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ON THE VALUE DISTRIBUTION OF $f^2 f^{(k)}$

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Abstract

In this paper, we prove that for a transcendental meromorphic function $f(z)$ on the complex plane, the inequality $T(r, f) < 6N(r, 1/(f^2 f^{(k)} - 1)) + S(r, f)$ holds, where k is a positive integer. Moreover, we prove the following normality criterion: Let $\mathscr{F}$ be a family of meromorphic functions on a domain D and let k be a positive integer. If for each $f \in \mathscr{F}$, all zeros of f are of multiplicity at least k, and $f^2 f^{(k)} \neq 1$ for $z \in D$, then $\mathscr{F}$ is normal in the domain D. At the same time we also show that the condition on multiple zeros of f in the normality criterion is necessary.

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1. Introduction

In 1979 Mues [1] proved that for a transcendental meromorphic function $f(z)$ in the open plane, $f^2 f' - 1$ has infinitely many zeros. This is a qualitative result. Later, Zhang [2] obtained a quantitative result, proving that the inequality $T(r, f) < 6N(r, 1/(f^2 f' - 1)) + S(r, f)$ holds. Naturally, we ask whether the above inequality is still true when $N(r, 1/(f^2 f' - 1))$ is replaced by $N(r, 1/(f^2 f^{(k)} - 1))$. In this paper, we solve this problem and obtain

THEOREM 1. *Let $f(z)$ be a transcendental function in the complex plane and let k be a positive integer. Then*

$$T(r, f) < 6N\left(r, \frac{1}{f^2 f^{(k)} - 1}\right) + S(r, f).$$

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From Theorem 1, we have at once:

COROLLARY. *Let $f(z)$ be a transcendental meromorphic function and let k be a positive integer. Then $f^2 f^{(k)} - 1$ assumes every non-zero finite value infinitely often.*

Using Mues' result, Pang [2] proved:

THEOREM A ([2]). *Let $\mathscr{F}$ be a family of meromorphic function on a domain D. If each $f \in \mathscr{F}$ satisfies $f^2 f' \neq 1$, then $\mathscr{F}$ is normal on domain D.*

Now, utilizing Theorem 1 we also can obtain the following theorem:

THEOREM 2. *Let $\mathscr{F}$ be a family of meromorphic functions on a domain D and let k be a positive integer. If for each $f \in \mathscr{F}$, f has only zeros of multiplicity at least k and $f^2 f^{(k)} \neq 1$, then $\mathscr{F}$ is normal on domain D.*

The following example shows that the condition on multiple zeros of f in Theorem 2 is necessary.

EXAMPLE. Let $k \geq 2$ be a positive integer and $\mathscr{F} = \{nz^{k-1} : n = 1, 2, \ldots\}$. So, each $f \in \mathscr{F}$ satisfies $f^2 f^{(k)} \neq 1$. But $\mathscr{F}$ is not normal at the origin.

2. Some lemmas

LEMMA 1. *Let $f(z)$ be a transcendental function. Then $f^2 f^{(k)}$ is not identically constant.*

PROOF. Suppose that $f^2 f^{(k)} \equiv C$. Obviously, $C \neq 0$. So $f \neq 0$ and $1/f^3 = C^{-1} f^{(k)}/f$. Hence we obtain

$$3T(r, f) = m\left(r, \frac{1}{f^3}\right) + O(1) = O(1)\left\{m\left(r, \frac{f^{(k)}}{f}\right) + 1\right\} = S(r, f).$$

This contradicts the assumption that $f(z)$ is a transcendental function. □

LEMMA 2. *Let $f(z)$ be a transcendental meromorphic function, $g(z) = f^2 f^{(k)} - 1$ and $h(z) = g'/f = f f^{(k+1)} + 2 f' f^{(k)}$. Then*

$$3T(r, f) < \overline{N}(r, f) + 2N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g}\right) - N\left(r, \frac{1}{h}\right) + S(r, f) \tag{2.1}$$

$$[N(r, f) - \overline{N}(r, f)] + m(r, f) + 2m\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{h}\right) < N\left(r, \frac{1}{g}\right) + S(r, f). \tag{2.2}$$

PROOF. By Lemma 1, we know $g \not\equiv C$ and $h \not\equiv 0$. Set

$$\frac{1}{f^3} = \frac{f^2 f^{(k)}}{f^3} - \frac{g'}{f^3}\frac{g}{g'},$$

so

$$\begin{aligned} 3m(r, f) &< m\left(r, \frac{g}{g'}\right) + S(r, f) < N\left(r, \frac{g'}{g}\right) - N\left(r, \frac{g}{g'}\right) + S(r, f) \\ &= \overline{N}(r, g) + N\left(r, \frac{1}{g}\right) - N\left(r, \frac{1}{g'}\right) + S(r, f) \\ &= \overline{N}(r, g) + N\left(r, \frac{1}{g}\right) - N\left(r, \frac{1}{fh}\right) + S(r, f) \\ &= \overline{N}(r, g) + N\left(r, \frac{1}{g}\right) - N\left(r, \frac{1}{f}\right) - N\left(r, \frac{1}{h}\right) + S(r, f). \end{aligned}$$

Hence

$$\begin{aligned} 3T(r, f) &= 3m\left(r, \frac{1}{f}\right) + 3N\left(r, \frac{1}{f}\right) + O(1) \\ &< \overline{N}(r, f) + 2N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g}\right) - N\left(r, \frac{1}{h}\right) + S(r, f). \end{aligned}$$

Thus the inequality (2.1) is proved. Since

$$3T(r, f) = m(r, f) + N(r, f) + 2m\left(r, \frac{1}{f}\right) + 2N\left(r, \frac{1}{f}\right) + O(1),$$

the inequality (2.2) can be obtained. □

LEMMA 3. *Let $f(z), g(z), h(z)$ $(k \geq 2)$ be as stated above and let*

$$a_1 = 2(k+1)^2 - \frac{(3k+7)(k^2-4k-29)}{(k+3)}, \qquad a_3 = 2(k+2)(k+3)(k+5),$$
$$a_4 = -4(k+3)(k+1),$$
$$a_2 = -(k+5)(k^2-4k-29), \qquad a_5 = 4(k^2-4k-29),$$

and

$$F(z) = a_1\left(\frac{g'(z)}{g(z)}\right)^2 + a_2\left(\frac{g'(z)}{g(z)}\right)' + a_3\left(\frac{h'(z)}{h(z)}\right)' + a_4\left(\frac{h'(z)}{h(z)}\right)^2 + a_5\left(\frac{g'(z)}{g(z)}\frac{h'(z)}{h(z)}\right). \tag{2.3}$$

Then $F \not\equiv 0$.

PROOF. Suppose that $F(z) \equiv 0$, we claim that

(i) $g(z) \neq 0$;

(ii) $h(z) \neq 0$;
(iii) all zeros of $f(z)$ are simple.

Suppose first that z_1 is a zero of $g(z)$ of multiplicity l ($l \geq 1$). From $g(z_1) = 0$ and $g = f^2 f^{(k)} - 1$ we can get $f(z_1) \neq 0, \infty$. Since z_1 is a zero of order $(l-1)$ of $g' = fh$ we have that z_1 be a zero of $h(z)$ of multiplicity $l-1$. Using the Laurent series of $F(z)$ at the point z_1, we can get the coefficient of $(z-z_1)^{-2}$:

$$A(l) = (a_1 + a_4 + z_5)l^2 - (a_2 + a_3 + 2a_4 + a_5)l + (a_3 + a_4).$$

From the definition of a_i, $i = 1, \ldots, 5$, we have

$$A(l) = -\frac{(k+5)^2(k+7)}{k+3}l^2 - (k+1)(k+5)(k+7)l + 2(k+1)^2(k+3).$$

Obviously, $A(l) \neq 0$ for all positive integers l. So the point z_1 is a pole of $F(z)$ which contradicts $F(z) \equiv 0$. Hence conclusion (i) $g(z) \neq 0$ holds.

Suppose next that z_2 is a zero of $h(z)$ of order l ($l \geq 1$). By (i) we have $g(z_2) \neq 0, \infty$. Using the Laurent series of $F(z)$ at the point z_2, we can get the coefficient of $(z-z_2)^{-2}$ as $B(l) = -a_3 l + a_4 l^2$. From the definition of a_i, $i = 1, \ldots, 5$, we have

$$B(l) = -2(k+1)(k+3)(k+5)l - 4(k+1)(k+3)l^2 < 0,$$

so that the point z_2 is a pole of $F(z)$ which contradicts $F(z) \equiv 0$. Hence conclusion (ii) $h(z) \neq 0$ holds.

Using $h(z) = ff^{(k+1)} + 2f'f^{(k)}$ and (ii) ($h(z) \neq 0$), we can get (iii).

Set $\phi(z) = h(z)/g(z)$, we can deduce that $\phi(z)$ is an entire function, all zeros of $\phi(z)$ can occur only at multiple poles of $f(z)$ and the following expressions hold:

$$\frac{g'}{g} = \frac{fh}{g} = f\phi, \qquad \frac{h'}{h} = \frac{g'}{g} + \frac{\phi'}{\phi} = f\phi + \frac{\phi'}{\phi}.$$

Substituting the above two equalities in the expression (2.3) for $F(z)$, we get

$$(a_1 + a_4 + a_5)f^2\phi^2 + (a_2 + a_3 + 2a_4 + a_5)f\phi' + \left[a_3\left(\frac{\phi'}{\phi}\right)' + a_4\left(\frac{\phi'}{\phi}\right)^2\right] + (a_2 + a_3)f'\phi \equiv 0. \tag{2.4}$$

Obviously, $a_2 + a_3 = (k+5)^2(k+7) \neq 0$ and $\phi \not\equiv 0$, otherwise $g'/g = f\phi \equiv 0$, that is, $g \equiv C$ which contradicts the result of Lemma 1.

Thus, by the equality (2.4), we have

$$f' \equiv \frac{1}{\phi}l_{11}(z) + fl_{12}(z) + f^2\phi l_{13}(z), \tag{2.5}$$

where $l_{1i}(z)$ $(i = 1, 2, 3)$ are differential monomials of (ϕ'/ϕ). Differentiating both sides of (2.5), we have

$$f'' = -\frac{1}{\phi}\frac{\phi'}{\phi}l_{11}(z) + \frac{1}{\phi}l'_{11}(z) + f'l_{12}(z) + fl'_{12}(z) + 2ff'\phi l_{13}(z) + f^2\phi\left[\frac{\phi'}{\phi}l_{13}(z) + l'_{13}(z)\right].$$

Using the above equality and (2.5), we get

$$f'' = \frac{1}{\phi}l_{21}(z) + fl_{22}(z) + f^2\phi l_{23}(z) + f^3\phi^2 l_{24}(z),$$

where $l_{2i}(z)$ $(i = 1, \ldots, 4)$ are differential monomials of (ϕ'/ϕ). Continuing the above process we obtain

$$f^{(k)} = \frac{1}{\phi}l_{k1}(z) + fl_{k2}(z) + f^2\phi l_{k3}(z) + \cdots + f^{k+1}\phi^k l_{kk+2}(z), \tag{2.6}$$

where $l_{ki}(z)$ $(i = 1, \ldots, k = 2)$ are differential monomials of (ϕ'/ϕ).

Now, suppose z_3 is a zero of f. Combining (2.5), (2.6) and $\phi(z_3) \neq 0, \infty$, we have

$$f'(z_3) = \frac{1}{\phi(z_3)}l_{11}(z_3), \qquad f^{(k)}(z_3) = \frac{1}{\phi(z_3)}l_{k1}(z_3).$$

Further, by the above two equalities and the expression for $g(z)$ and $h(z)$ in Lemma 2, we have

$$g(z_3) = -1, \quad h(z_3) = 2f'(z_3)f^{(k)}(z_3) = \frac{2}{\phi^2(z_3)}l_{11}(z_3)l_{k1}(z_3).$$

Substituting the above equality in the expression for $\phi(z) = h(z)/g(z)$ we have

$$\phi^3(z_3) = -2l_{11}(z_3)l_{k1}(z_3). \tag{2.7}$$

Set $G(z) = \phi^3(z) + 2l_{11}(z)l_{k1}(z)$. We distinguish two cases.

Case 1. $G(z) \not\equiv 0$. By (2.7) and (iii) we have

$$N\left(r, \frac{1}{f}\right) = \overline{N}\left(r, \frac{1}{f}\right) \leq N\left(r, \frac{1}{G}\right) < T(r, G) + O(1) < O\{T(r, \phi)\} + O(1), \tag{2.8}$$

$$T(r, \phi) = m(r, \phi) = m\left(r, \frac{h}{g}\right) = m\left(r, \frac{g'}{g}\frac{1}{f}\right) \leq m\left(r, \frac{1}{f}\right) + S(r, f). \tag{2.9}$$

Applying (2.2) of Lemma 2, and combining with $N(r, 1/G) = 0$ we have

$$m(r, 1/f) = S(r, f). \tag{2.10}$$

By (2.10), (2.9) and (2.8), we have

$$N(r, 1/f) = S(r, f). \tag{2.11}$$

Combining (2.10) and (2.11) we get $T(r, f) = T(r, 1/f) + O(1) = S(r, f)$. This gives a contradiction, since f is a transcendental meromorphic function.

Case 2. $G(z) \equiv 0$. Using the expression for $G(z)$, and noting that $l_{11}(z)$, $l_{k1}(z)$ are differential monomials of (ϕ'/ϕ) we deduce that

$$T(r, \phi) = m(r, \phi) = S(r, \phi). \tag{2.12}$$

Again, using the expression for $G(z)$ and the fact that $G(z) \equiv 0$ we have

$$\phi^3 \equiv -2l_{11}(z)l_{k1}(z). \tag{2.13}$$

From (2.12), we deduce that $\phi(z)$ is a polynomial or a constant. If ϕ is a polynomial, then the right-hand side of (2.13) is a constant or rational function and the left-hand side of (2.13) is a polynomial, and this gives a contradiction. So ϕ is a constant. If $\phi \equiv 0$, using $g'/g = f\phi \equiv 0$, we deduce that g is a constant, which contradicts Lemma 1.

Hence, $\phi(z) \equiv C$, where $C \neq 0$. Substituting this equality in (2.4), we have

$$(a_1 + a_4 + a_5)C^2 f^2 + (a_2 + a_3)Cf' \equiv 0,$$

so $f' = C_1 f^2$, that is, $(1/f)' \equiv -C_1$, where $C_1 \neq 0$ is a constant. Then we deduce that $f(z)$ is a rational function, but this is impossible. This completes the proof. □

LEMMA 4. *Let $f(z), g(z), h(z), k \geq 2, F(z)$ be stated as above. Then all simple poles of $f(z)$ are zeros of $F(z)$.*

PROOF. Suppose z_0 is a simple pole of $f(z)$, then

$$f(z) = \frac{a}{(z - z_0)}\{1 + b_0(z - z_0) + b_1(z - z_0)^2 + O((z - z_0)^3)\},$$

where $a \neq 0$, b_0, b_1 are constants. Since $k \geq 2$, we have

$$\begin{aligned} g(z) &= f^2 f^{(k)} - 1 \\ &= \frac{(-1)^k k! a^3}{(z - z_0)^{k+3}}\{1 + 2b_0(z - z_0) + (b_0^2 + 2b_1)(z - z_0)^2 + O((z - z_0)^3)\}, \end{aligned}$$

$$h(z) = \frac{g'}{f} = \frac{(-1)^{k+1}k!a^2}{(z-z_0)^{k+3}}\big\{(k+3) + (k+1)b_0(z-z_0) + (k-1)b_1(z-z_0)^2 + O((z-z_0)^3)\big\}.$$

Hence, we have

$$\begin{aligned}
\frac{g'}{g} &= \frac{(-1)}{(z-z_0)}\big\{(k+3) - 2b_0(z-z_0) + (2b_0^2 - 4b_1)(z-z_0)^2 + O((z-z_0)^3)\big\},\\
\frac{h'}{h} &= \frac{(-1)}{(z-z_0)}\frac{1}{k+3}\Bigg\{(k+3)^2 - (k+1)b_0(z-z_0)\\
&\quad + \left[\frac{(k+1)^2}{k+3}b_0^2 - 2(k-1)b_1\right](z-z_0)^2 + O((z-z_0)^3)\Bigg\},\\
\left(\frac{g'}{g}\right)^2 &= \frac{1}{(z-z_0)^2}\big\{(k+3)^2 - 4(k+3)b_0(z-z_0)\\
&\quad + \big[4(k+4)b_0^2 - 8(k+3)b_1\big](z-z_0)^2 + O((z-z_0)^3)\big\},\\
\left(\frac{g'}{g}\right)' &= \frac{1}{(z-z_0)^2}\big\{(k+3) - (2b_0^2 - 4b_1)(z-z_0)^2 + O((z-z_0)^3)\big\},\\
\left(\frac{h'}{h}\right)' &= \frac{1}{(z-z_0)^2}\frac{1}{k+3}\Bigg\{(k+3)^2 - \left[\frac{(k+1)^2}{k+3}b_0^2 - 2(k-1)b_1\right](z-z_0)^2\\
&\quad + O((z-z_0)^3)\Bigg\},\\
\left(\frac{h'}{h}\right)^2 &= \frac{1}{(z-z_0)^2}\frac{1}{(k+3)^2}\big\{(k+3)^4 - 2(k+1)(k+3)^2b_0(z-z_0)\\
&\quad + \big[(k+1)^2(2k+7)b_0^2 - 4(k-1)(k+3)^2b_1\big](z-z_0)^2 + O((z-z_0)^3)\big\},\\
\frac{g'}{g}\frac{h'}{h} &= \frac{1}{(z-z_0)^2}\big\{(k+3)^2 - (3k+7)b_0(z-z_0)\\
&\quad + [(3k+7)b_0^2 - 2(3k+5)b_1](z-z_0)^2 + O((z-z_0)^3)\big\}.
\end{aligned}$$

By substituting all of the above equalities in the expression (2.3) of $F(z)$ and performing some easy calculations we obtain that $F(z) = O((z-z_0))$. So, z_0 is the zero of $F(z)$. This completes the proof. □

LEMMA 5 ([3]). *Let $\mathscr{F}$ be a family of meromorphic functions on the unit disc Δ such that all zeros of functions in $\mathscr{F}$ have multiplicity at least k. Let α be a real number satisfying $0 \le \alpha < k$. Then $\mathscr{F}$ is not normal in any neighbourhood of $z_0 \in \Delta$ if and only if there exist*

(i) *points $z_k \in \Delta$, $z_k \to z_0$;*
(ii) *positive numbers ρ_k, $\rho_k \to 0$; and*

(iii) *functions* $f_k \in \mathscr{F}$

such that $\rho_k^{-\alpha} f_k(z_k + \rho_k \xi) \to g(\xi)$ *spherically uniformly on compact subsets of* $\mathbb{C}$, *where* g *is a nonconstant meromorphic function.*

3. Proof of theorems

PROOF OF THEOREM 1. When $k = 1$, this is the result of Zhang [4]. So we assume that $k \geq 2$. By Lemma 3, $F(z) \not\equiv 0$. Thus by Lemma 4 we have

$$N_1(r, f) \leq N(r, 1/F) \leq T(r, F) + O(1), \tag{3.1}$$

where in $N_1(r, f)$ only simple poles of $f(z)$ are to be considered. By (2.3), we know that the poles of $F(z)$ can occur only at multiple poles of $f(z)$ or zeros of $g(z)$, or zeros of $h(z)$, and all poles of $F(z)$ are of multiplicity at most 2. So

$$N(r, F) \leq 2\overline{N}_{(2}(r, f) + 2N(r, 1/g) + 2N(r, 1/h) + S(r, f), \tag{3.2}$$

where in $\overline{N}_{(2}(r, f)$ only multiple poles of $f(z)$ are to be considered, and each pole is counted only once. Obviously, we have

$$m(r, F) = S(r, f). \tag{3.3}$$

By (3.1), (3.2) and (3.3), we have

$$N_1(r, f) \leq 2\overline{N}_{(2}(r, f) + 2N(r, 1/g) + 2N(r, 1/h) + S(r, f). \tag{3.4}$$

Combining Lemma 2, (2.1) and (3.4) gives

$$3T(r, f) < 3\overline{N}_{(2}(r, f) + 2N(r, 1/f) + 3N(r, 1/g) + N(r, 1/h) + S(r, f). \tag{3.5}$$

On the other hand, using Lemma 2 and (2.2), we have

$$\begin{aligned} 3\overline{N}_{(2}(r, f) + N(r, 1/h) &\leq 3[N(r, f) - \overline{N}(r, f)] + N(r, 1/h) \\ &< 3N(r, 1/g) + S(r, f). \end{aligned} \tag{3.6}$$

Thus, by (3.5) and (3.6), we obtain

$$\begin{aligned} 3T(r, f) &< 6N(r, 1/g) + 2N(r, 1/f) + S(r, f) \\ &< 6N(r, 1/g) + 2T(r, f) + S(r, f), \end{aligned}$$

that is, $T(r, f) < 6N(r, 1/g) + S(r, f)$. This completes the proof of Theorem 1. □

PROOF OF THEOREM 2. We may assume that $D = \Delta$. Suppose that $\mathscr{F}$ is not normal on Δ. Then, taking $\alpha = k/3$ and applying Lemma 5, we can find $f_n \in \mathscr{F}$, $z_n \in \Delta$ and $\rho_n \to 0+$ such that

$$\frac{f_n(z_n + \rho_n \xi)}{\rho_n^{\alpha}} = g_n(\xi) \to g(\xi)$$

locally uniformly with respect to the spherical metric, where g is a nonconstant meromorphic function on $\mathbb{C}$. By the assumption,

$$\begin{aligned} g_n^2(\xi)(g_n(\xi))^{(k)} - 1 &= \rho_n^{k-3\alpha} f_n^2(z_n + \rho_n \xi) f_n^{(k)}(z_n + \rho_n \xi) - 1 \\ &= f_n^2(z_n + \rho_n \xi) f_n^{(k)}(z_n + \rho_n \xi) - 1 \\ &\neq 0. \end{aligned}$$

So

$$g^2(\xi) g^{(k)}(\xi) - 1 \neq 0 \quad \text{or} \quad g^2(\xi) g^{(k)}(\xi) - 1 \equiv 0. \tag{3.7}$$

By Hurwitz's theorem, all zeros of $g(\xi)$ are of multiplicity at least k and it is easy to see that $g^2(\xi)g^{(k)}(\xi) \not\equiv 0$. Hence, $g^2(\xi)g^{(k)}(\xi) - 1 \neq 0$. According to Mues's result ($k = 1$) and Theorem 1 ($k \geq 2$) we find that $g(\xi)$ is not a transcendental meromorphic function. If $g(\xi)$ is a polynomial, then its degree is at most $k - 1$ which contradicts the fact that the zeros of $g(\xi)$ are of multiplicity at least k. If $g(\xi)$ is a nonconstant rational function, we set $g(\xi) = Q(\xi)/P(\xi)$, where $Q(\xi)$ and $P(\xi)$ are two prime polynomials and set $p = \deg(P)$ and $q = \deg(Q)$. From (3.7) we deduce that there exists a polynomial $h(\xi)$ such that

$$g^2(\xi) g^{(k)}(\xi) = \frac{h(\xi) + 1}{h(\xi)}. \tag{3.8}$$

It is easy to verify that the difference between the degree of the numerator of $g^2(\xi)g^{(k)}(\xi)$ and the degree of the denominator of $g^2(\xi)g^{(k)}(\xi)$ is $3(q - p) - k$. It follows from (3.8) that $k = 3(q - p)$ and $(q - p) \geq 1$.

We set $n = (q - p)$ and $g(\xi) = a_0\xi^n + \cdots + a_n + R(\xi)/P(\xi)$, where $R(\xi)$ and $P(\xi)$ are two prime polynomials and $\deg(P) - \deg(R) > 0$. Noting that $g^{(k)}(\xi) = (R(\xi)/P(\xi))^{(k)}$, it follows from (3.8) that $\deg(P) - \deg(R) = -n$, which contradicts $\deg(P) - \deg(R) > 0$. Thus, we obtain our result. $\square$

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