$$\psi_S^{p^k}(J_r) = (p - (p - r)\beta^{r-1})(J_1 - J_p) + \alpha_r J_p.$$

If r is even,

$$\psi_S^{p^k}(J_r) = \begin{cases} p(J_1 - J_p) + (p - r)\beta^r J_{p-1} + \alpha_r J_p & \text{if } k \text{ is odd;} \\ p(J_1 - J_p) + (p - r)\beta^{r+p-1} J_{p-1} + \alpha_r J_p & \text{if } k \text{ is even.} \end{cases}$$

PROOF. This holds for both p odd and $p = 2$. It follows by straightforward calculations from Lemma 4.3, Theorem 4.2 and Lemma 4.1. □

LEMMA 4.5. *For all k, i and r, $\psi_S^{p^k}(\alpha^i J_r) = \alpha^i \psi_S^{p^k}(J_r)$.*

PROOF. This follows from Lemma 2.5, since $\alpha^{ip^k} = \alpha^i$. □

The following lemma is proved by direct calculation from Theorem 4.4 and Lemma 4.5, using the linearity of ψ_S^p and $\psi_S^{p^2}$.

LEMMA 4.6. *Let $r \in \{1, \ldots, p\}$. If r is odd,*

$$(\psi_S^p \circ \psi_S^p)(J_r) = (\psi_S^{p^2} \circ \psi_S^{p^2})(J_r) = (\psi_S^{p^2} \circ \psi_S^p)(J_r) = (\psi_S^p \circ \psi_S^{p^2})(J_r)$$
$$= (p - p^2 + (p-1)(p-r)\beta^{r-1} + p\alpha_r)(J_1 - J_p) + \alpha_r J_p.$$

If r is even,

$$(\psi_S^p \circ \psi_S^p)(J_r) = p(1 - p + (p-r)\beta^r + \alpha_r)(J_1 - J_p) + (p-r)\beta^{r+p-1} J_{p-1} + \alpha_r J_p,$$
$$(\psi_S^{p^2} \circ \psi_S^{p^2})(J_r) = p(1 - p + (p-r)\beta^{r+p-1} + \alpha_r)(J_1 - J_p) + (p-r)\beta^{r+p-1} J_{p-1} + \alpha_r J_p,$$
$$(\psi_S^{p^2} \circ \psi_S^p)(J_r) = p(1 - p + (p-r)\beta^r + \alpha_r)(J_1 - J_p) + (p-r)\beta^r J_{p-1} + \alpha_r J_p,$$
$$(\psi_S^p \circ \psi_S^{p^2})(J_r) = p(1 - p + (p-r)\beta^{r+p-1} + \alpha_r)(J_1 - J_p) + (p-r)\beta^r J_{p-1} + \alpha_r J_p.$$

The remaining lemma of this section follows easily from Theorem 4.4 and Lemma 4.6. It is required for the calculations in Section 5.

LEMMA 4.7. *Let $r \in \{1, \ldots, p\}$. If r is odd,*

$$(-\psi_S^p + \psi_S^{p^2} \circ \psi_S^p + p\psi_S^{p^2})(J_r) = (-\psi_S^{p^2} + \psi_S^p \circ \psi_S^p + p\psi_S^p)(J_r) = p\alpha_r J_1,$$
$$(-\psi_S^{p^2} + \psi_S^{p^2} \circ \psi_S^{p^2} + p\psi_S^p)(J_r) = (-\psi_S^p + \psi_S^p \circ \psi_S^{p^2} + p\psi_S^{p^2})(J_r) = p\alpha_r J_1.$$

If r is even,

$$(-\psi_S^p + \psi_S^{p^2} \circ \psi_S^p + p\psi_S^{p^2})(J_r) = p(p-r)\beta^r (J_1 + \beta^{p-1} J_{p-1} - J_p) + p\alpha_r J_1,$$
$$(-\psi_S^{p^2} + \psi_S^p \circ \psi_S^p + p\psi_S^p)(J_r) = p(p-r)\beta^r (J_1 + J_{p-1} - J_p) + p\alpha_r J_1,$$
$$(-\psi_S^{p^2} + \psi_S^{p^2} \circ \psi_S^{p^2} + p\psi_S^p)(J_r) = p(p-r)\beta^{r+p-1} (J_1 + \beta^{p-1} J_{p-1} - J_p) + p\alpha_r J_1,$$
$$(-\psi_S^p + \psi_S^p \circ \psi_S^{p^2} + p\psi_S^{p^2})(J_r) = p(p-r)\beta^{r+p-1} (J_1 + J_{p-1} - J_p) + p\alpha_r J_1.$$

5. The key special case

Let K be a field of prime characteristic p, and let Q be a group of order $p(p-1)$ generated by elements a and b with relations $a^p = 1$, $b^{p-1} = 1$ and $b^{-1}ab = a^l$, where l is a positive integer such that the image of l in K has multiplicative order $p-1$. In other words, Q is isomorphic to the holomorph of a group of order p. In this section we shall prove Theorem 1.1 for Q by proving the following result.

THEOREM 5.1. *Let K be a field of prime characteristic p and let Q be isomorphic to the holomorph of a group of order p. Then $\Phi^n_{KQ} = \zeta^n_{KQ}$ for all n.*

The KQ-modules $J_1, \ldots, J_p$ and α are defined as in Section 3. When convenient we also use β such that $\beta^2 = \alpha$, as in Section 3. There are, up to isomorphism, precisely $p(p-1)$ indecomposable KQ-modules. In [6, Section 4] these were denoted by $J_{i,r}$, for $i = 0, \ldots, p-2$ and $r = 1, \ldots, p$, and further details can be found there. It is easily checked that, in the notation of the present paper, $J_{i,r} = \alpha^i J_r$.

By [6, Theorem 4.4] with $i = 0$, combined with [6, Lemma 4.1], we have

$$\sum_{d|n} (\Phi^d_{KQ} \circ \psi^{n/d}_S)(J_r) = \begin{cases} J_r & \text{for } n = 1; \\ -p(J_p - \alpha J_{p-1} - J_1) & \text{for } n = p; \\ 0 & \text{for } n \neq 1, p, \end{cases} \tag{5.1}$$

for $r = 2, \ldots, p$. Also, by Lemma 2.5,

$$\Phi^n_{KQ}(J_1) = \mu(n) J_1, \qquad \text{for all } n, \quad \text{and} \tag{5.2}$$

$$\Phi^n_{KQ}(\alpha^i J_r) = \alpha^{ni} \Phi^n_{KQ}(J_r), \qquad \text{for all } n, i \text{ and } r. \tag{5.3}$$

Equations (5.2)–(5.3) yield $\Phi^n_{KQ}(\alpha^i J_1)$ for all n and all i. For $r \geq 2$, (5.1) and (5.3) yield $\Phi^n_{KQ}(\alpha^i J_r)$ in terms of Adams operations and values of the functions Φ^d_{KQ} for proper divisors d of n. Thus $\Phi^1_{KQ}, \Phi^2_{KQ}, \ldots$ are the unique linear functions on R_{KQ} satisfying (5.1)–(5.3).

LEMMA 5.2. *If $n = p^k m$ where $p \nmid m$, then $\Phi^n_{KQ} = \Phi^{p^k}_{KQ} \circ \mu(m)\psi^m_S$.*

PROOF. By [6, Theorem 4.4, Lemma 4.6 and Lemma 5.1 (ii)], we have $\Phi^n_{KQ} = \Phi^{p^k}_{KQ} \circ \Phi^m_{KQ}$. The result follows by (1.3). □

By (5.1) with $n = p$, $\psi^p_S(J_r) + \Phi^p_{KQ}(J_r) = -p(J_p - \alpha J_{p-1} - J_1)$, for all $r \geq 2$. However, $\zeta^p_{KQ} = \Phi^p_{KQ}$, by the definition of ζ^p_{KQ}. Thus, for all $r \geq 2$,

$$\zeta^p_{KQ}(J_r) = pJ_1 + p\alpha J_{p-1} - pJ_p - \psi^p_S(J_r). \tag{5.4}$$

Also, by Lemma 2.5,

$$\zeta^p_{KQ}(J_1) = -J_1. \tag{5.5}$$

From the definition of ζ^n_{KQ}, if $n = p^k m$ where $p \nmid m$, then

$$\zeta^n_{KQ} = \zeta^{p^k}_{KQ} \circ \mu(m)\psi^m_S. \tag{5.6}$$

The following result is easily obtained from (5.5), (5.4) and Theorem 4.4. (Recall that $\beta^2 = \alpha$ and $\bar{\alpha} = 1 + \alpha + \cdots + \alpha^{p-2}$.)

LEMMA 5.3. *We have $\zeta^p_{KQ}(J_1) = -J_1$ and $\zeta^p_{KQ}(J_p) = p\alpha J_{p-1} - (1+\bar{\alpha})J_p$. Also, for p odd, $\zeta^p_{KQ}(J_{p-1}) = (p\alpha - \beta^{p-1})J_{p-1} - \bar{\alpha}J_p$.*

Since R_{KQ} is spanned by the modules $\alpha^i J_r$, Theorem 4.4 and Lemma 4.5 give

$$\psi^p_S = \psi^{p^3}_S = \psi^{p^5}_S = \cdots \quad \text{and} \quad \psi^{p^2}_S = \psi^{p^4}_S = \psi^{p^6}_S = \cdots \quad \text{on } R_{KQ}. \tag{5.7}$$

LEMMA 5.4. *Let m be a positive integer, where $m \geq 3$. Then*

$$-\psi^{p^m}_S + \psi^{p^2}_S \circ \psi^{p^{m-2}}_S + p\psi^{p^{m-1}}_S + \zeta^p_{KQ} \circ \left(-\psi^{p^{m-1}}_S + \psi^p_S \circ \psi^{p^{m-2}}_S + p\psi^{p^{m-2}}_S\right) = 0.$$

PROOF. Let χ and χ' be the linear functions on R_{KQ} defined by

$$\chi = -\psi^p_S + \psi^{p^2}_S \circ \psi^p_S + p\psi^{p^2}_S + \zeta^p_{KQ} \circ (-\psi^{p^2}_S + \psi^p_S \circ \psi^p_S + p\psi^p_S),$$
$$\chi' = -\psi^{p^2}_S + \psi^{p^2}_S \circ \psi^{p^2}_S + p\psi^p_S + \zeta^p_{KQ} \circ (-\psi^p_S + \psi^p_S \circ \psi^{p^2}_S + p\psi^{p^2}_S).$$

By (5.7), it suffices to prove that $\chi = \chi' = 0$. By Lemma 4.5, $\psi^{p^k}_S(\alpha^i J_r) = \alpha^i \psi^{p^k}_S(J_r)$ for all k, i and r. Similarly, by Lemma 2.5, $\zeta^p_{KQ}(\alpha^i J_r) = \alpha^i \zeta^p_{KQ}(J_r)$. Hence it suffices to show that $\chi(J_r) = \chi'(J_r) = 0$ for all r. This follows by direct calculation from Lemmas 4.7 and 5.3. □

COROLLARY 5.5. *For all $k \geq 3$, $\zeta^{p^k}_{KQ} = p\zeta^{p^{k-1}}_{KQ}$.*

PROOF. By (5.7) and the definition of $\zeta^{p^k}_{KQ}$, we have $\zeta^{p^k}_{KQ} = p^2\zeta^{p^{k-2}}_{KQ}$ for all $k \geq 4$. Thus it suffices to prove that $\zeta^{p^3}_{KQ} = p\zeta^{p^2}_{KQ}$. However,

$$\begin{aligned}
\zeta^{p^3}_{KQ} - p\zeta^{p^2}_{KQ} &= -\psi^{p^3}_S - \zeta^p_{KQ} \circ \psi^{p^2}_S - \zeta^{p^2}_{KQ} \circ \psi^p_S - p\zeta^{p^2}_{KQ} \\
&= -\psi^{p^3}_S - \zeta^p_{KQ} \circ \psi^{p^2}_S + (\psi^{p^2}_S + \zeta^p_{KQ} \circ \psi^p_S) \circ \psi^p_S + p(\psi^{p^2}_S + \zeta^p_{KQ} \circ \psi^p_S) \\
&= -\psi^{p^3}_S + \psi^{p^2}_S \circ \psi^p_S + p\psi^{p^2}_S + \zeta^p_{KQ} \circ (-\psi^{p^2}_S + \psi^p_S \circ \psi^p_S + p\psi^p_S).
\end{aligned}$$

This is equal to 0, by Lemma 5.4. Therefore $\zeta^{p^3}_{KQ} = p\zeta^{p^2}_{KQ}$. □

LEMMA 5.6. *For $k \geq 2$, $\sum_{j=0}^k \zeta^{p^j}_{KQ} \circ \psi^{p^{k-j}}_S = 0$.*

PROOF. For $k = 2$, the result follows from the definition of $\zeta_{KQ}^{p^2}$. Suppose that $m \geq 3$ and that the result holds for $k = m - 1$. Then, by Corollary 5.5,

$$\begin{aligned}\sum_{j=0}^{m} \zeta_{KQ}^{p^j} \circ \psi_S^{p^{m-j}} &= \psi_S^{p^m} + \zeta_{KQ}^{p} \circ \psi_S^{p^{m-1}} + \zeta_{KQ}^{p^2} \circ \psi_S^{p^{m-2}} + \sum_{j=3}^{m} \zeta_{KQ}^{p^j} \circ \psi_S^{p^{m-j}} \\ &= \psi_S^{p^m} + \zeta_{KQ}^{p} \circ \psi_S^{p^{m-1}} + \zeta_{KQ}^{p^2} \circ \psi_S^{p^{m-2}} + p \sum_{j=2}^{m-1} \zeta_{KQ}^{p^j} \circ \psi_S^{p^{m-1-j}} \\ &= \psi_S^{p^m} + \zeta_{KQ}^{p} \circ \psi_S^{p^{m-1}} + \zeta_{KQ}^{p^2} \circ \psi_S^{p^{m-2}} - p(\psi_S^{p^{m-1}} + \zeta_{KQ}^{p} \circ \psi_S^{p^{m-2}}).\end{aligned}$$

By definition, $\zeta_{KQ}^{p^2} = -(\psi_S^{p^2} + \zeta_{KQ}^{p} \circ \psi_S^{p})$. Therefore $\sum_{j=0}^{m} \zeta_{KQ}^{p^j} \circ \psi_S^{p^{m-j}}$ is equal to

$$-\left(-\psi_S^{p^m} + \psi_S^{p^2} \circ \psi_S^{p^{m-2}} + p\psi_S^{p^{m-1}} + \zeta_{KQ}^{p} \circ \left(-\psi_S^{p^{m-1}} + \psi_S^{p} \circ \psi_S^{p^{m-2}} + p\psi_S^{p^{m-2}}\right)\right).$$

This is equal to 0, by Lemma 5.4. Hence the result holds for $k = m$. By induction, the result holds for all $k \geq 2$. □

PROOF OF THEOREM 5.1. We need to prove that $\Phi_{KQ}^{n} = \zeta_{KQ}^{n}$ for all n. By (5.6) and Lemma 5.2, it suffices to prove that $\Phi_{KQ}^{p^k} = \zeta_{KQ}^{p^k}$ for all $k \geq 0$. We consider (5.1)–(5.3) restricted to values of n which are powers of p. These equations uniquely determine the linear functions $\Phi_{KQ}^{1}, \Phi_{KQ}^{p}, \Phi_{KQ}^{p^2}, \ldots$. Hence it suffices to show that the functions $\zeta_{KQ}^{1}, \zeta_{KQ}^{p}, \zeta_{KQ}^{p^2}, \ldots$ satisfy the same equations. Equations (5.2) and (5.3) for the $\zeta_{KQ}^{p^k}$ are given by Lemma 2.5. This leaves (5.1). For $n = 1$ the required result is clear. For $n = p$ it is given by (5.4). Finally, for $n = p^k$ with $k \geq 2$, the result is given by Lemma 5.6. □

6. Normal Sylow subgroup

In this section we prove Theorem 1.1 for the case in which the Sylow p-subgroup of G has order p and is normal. It suffices to prove the following result.

THEOREM 6.1. *Let K be a field of prime characteristic p and let G be a finite group with a normal Sylow p-subgroup of order p. Then $\Phi_{KG}^{n} = \zeta_{KG}^{n}$ for all n.*

We use the notation of Section 3. In particular, $G = HP$, where P is the Sylow p-subgroup of G and H is a p'-group. We consider the KG-modules $J_1, \ldots, J_p$ and α. When convenient we also use $\widehat{K}$, $\widetilde{G}$, β and λ, as in Section 3.

LEMMA 6.2. *The isomorphism classes of finite-dimensional indecomposable KG-modules are represented by the modules $I \otimes J_r$, where $1 \leq r \leq p$ and I ranges over* *a set of representatives of the isomorphism classes of irreducible KH-modules, these being regarded as KG-modules through the projection $G \to H$.*

PROOF. This is given by [14, Proposition 1.1], where it is not necessary to assume that the field is a splitting field. See also [16, Proposition 4.4]. □

LEMMA 6.3. *Let U and V be elements of R_{KG} such that $U\downarrow_{H_0P} = V\downarrow_{H_0P}$ for every cyclic subgroup H_0 of H. Then $U = V$.*

PROOF. This is given by [16, Corollary 4.4]. It can be obtained by applying Lemma 6.2 to G and to the subgroups H_0P. □

LEMMA 6.4. *Let U be a finite-dimensional KH-module, regarded as a KG-module. Then, for $r = 1, \ldots, p$ and every positive integer n,*

$$\psi^n_\wedge(UJ_r) = \psi^n_\wedge(U)\psi^n_\wedge(J_r), \qquad \psi^n_S(UJ_r) = \psi^n_S(U)\psi^n_S(J_r),$$
$$\Phi^n_{KG}(UJ_r) = \psi^n_\wedge(U)\Phi^n_{KG}(J_r), \qquad \zeta^n_{KG}(UJ_r) = \psi^n_\wedge(U)\zeta^n_{KG}(J_r).$$

PROOF. By Lemma 2.4, we may assume that K is algebraically closed. By Lemmas 6.3 and 2.3 it suffices to prove the corresponding results for the subgroups H_0P, where H_0 is a cyclic subgroup of H. Thus we may assume that H is cyclic. Therefore U is isomorphic to the direct sum of one-dimensional modules, and it suffices to consider the case where U is one-dimensional. Let ψ^n denote either $\psi^n_\wedge$, ψ^n_S, Φ^n_{KG} or ζ^n_{KG}. Thus, by Lemma 2.5, $\psi^n(UJ_r) = U^n\psi^n(J_r)$ and $U^n = \psi^n_\wedge(U) = \psi^n_S(U)$. The result follows. □

LEMMA 6.5. *For $r = 1, \ldots, p$ and all n, $\Phi^n_{KG}(J_r) = \zeta^n_{KG}(J_r)$.*

PROOF. Let Q be the holomorph of P, identified with the group Q of Section 5. Thus $Q = \mathrm{Aut}(P)P$ where P is generated by a and $\mathrm{Aut}(P)$ is generated by b. The action of H on P by conjugation gives a homomorphism $H \to \mathrm{Aut}(P)$. This extends to a homomorphism $\tau : G \to Q$ which is the identity on P and gives a homomorphism $\tau^* : R_{KQ} \to R_{KG}$. It is easy to check that $\tau^*(J_r) = J_r$ (using the same notation J_r in connection with both Q and G). By Theorem 5.1, $\Phi^n_{KQ}(J_r) = \zeta^n_{KQ}(J_r)$. Hence $\tau^*(\Phi^n_{KQ}(J_r)) = \tau^*(\zeta^n_{KQ}(J_r))$. Therefore $\Phi^n_{KG}(J_r) = \zeta^n_{KG}(J_r)$, by Lemma 2.3. □

PROOF OF THEOREM 6.1. By Lemma 6.2, it suffices to show that we have

$$\Phi^n_{KG}(IJ_r) = \zeta^n_{KG}(IJ_r)$$

for $r = 1, \ldots, p$ and all irreducible KH-modules I. However, by Lemma 6.4, $\Phi^n_{KG}(IJ_r) = \psi^n_\wedge(I)\Phi^n_{KG}(J_r)$ and $\zeta^n_{KG}(IJ_r) = \psi^n_\wedge(I)\zeta^n_{KG}(J_r)$. Thus the result follows from Lemma 6.5. □

If we wish to apply Theorem 1.1 for our group G with a normal Sylow p-subgroup we need to know the Adams operations on R_{KG} and the functions $\zeta_{KG}^{p^k}$ (or, at least, ζ_{KG}^{p}). By Lemmas 6.2 and 6.4, these can be obtained from the Adams operations on R_{KH} and the values of the Adams operations and the functions $\zeta_{KG}^{p^k}$ on the modules J_r. These values of $\zeta_{KG}^{p^k}$ are given by the following result, in the notation of Section 3. (Recall that $\beta^2 = \alpha$ and $\alpha_r = 1 + \alpha + \cdots + \alpha^{r-1}$.)

LEMMA 6.6. *We have $\zeta_{KG}^{p}(J_1) = -J_1$ and $\zeta_{KG}^{p^2}(J_1) = 0$. For $r \geq 2$,*

$$\zeta_{KG}^{p}(J_r) = \begin{cases} p\alpha J_{p-1} + (p-r)\beta^{r-1}(J_1 - J_p) - \alpha_r J_p & \text{if } r \text{ is odd;} \\ p\alpha J_{p-1} - (p-r)\beta^{r} J_{p-1} - \alpha_r J_p & \text{if } r \text{ is even,} \end{cases}$$

$$\zeta_{KG}^{p^2}(J_r) = \begin{cases} p\alpha(p - (p-r)\beta^{r-1} - \alpha_r) J_{p-1} & \text{if } r \text{ is odd;} \\ p\alpha(p - (p-r)\beta^{r} - \alpha_r) J_{p-1} & \text{if } r \text{ is even.} \end{cases}$$

Furthermore, $\zeta_{KG}^{p^k}(J_r) = p\zeta_{KG}^{p^{k-1}}(J_r)$ for all r and $k \geq 3$.

PROOF. We use the homomorphism $\tau^* : R_{KQ} \to R_{KG}$, as in the proof of Lemma 6.5. As observed there, $\tau^*(J_r) = J_r$. It is also easy to verify that $\tau^*(\alpha) = \alpha$ (using the same notation α in connection with both Q and G). The powers of β in the formulae of the lemma are actually powers of α, since $\beta^2 = \alpha$. Thus, by Lemma 2.3, it suffices to prove these formulae for Q instead of G. The results for ζ_{KQ}^{p} are obtained by straightforward calculations from (5.4), (5.5) and Theorem 4.4. Also, by definition, $\zeta_{KQ}^{p^2}(J_r) = -\psi_S^{p^2}(J_r) - \zeta_{KQ}^{p}(\psi_S^{p}(J_r))$. This allows the calculation of $\zeta_{KQ}^{p^2}$. The last statement of the lemma is given by Corollary 5.5. □

As far as Adams operations on R_{KG} are concerned, we only need finitely many because of the periodicity given by the following result.

LEMMA 6.7. *Let $q = |H/C_H(P)|$ and let e be the least common multiple of $2pq$ and the orders of the elements of H. Then, for all n, $\psi_\wedge^n = \psi_\wedge^{n+e}$ and $\psi_S^n = \psi_S^{n+e}$.*

PROOF. This was proved in [16, Proposition 4.7], using results for GL(2, p). We sketch an independent proof.

By Lemma 6.2 it suffices to show that we have $\psi_\wedge^n(IJ_r) = \psi_\wedge^{n+e}(IJ_r)$ and $\psi_S^n(IJ_r) = \psi_S^{n+e}(IJ_r)$ for $r = 1, \ldots, p$ and all irreducible KH-modules I. By Lemma 2.6 and the choice of e, the elements $\psi_\wedge^n(I)$, $\psi_\wedge^{n+e}(I)$, $\psi_S^n(I)$ and $\psi_S^{n+e}(I)$ of R_{KH} have the same Brauer character. Thus they are equal, by Lemma 2.9. Therefore, by Lemma 6.4, it suffices to prove that $\psi_\wedge^n(J_r) = \psi_\wedge^{n+e}(J_r)$ and $\psi_S^n(J_r) = \psi_S^{n+e}(J_r)$. In fact we prove the stronger result that, for all n, $\psi_\wedge^n(J_r) = \psi_\wedge^{n+2pq}(J_r)$ and

$\psi_S^n(J_r) = \psi_S^{n+2pq}(J_r)$. For this we may assume that $K = \widehat{K}$ and $G = \widetilde{G}$, in the notation of Section 3. By (4.1),

$$\psi_\wedge^n(J_r) = \beta^{(r-1)n} \sum_{j=0}^{r-1} \lambda^{(r-1-2j)n}, \quad \psi_\wedge^{n+2pq}(J_r) = \beta^{(r-1)(n+2pq)} \sum_{j=0}^{r-1} \lambda^{(r-1-2j)(n+2pq)}.$$

However, $\beta^{(r-1)n} = \beta^{(r-1)(n+2pq)}$, since $\beta^{2q} = 1$. Also, from the formula for J_p given by (3.1), $\lambda^{2p} - 1 = (\lambda^2 - 1)\lambda^{p-1}\beta^{-p+1}J_p \in \Omega$, where Ω is the ideal of $R_{KG}[\lambda]$ generated by J_p. Therefore $\psi_\wedge^{n+2pq}(J_r) = \psi_\wedge^n(J_r) + U$, where $U \in \Omega \cap R_{KG}$. However, $\Omega \cap R_{KG} = R_{KG} J_p$. Thus $U \in (R_{KG})_{\text{proj}}$, in the notation at the end of Section 2. Also, by Lemma 2.6, $\text{Br}(\psi_\wedge^{n+2pq}(J_r)) = \text{Br}(\psi_\wedge^n(J_r))$. Thus $\psi_\wedge^{n+2pq}(J_r) = \psi_\wedge^n(J_r)$ by Lemma 2.9. From this we obtain $\psi_S^{n+2pq}(J_r) = \psi_S^n(J_r)$ by Lemmas 4.3 and 4.1. □

The values of the Adams operations on the J_r can, at least in principle, be calculated using (4.1) and Lemma 4.3. (See [1] for corresponding calculations for the group of order p.)

7. The general case

Let K be a field of prime characteristic p. If G is a finite p'-group then $\Phi_{KG}^n = \zeta_{KG}^n$ for all n, by Lemmas 2.8 and 2.9. (Indeed, we also have $\Phi_{KG}^n = \mu(n)\psi_S^n$ by Lemmas 2.6 and 2.8). Thus, to complete the proof of Theorem 1.1, we only need consider the case where G is a finite group with a Sylow p-subgroup P of order p. We write N for the normalizer of P in G. Thus N is a finite group with a normal Sylow p-subgroup of order p, and the results of Sections 3–6 apply (with N replacing G). We write $N = HP$, where H is a p'-group.

The subgroup P of G is a trivial-intersection set, so a simple form of the Green correspondence applies (see [2, Theorem 10.1], where the field does not need to be algebraically closed): there is a one-one correspondence between finite-dimensional non-projective indecomposable KG-modules and finite-dimensional non-projective indecomposable KN-modules. Here, if V corresponds to V^* then $V{\downarrow_N}$ is the direct sum of V^* and a projective module. It follows that if $V, V' \in R_{KG}$ and $V{\downarrow_N} = V'{\downarrow_N}$ then $V_{\text{nonp}} = V'_{\text{nonp}}$. The proof of Theorem 1.1 is completed by the following result.

THEOREM 7.1. *Let K be a field of prime characteristic p and let G be a finite group with a Sylow p-subgroup of order p. Then $\Phi_{KG}^n = \zeta_{KG}^n$ for all n.*

PROOF. Let V be a finite-dimensional KG-module. Then, by Theorem 6.1 and Lemma 2.3, $\Phi_{KG}^n(V){\downarrow_N} = \zeta_{KG}^n(V){\downarrow_N}$. Hence, by the Green correspondence, $\Phi_{KG}^n(V)_{\text{nonp}} = \zeta_{KG}^n(V)_{\text{nonp}}$. However, $\text{Br}(\Phi_{KG}^n(V)) = \text{Br}(\zeta_{KG}^n(V))$, by Lemma 2.8. Therefore $\Phi_{KG}^n(V) = \zeta_{KG}^n(V)$, by Lemma 2.9. This gives the required result. □

By Theorem 1.1 we can calculate all Lie powers $L^n(V)$ if we can find tensor powers, Adams operations and the pth Lie powers of all indecomposables. By the next result, only finitely many Adams operations need to be found. With H as defined above, let $q = |H/C_H(P)|$ and let e be the least common multiple of $2pq$ and the orders of the p'-elements of G.

THEOREM 7.2. *Let K be a field of prime characteristic p and let G be a finite group with a Sylow p-subgroup of order p. Let e be as defined above. Then, for every positive integer n, $\psi^n_\wedge = \psi^{n+e}_\wedge$ and $\psi^n_S = \psi^{n+e}_S$.*

PROOF. (For $G = \mathrm{GL}(2, p)$, this is given by [15, Proposition 3.5].) Let V be a finite-dimensional KG-module. Then, by Lemma 6.7, $\psi^n_\wedge(V){\downarrow_N} = \psi^{n+e}_\wedge(V){\downarrow_N}$. Hence, by the Green correspondence, $\psi^n_\wedge(V)_{\mathrm{nonp}} = \psi^{n+e}_\wedge(V)_{\mathrm{nonp}}$. However, by Lemma 2.6 and the definition of e, $\mathrm{Br}(\psi^n_\wedge(V)) = \mathrm{Br}(\psi^{n+e}_\wedge(V))$. Thus, by Lemma 2.9, $\psi^n_\wedge(V) = \psi^{n+e}_\wedge(V)$. Similarly, $\psi^n_S(V) = \psi^{n+e}_S(V)$. This gives the result. □

If we have detailed information about the indecomposable KG-modules and KN-modules, the Green correspondence, and the Brauer characters of G, we can hope to find the Lie powers of a finite-dimensional KG-module V from Lie powers of KN-modules as follows. Since $L^n(V){\downarrow_N} = L^n(V{\downarrow_N})$, by Lemma 2.2, $L^n(V){\downarrow_N}$ can be calculated by the methods described at the end of Section 6. Thus, by the Green correspondence, we can determine $L^n(V)_{\mathrm{nonp}}$ and hence $\mathrm{Br}(L^n(V)_{\mathrm{nonp}})$. However, $\mathrm{Br}(L^n(V))$ is given by Brandt's character formula (Lemma 2.7). Thus we can find $\mathrm{Br}(L^n(V)_{\mathrm{proj}})$. Therefore $L^n(V)_{\mathrm{proj}}$ can be found, at least in principle, by the modular orthogonality relations. Hence we can find $L^n(V)$.

The connection between Lie powers of KG-modules and Lie powers of KN-modules was a key factor in obtaining the results of [8, 17] and [10]. The following theorem generalises one of the main qualitative results of [10]. Recall that the $(p-1)$-dimensional KN-module J_{p-1} is as defined in Section 3.

THEOREM 7.3. *Let K be a field of prime characteristic p and let G be a finite group with a Sylow p-subgroup of order p. Let V be a finite-dimensional KG-module and let n be a positive integer. Then, in the notation established above, every non-projective indecomposable summand of $L^n(V)$ is either a summand of the nth tensor power V^n or is the Green correspondent of a KN-module of the form $I \otimes J_{p-1}$, where I is an irreducible KH-module.*

PROOF. We give a sketch only. Note that $L^n(V){\downarrow_N} = L^n(V{\downarrow_N})$ and $V^n{\downarrow_N} = (V{\downarrow_N})^n$. By the Green correspondence it suffices to show that every non-projective indecomposable summand of $L^n(V{\downarrow_N})$ is either a summand of $(V{\downarrow_N})^n$ or has the form $I \otimes J_{p-1}$, where I is an irreducible KH-module. Thus we may assume that $G = N = HP$.

Write $n = p^k m$ where $p \nmid m$. By Theorem 1.1 and Corollary 1.2,

$$L^n(V) = \frac{1}{p^k} \sum_{i=0}^{k} \zeta_{KG}^{p^i}(L^m(V^{p^{k-i}})).$$

However, for $i = 0, \ldots, k$, $L^m(V^{p^{k-i}})$ is a summand of $V^{mp^{k-i}}$, since $p \nmid m$ (see, for example, [13, Section 3.1]). Hence it suffices to show, for $i \geq 0$, that if Y is a finite-dimensional indecomposable KG-module then $\zeta_{KG}^{p^i}(Y)$ is a linear combination of projective KG-modules, summands of Y^{p^i}, and modules of the form $I \otimes J_{p-1}$, where I is an irreducible KH-module. By Lemma 6.2, $Y \cong U \otimes J_r$ where $1 \leq r \leq p$ and U is an irreducible KH-module. By Lemma 6.4, $\zeta_{KG}^{p^i}(Y) = \psi_{\wedge}^{p^i}(U)\zeta_{KG}^{p^i}(J_r)$. However, by (2.2) or (2.3), $\psi_{\wedge}^{p^i}(U)$ is a linear combination of modules which are homomorphic images of U^{p^i}. Thus, since H is a p'-group, $\psi_{\wedge}^{p^i}(U)$ is a linear combination of summands of U^{p^i}. It therefore suffices to prove that $\zeta_{KG}^{p^i}(J_r)$ is a linear combination of projective modules, summands of $J_r^{p^i}$, and modules of the form $I \otimes J_{p-1}$. This is trivial for $i = 0$ and, by Lemma 6.6, it is clear for $i \geq 2$. Suppose then that $i = 1$. By Lemma 6.6, the result is clear for r even, $r = 1$ and $r = p$. By the same lemma, it is true for r odd with $1 < r < p$ provided that $\beta^{r-1} J_1$ is a summand of J_r^p. This can be proved as follows, using the notation of Section 3.

It is sufficient to consider the case where $K = \widehat{K}$ and $G = \widetilde{G}$. Let Ω' be the ideal of $R_{KG}[\lambda]$ generated by $pR_{KG}[\lambda]$ and J_p. Then, as in the proof of Lemma 6.7, $\lambda^{2p} - 1 \in \Omega'$. Also, $\beta^{(r-1)p} = \beta^{r-1}$. However, by (3.1),

$$J_r^p \equiv \beta^{(r-1)p} \sum_{j=0}^{r-1} \lambda^{(r-1-2j)p} \pmod{\Omega'}.$$

Hence $J_r^p \equiv r\beta^{r-1} J_1 \pmod{\Omega' \cap R_{KG}}$. However, $\Omega' \cap R_{KG} = pR_{KG} + R_{KG} J_p$. Since r is not divisible by p it follows that $\beta^{r-1} J_1$ is a summand of J_r^p. □

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