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HEATH g-FUNCTIONS AND METRIZATION

ABDUL M. MOHAMAD

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Abstract

In this paper, we present some new metrization theorems in terms of Heath g-functions.

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1. Introduction

In this short note we characterize metrizability in terms of Heath g-functions.

Heath in [3] introduced a method of describing a generalized metric property of a topological space (X, τ) by means of a function $g : \mathbb{N} \times X \to \tau$. Hodel, Fletcher, Lindgren and Nagata have modified this method to obtain important new classes of spaces.

A *Heath g-function* [COC-map (= countable open covering map)] for a topological space X is a function g from $\mathbb{N} \times X$ into the topology of X such that for every $x \in X$ and $n \in \mathbb{N}$, $x \in g(n, x)$ and $g(n+1, x) \subseteq g(n, x)$.

It is well known that many important classes of generalized metrizable spaces can be characterized in terms of a Heath g-function. In particular, X is developable [3] ($w\Delta$-space) if and only if X has a Heath g-function g such that if $\{p, x_n\} \subseteq g(n, y_n)$ for all n, then p is a cluster point of the sequence $\langle x_n \rangle$ (then $\langle x_n \rangle$ has a cluster point).

A space X is called a *wM-space* [4] if and only if X has a Heath g-function g such that if $x \in g(n, z_n)$, $g(n, z_n) \cap g(n, y_n) \neq \emptyset$ and $x_n \in g(n, y_n)$ for all n then $\langle x_n \rangle$ has a cluster point. Let $\mathcal{G}$ be a collection of sets. We define $st(x, \mathcal{G}) = \bigcup\{G \in \mathcal{G} : x \in G\}$ and $st^2(x, \mathcal{G}) = \bigcup_{y \in st(x, \mathcal{G})} st(y, \mathcal{G})$.

In this paper all spaces will be T_0, unless we state otherwise.

2. Main results

First we consider what additional conditions need to be attached to a space already known to be *stratifiable* (conditions (1)–(3) in the next theorem are the axioms of stratifiability [2, Theorem 5.8]) to make it metrizable. To do this, the function $g : \omega \times X \to \mathscr{T}$ can be strengthened to give it some sort of symmetry as shown by the next theorem.

THEOREM 2.1. *A space X is metrizable if and only if there exists a function $g : \omega \times X \to \mathscr{T}$ such that*

(1) $\{x\} = \bigcap_{n\in\omega} g(n, x)$;
(2) *if $y \in g(n, x_n)$ for all n then $x_n \to y$;*
(3) *for any $y \notin H$ closed, $y \notin \overline{\bigcup\{g(x, n) : x \in H\}}$ for some $n \in \omega$;*
(4) *if $y \in g(n, x)$ then $x \in g(n, y)$.*

PROOF. For any metric space we can define g to satisfy the axioms of stratifiability as given in [2, Theorem 5.8]. The fourth condition of the theorem holds because of the symmetry of a metric.

To prove the converse, we assume that without loss of generality,

$$g(n, x) \subseteq g(n + 1, x) \quad \text{for any} \;\; x \in X.$$

If this was not the case, we define the function $g'(n, x) = \bigcap_{k\leqslant n} g(k, x)$. Certainly each $g'(n, x)$ is open as the finite intersection of open sets and the axioms for stratifiability still hold since $g'(n, x) \subseteq g(n, x)$ for each $n \in \omega$ and $x \in X$. Notice also that condition (4) remains true when considering these new open sets.

X can be shown to be a T_1 space by showing $\{x\}$ is closed for each $x \in X$. Suppose $y \notin \{x\}$; that is, $y \neq x$. Then we must have some $n \in \omega$ such that $x \notin g(n, y)$, otherwise $x \in \bigcap_{n\in\omega} g(n, y) = \{y\}$ and so the points are not distinct. Hence there is an open neighbourhood of y which does not meet $\{x\}$ and so $\{x\}$ is closed.

For each $n \in \omega$, we define an open cover $\mathscr{G}_n = \{g(n, x) : x \in X\}$. Suppose that x is in some open set U. If we can show that there exists some $n \in \omega$ such that $st^2(x, \mathscr{G}_n) \subseteq U$ then since X is T_0, the space will be metrizable by the Moore Metrization theorem [1].

Firstly we notice that there must exist some $n_0 \in \omega$ such that $g(n_0, x) \subseteq U$, otherwise we can define a sequence of points x_n such that $x_n \in g(n, x) \setminus U$ for each $n \in \omega$. Then by our new symmetry condition, $x \in g(n, x_n)$ for each $n \in \omega$, hence $x_n \to x$, contradicting the fact that $x \in U$ since the points x_n all lie in the closed set $X \setminus U$ and so their limit must also lie in $X \setminus U$. Define $U_1 = g(n_0, x)$ and notice that

$$x \notin X \setminus U_1 \Rightarrow x \notin \overline{\bigcup\{g(n_1, y) : y \in X \setminus U_1\}} = X \setminus U_2, \quad \text{some} \;\; n_1 \in \omega,$$

$$x \notin X \setminus U_2 \Rightarrow x \notin \overline{\bigcup\{g(n_2, y) : y \in X \setminus U_2\}} = X \setminus U_3, \quad \text{some } n_2 \in \omega,$$
$$x \notin X \setminus U_3 \Rightarrow x \notin \overline{\bigcup\{g(n_3, y) : y \in X \setminus U_3\}} = X \setminus U_4, \quad \text{some } n_3 \in \omega,$$
$$x \notin X \setminus U_4 \Rightarrow x \notin \overline{\bigcup\{g(n_4, y) : y \in X \setminus U_4\}} = X \setminus U_5, \quad \text{some } n_4 \in \omega.$$

Let $n = \max\{n_0, n_1, n_2, n_3, n_4\}$. We now show that $st(x, \mathcal{G}_n) \subseteq U_3$. If x_2 is any point in $st(x, \mathcal{G}_n)$ then there is some x_1 such that $x \in g(n, x_1)$ and $x_2 \in g(n, x_1)$, hence $x_1 \in g(n, x_2)$ and $x_1 \in g(n_3, x_2)$. If we assume that $x_2 \notin U_3$, then $x_2 \in X \setminus U_3$, so $x_1 \in \overline{\bigcup\{g(n_3, y) : y \in X \setminus U_3\}} = X \setminus U_4$. Similarly, since we have $x_1 \notin U_4$ and $x \in g(n, x_1)$ (hence $x \in g(n_4, x_1)$), then $x \in \overline{\bigcup\{g(n_4, y) : y \in X \setminus U_4\}} = X \setminus U_5$ which contradicts the fact that $x \in U_5$. This means that $x_2 \in U_3$ and so $st(x, \mathcal{G}_n) \subseteq U_3$.

The final stage of the proof is to show that $st^2(x, \mathcal{G}_n) \subseteq U$ by showing that $st^2(x, \mathcal{G}_n) \subseteq U_1$. Consider $x_4 \in st^2(x, \mathcal{G}_n)$. This means we have some point x_3 such that $x_2 \in g(n, x_3)$ and $x_4 \in g(n, x_3)$ (for some $x_2 \in st(x, \mathcal{G}_n)$), hence $x_3 \in g(n, x_4)$ and $x_3 \in g(n_1, x_4)$. If we assume that $x_4 \notin U_1$, then $x_3 \in \overline{\bigcup\{g(n_1, y) : y \in X \setminus U_1\}} = X \setminus U_2$. Similarly, since we have $x_3 \notin U_2$ and $x_2 \in g(n, x_3)$ (hence $x_2 \in g(n_2, x_3)$), then $x_2 \in \overline{\bigcup\{g(n_2, y) : y \in X \setminus U_2\}} = X \setminus U_3$ which contradicts the fact that $x_2 \in U_3$. This means that $x_4 \in U_1$ and so $st^2(x, \mathcal{G}_n) \subseteq U_1 \subseteq U$. □

We now consider some similar results where, instead of requiring convergence of sequences, we only require clustering.

THEOREM 2.2. *A space X is metrizable if and only if there is a Heath g-function g such that*

(1) *if $x \in g(n, y)$ then $y \in g(n, x)$;*
(2) *if $\{x, x_n\} \subset g(n, y_n)$ for all n then x is a cluster point of the sequence $\langle x_n \rangle$.*

PROOF. Necessity is clear. For sufficiency: since the condition (2) gives developability to the space X, we need only to prove that X is a regular and wM-space (every regular, developable, wM-space is metrizable [5]). We first prove X is regular. Let $x \in U$ be open in X. Suppose $x_n \in \overline{g(n, x)} - U$ for all $n \in \mathbb{N}$. Then $y_n \in g(n, x) \cap g(n, x_n)$ for each n. So $x \in g(n, y_n)$ and $x_n \in g(n, y_n)$. Therefore, we have $\{x, x_n\} \subset g(n, y_n)$, so x is a cluster point of the sequence $\langle x_n \rangle$. But $x \in U$ is open and $x_n \notin U$ for each n, which contradicts that x is a cluster point of the sequence $\langle x_n \rangle$. Therefore, $\overline{g(n, x)} \subset U$ for some n and X is regular.

Finally, we prove X is a wM-space. Let $x \in g(n, z_n), g(n, z_n) \cap g(n, y_n) \neq \emptyset$ and $x_n \in g(n, y_n)$. Now we want to show that $\langle x_n \rangle$ has a cluster point. Let $p_n \in g(n, z_n) \cap g(n, y_n)$. Since $p_n \in g(n, z_n)$ and $x \in g(n, z_n)$, $\{x, p_n\} \subset g(n, z_n)$. Therefore, x is a cluster point of the sequence $\langle p_n \rangle$. There is a subsequence $\langle m(n) \rangle$ of the sequence $\langle n \rangle$ such that $p_{m(n)} \in g(n, x)$, which implies that $x \in g(n, p_{m(n)})$. We have $p_{m(n)} \in g(n, y_{m(n)})$, so $y_{m(n)} \in g(m(n), p_{m(n)}) \subset g(n, p_{m(n)})$. Now $\{x, y_{m(n)}\} \subset g(n, p_{m(n)})$, so x is a cluster point of the sequence $\langle y_{m(n)} \rangle$. Therefore, there is a subsequence $\langle m(n)(k) \rangle$ of the sequence $\langle m(n) \rangle$ such that $y_{m(n)(k)} \in g(k, x)$ for all k and hence $x \in g(k, y_{m(n)(k)})$ for all k. Since

$$x_{m(n)(k)} \in g(m(n)(k), y_{m(n)(k)}) \subset g(k, y_{m(n)(k)}),$$

$\{x, x_{m(n)(k)}\} \subset g(n, y_{m(n)(k)})$ for all k and hence x is the cluster point of the sequence $\langle x_{m(n)(k)} \rangle$. Therefore, x is the cluster point of the sequence $\langle x_n \rangle$. ☐

We define $g^1(n, x) = g(n, x)$, and $g^{k+1}(n, x) = \bigcup\{g(n, y) : y \in g^k(n, x)\}$ for $k \geq 1$.

THEOREM 2.3. *A space X is metrizable if and only if there is a Heath g-function g such that*

(1) *if $x \in g(n, y)$ then $y \in g(n, x)$;*
(2) *if $x \in g^2(n, x_n)$ for all n then x is a cluster point of the sequence $\langle x_n \rangle$.*

PROOF. Let X be metrizable space with a sequence $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ of open covers of X satisfying that $\{st^2(x, \mathscr{G}_n)\}$ is a local base at x for all $x \in X$. Put $g(n, x) = st(x, \mathscr{G}_n)$ for each $x \in X$ and for each n. Then g is a COC-map which satisfies (1) and (2), because $g^2(n, x_n) = st^2(x, \mathscr{G}_n)$.

For the converse, we can prove by induction on k that if $\langle x_n \rangle$ is a sequence in X and $x \in X$ with $x_n \in g^k(n, x)$ for all n then x is a cluster point of $\langle x_n \rangle$. From this it follows that if U open with $x \in U$ then there is some n with $g^4(n, x) \subset U$. Put $\mathscr{G}_n = \{g(n, x) : x \in X\}$ for $n \in \mathbb{N}$. Then $\{st^2(x, \mathscr{G}_n)\} = g^4(n, x)$, so $\{\mathscr{G}_n\}_{n\in\mathbb{N}}$ is a sequence of open covers such that $\{st^2(x, \mathscr{G}_n) : n \in \mathbb{N}\}$ is a local base at x for all $x \in X$. Hence, by By the Moore Metrization theorem [1], X is metrizable. This completes the proof. ☐

COROLLARY 2.4. *A space X is metrizable if and only if there is a Heath g-function g such that*

(1) *if $x \in g(n, y)$ then $y \in g(n, x)$;*
(2) *$\{g^2(n, x_n) : n \in \mathbb{N}\}$ is a local basis at x for all $x \in X$.*

THEOREM 2.5. *A space X is metrizable if and only if there is a Heath g-function g such that*

(1) *if $x \in g(n, y)$ then $y \in g(n, x)$;*
(2) *$\bigcap_{n\in\mathbb{N}} \overline{g^2(n, x)} = \{x\}$;*
(3) *if $\{x, x_n\} \subset g(n, y_n)$ then the sequence $\langle x_n \rangle$ has a cluster point.*

PROOF. It is easy to prove necessity. To prove sufficiency, we need to prove that x is a cluster point of the sequence $\langle x_n \rangle$. Let q be a cluster point of $\langle x_n \rangle$. Suppose that $q \neq x$. Then there are infinitely many integer $m \geq n$ such that $x_m \in g(n, q)$. Now we have $\{x, x_m\} \subset g(n, y_m)$. By conditions (1) and (2) we get $x \in g(n, y_m)$ and $y_m \in g(n, x_m)$. Therefore, $\{x_m : m \geq n\} \subset g^2(n, x)$, so $q \in \overline{\{x_m : m \geq n\}} \subset \overline{g^2(n, x)}$. Thus $q \in \bigcap_{n \in \mathbb{N}} \overline{g^2(n, x)} = \{x\}$. Hence $q = x$, as required. □

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Department of Mathematics and Statistics
College of Science
Sultan Qaboos University
P. O. Box 36, Al-Khodh 123
Muscat
Oman
e-mail: mohamad@squ.edu.om