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BOUNDEDNESS OF SIGN-PRESERVING CHARGES, REGULARITY, AND THE COMPLETENESS OF INNER PRODUCT SPACES

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Abstract

We introduce sign-preserving charges on the system of all orthogonally closed subspaces, $F(S)$, of an inner product space S, and we show that it is always bounded on all the finite-dimensional subspaces whenever $\dim S = \infty$. When S is finite-dimensional this is not true. This fact is used for a new completeness criterion showing that S is complete whenever $F(S)$ admits at least one non-zero sign-preserving regular charge. In particular, every such charge is always completely additive.

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1. Introduction

Gleason [4] characterised the set of all σ-additive states on the system $L(H)$ of all closed subspaces of a real, complex or quaternion separable Hilbert space, H, showing that there is a one-to-one correspondence among σ-additive states, s, on $L(H)$, $3 \leq \dim H \leq \aleph_0$, and positive trace operators with unit trace, T, on H given by

$$s(M) = \mathrm{tr}(TP_M), \quad M \in L(H), \tag{1.1}$$

where P_M is the orthogonal projector from H onto M.

In the paper [4], there is an example (see (2.1) below) showing that for any finite-dimensional Hilbert space H of dimension at least three, $L(H)$ admits many unbounded charges (= signed measures). The result of Dorofeev and Sherstnev [1] that every σ-additive measure on $L(H)$ with $\dim H = \infty$ is bounded was therefore very surprising.

In what follows, we show that an analogical result can be extended to sign-preserving charges on $F(S)$ with $\dim S = \infty$, that is, for charges m satisfying that if $m(M_i)$ is strictly positive (negative) for a sequence of mutually orthogonal finite-dimensional subspaces $\{M_i\}$, then $m(\bigvee_i M_i)$ is not negative (not positive).

We recall that if S is an inner product space over real, complex or quaternion numbers, we can define two families of closed subspaces of S.

Let us denote by $F(S)$ the set of all *orthogonally closed subspaces* of S, that is,

$$F(S) = \{M \subseteq S : M^{\perp\perp} = M\},$$

where $M^{\perp} = \{x \in S : (x, y) = 0 \text{ for all } y \in M\}$. Then $F(S)$ is a complete lattice with respect to the set-theoretical inclusion [7, 2].

Let us denote by $E(S)$ the set of all *splitting* subspaces of S, that is,

$$E(S) = \{M \subseteq S : M + M^{\perp} = S\}.$$

Thus, $E(S)$ is the collection of all subspaces M of S where the projection theorem holds. Observe that every complete subspace is splitting, and $E(S) \subseteq F(S)$. In fact, S is complete if and only if $E(S) = F(S)$ (see [2]).

The paper is organised as follows. A charge on $F(S)$ is a finitely additive mapping. A charge is regular if the value of $m(M)$ for $M \in F(S)$ can be approximated by values on finite-dimensional subspaces of M. In Section 2 we characterise $P_1(S)$-bounded charges on $F(S)$—charges bounded on one-dimensional subspaces. In Section 3 we introduce sign-preserving charges, and we show that these are always bounded on all the finite-dimensional subspaces of S whenever $\dim S = \infty$.

In Section 4 we apply this result to obtain a new completeness criterion showing that S is complete if and only if $F(S)$ admits at least one non-zero sign-preserving regular charge. In addition, every such charge is of the form (1.1) for some Hermitian trace operator T (not necessary positive and of trace one), and moreover, such a regular charge is even bounded.

We recall that our completion criterion is not valid for sign-preserving charges on $E(S)$, because every $E(S)$ (also for incomplete S) admits many regular charges.

2. $P_1(S)$-bounded charges on $F(S)$

A *charge* on $F(S)$ is any mapping $m : F(S) \to \mathbb{R}$ such that

$$(*) \qquad m(M \vee N) = m(M) + m(N)$$

whenever $M, N \in F(S)$ and $M \perp N$. A positive valued charge m such that $m(S) = 1$ is said to be a *state*. A charge $m : F(S) \to \mathbb{R}$ is a *σ-additive measure* or a *completely additive measure* if $(*)$ holds for any sequence $\{M_n\}$ or any system $\{M_t\}$ of mutually orthogonal elements from $F(S)$. In a similar manner we can define a charge on $E(S)$.

We denote by $P(S)$ and $P_1(S)$ the set of all finite-dimensional and of all one-dimensional subspaces of S, respectively. We say that a charge m on $F(S)$ is

(i) *bounded* if $\sup\{|m(M)| : M \in F(S)\} < \infty$;
(ii) *$P(S)$-bounded* if $\sup\{|m(M)| : M \in P(S)\} < \infty$;
(iii) *$P_1(S)$-bounded* if $\sup\{|m(M)| : M \in P_1(S)\} < \infty$.

For example, let $\phi : \mathbb{R} \to \mathbb{R}$ be a discontinuous additive functional on $\mathbb{R}$ (see for example [5], or [2, Proposition 3.2.4]). Let us define the mapping, $m : L(H) \to \mathbb{R}$, by

$$m(M) := \phi(\operatorname{tr}(T P_M)), \quad M \in L(H), \tag{2.1}$$

where $O \neq T \neq kI$ is a Hermitian trace operator on H, $k \neq 0$. Then, for any H, $\dim H \geq 3$, m is an unbounded charge.

In a similar way, now let $0 \neq T \neq k I$ be a Hermitian trace operator on the completion $\overline{S}$ of S, where k is a non-zero real constant and I is the identity on $\overline{S}$. The mapping $m : E(S) \to \mathbb{R}$ defined by

$$m(M) = \phi(\operatorname{tr}(T P_{\overline{M}})), \quad M \in E(S), \tag{2.2}$$

is an unbounded charge on $E(S)$.

A mapping $f : \mathscr{S}(S) := \{x \in S : \|x\| = 1\} \to \mathbb{R}$ is said to be a *frame function* if there is a constant W (called the *weight* of f) such that $\sum_i f(x_i) = W$ holds for any maximal orthonormal system (MONS, for short) $\{x_i\}$ in S.

The mapping $f : \mathscr{S}(S) \to \mathbb{R}$ is said to be a *frame type function* on S if (i) for any orthonormal system (ONS, for short) $\{x_i\}$ in S, $\{f(x_i)\}$ is summable; and (ii) for any finite-dimensional subspace K of S, $f|\mathscr{S}(K)$ is a frame function on K.

The following result was originally proved for states in [6], where the first σ-additive state completeness criterion was presented, and then generalised for charges in [2, Lemma 4.2.1]. In order to be self-contained, we present the proof in details and in a little bit more general form—for $P_1(S)$-bounded charges.

LEMMA 2.1. (1) *For any $P_1(S)$-bounded charge m on $F(S)$ or $E(S)$*, $\dim S \neq 2$, *there exists a unique Hermitian operator $T = T_m : \overline{S} \to \overline{S}$ such that*

$$m(\operatorname{sp}(x)) = (Tx, x), \quad x \in \mathscr{S}(S). \tag{2.3}$$

(2) *Let v be a unit vector in the completion $\overline{S}$ of S*, $\dim S \neq 2$. *Then for any $\epsilon > 0$ and any $K > 0$, there exists a $\delta > 0$ such that the following statement holds*: *If $w \in S$*

is a unit vector such that $\|v-w\| < \delta$, *then for any* $P_1(S)$*-bounded charge* m *such that the norm of* $T = T_m$ *is less than* K, *and for each finite-dimensional* $A \subseteq S$ *satisfying the property* $v \perp A$, *we have the next inequality*

$$| m(A \vee \mathrm{sp}(w)) - m(A) - m(\mathrm{sp}(w)) | < \epsilon. \tag{2.4}$$

PROOF. (1) Suppose that m is a $P_1(S)$-bounded charge and define a function $f : \mathscr{S}(S) \to \mathbb{R}$ via $f(x) = m(\mathrm{sp}(x)),\ \|x\| = 1$. Then f is bounded on $\mathscr{S}(S)$.

Applying the Gleason theorem for finite-dimensional subspaces of S, see [2], there is a well-defined bounded bilinear form t such that $f(x) = t(x, x)$ for any $x \in \mathscr{S}(S)$. Hence, t may be uniquely extended to a bounded, bilinear form $\bar{t}$ defined on $\overline{S} \times \overline{S}$. Therefore, there is a unique Hermitian operator $T : \overline{S} \to \overline{S}$ such that (2.2) holds. We denote by $\|T\|$ the norm of T.

(2) Let $\epsilon > 0$ and $K > 0$ be given. By the continuity of the function $\rho(t) = (2 - 2(1-t^2)^{1/2})^{1/2}$ we can find a $\delta_1 > 0$ such that $\rho(t) < \epsilon/2K$ for any $t \in [0, \delta_1]$.

The continuity of the projection $P_{\mathrm{sp}(v)^\perp} : S \to \mathrm{sp}(v)^\perp$, allows us to find a $\delta \in (0,\ 1)$ such that the assumption $\|v - w\| < \delta$ implies $\|P_{\mathrm{sp}(v)^\perp}(w)\| < \delta_1$. Fix a $w \in S$ with $\|w\| = 1$, and suppose that A is any finite-dimensional subspace orthogonal to v. Then $\|P_A(w)\| = \|P_A P_{\mathrm{sp}(v)^\perp}(w)\| \le \|P_{\mathrm{sp}(v)^\perp}(w)\| \le \delta_1$. Thus, we obtain

$$\|(I - P_A)(w)/\ \|(I - P_A)(w)\| - w\| = \rho(\|P_A(w)\|) < \epsilon/2K.$$

Put $w' = (I-P_A)(w)/\|(I-P_A)(w)\|$. Then we have $\|w-w'\| < \epsilon/2K$, $A \vee \mathrm{sp}(w) = A \vee \mathrm{sp}(w')$ and $w' \perp A$. Calculate

$$\begin{aligned}
&|m(A \vee \mathrm{sp}(w)) - m(A) - m(\mathrm{sp}(w))| \\
&\quad = |m(A) + m(\mathrm{sp}(w')) - m(A) - m(\mathrm{sp}(w))| \\
&\quad = |m(\mathrm{sp}(w')) - m(\mathrm{sp}(w))| = |(Tw', w') - (Tw, w)| \\
&\quad \le |(Tw', w') - (Tw', w)| + |(Tw', w) - (Tw, w)| \\
&\quad \le 2\|T\|\, \|w - w'\| < \epsilon.
\end{aligned}$$

□

3. *P*(*S*)-boundedness of sign-preserving charges

In the present section we introduce a new kind of charges, sign-preserving charges, and we show that these are always $P(S)$-bounded. We recall that, in general, charges can be unbounded on $F(S)$, as an example below shows. This notion will be applied in the next section to obtain a new completeness criterion for inner product spaces.

We say that a charge m on $F(S)$ is *sign-preserving* (or we say also that m satisfies the *sign-preserving property*) if, for any sequence of mutually orthogonal finite-dimensional subspaces $\{M_i\}$ of S such that if $m(M_i) > 0$ for any i, we have $m\big(\bigvee_i M_i\big) \geq 0$, or $m(M_i) < 0$ for any i then $m\big(\bigvee_i M_i\big) \leq 0$.

It is easy to verify that if $m(M_i) > 0$ for any i, then

$$m\left(\bigvee_i M_i\right) \geq \sum_i m(M_i) > 0, \tag{3.1}$$

and if $m(M_i) < 0$ for any i, we have the opposite inequalities.

For example, every σ-additive measure m on $F(S)$ or every positive (negative) charge is sign-preserving. Let H be a separable infinite-dimensional Hilbert space and let m_1 and m_2 be two different states on $L(H)$ vanishing on all the finite-dimensional subspaces of H. Then $m = m_1 - m_2$ is a sign-preserving charge on $L(H)$, and m is neither positive (negative) nor σ-additive.

On the other hand, let H be a separable Hilbert space with an ONB $\{x_n\}_{n=1}^{\infty}$. Define the state $m_1(M) = \sum_{n=1}^{\infty} 1/2^n m_{x_n}(M)$, $M \in L(H)$, and let m_2 be any finitely additive state on $L(H)$ vanishing on all the finite-dimensional subspaces of H. Then $m =: m_1 - m_2$ is a bounded charge on $L(H)$ which is not sign-preserving. Indeed, let $M = \bigvee_{n=2}^{\infty} \mathrm{sp}(x_n)$. Then $m(\mathrm{sp}(x_n)) = 1/2^n$ for any n, but $m(M) = 1/2 - 1 = -1/2$. More general, if m_1 is a state defined by (1.1) and m_2 as above, then $m = m_1 - m_2$ is a bounded charge which is not sign-preserving.

Let now s be a state on $L(H)$ vanishing on all the finite-dimensional subspaces of H. According to [3], the range of s is the whole interval $[0, 1]$. Take an arbitrary discontinuous additive functional ϕ on $\mathbb{R}$. Then the mapping m on $L(H)$ defined by $m(M) = \phi(s(M))$, $M \in L(H)$, is a sign-preserving charge vanishing on all the finite-dimensional subspaces of H which is unbounded on $L(H)$.

We recall that according to [7, Lemma 33.3],

(1) $F(S)$ is an atomic, complete lattice with orthocomplementation satisfying the exchange axiom (that is, if M is an atom of $F(S)$, $N \in F(S)$, $M \not\subseteq N$, then $M \vee N$ covers N (that is, if $N \subseteq C \subseteq M \vee N$ for some $C \in F(S)$, then $C \in \{N, N \vee M\}$);
(2) if $M \in F(S)$ and $x \in S$ is a non-zero vector, then $M \vee \mathrm{sp}(x) = M + \mathrm{sp}(x) \in F(S)$;
(3) $\bigwedge M_i = \bigcap_i M_i$ for any system $\{M_i\}$ from $F(S)$.

LEMMA 3.1. *Let S be an inner product space and let N be a subspace of S,* $\dim N = n \geq 1$. *Then*

$$F(N^{\perp}) = \{A \in F(S) : A \subseteq N^{\perp}\}, \quad E(N^{\perp}) = \{A \in E(S) : A \subseteq N^{\perp}\}.$$

PROOF. If $X \subseteq S$, then $X^{\perp_{N^{\perp}}} := \{x \in N^{\perp} : x \perp X\}$. Let $\dim N = 1$ and suppose $A \in F(S)$ and $A \subseteq N^{\perp}$. Then $A^{\perp_{N^{\perp}} \perp_{N^{\perp}}} = (A^{\perp} \cap N^{\perp})^{\perp_{N^{\perp}}} = (A^{\perp} \cap N^{\perp})^{\perp} \cap N^{\perp} = (A^{\perp\perp} \vee N) \cap N^{\perp} = (A + N) \cap N^{\perp}$. Since N is an atom of $F(S)$ and $A \subseteq N^{\perp}$, $N \not\subseteq A$, we have that $(A + N) \cap N^\perp$ covers A, while $A \subseteq (A + N) \cap N^\perp \subseteq A + N$. Hence, $(A + N) \cap N^\perp = A$, that is, $A \in F(N^\perp)$.

Conversely, if $A \in F(N^\perp)$, then $A^{\perp_{N^\perp}\perp_{N^\perp}} = (A^{\perp\perp} + N) \cap N^\perp = A$. The exchange axiom implies $A^{\perp\perp} = (A^{\perp\perp} + N) \cap N^\perp = A$, that is, $A \in F(S)$.

The general case of $\dim N = n > 1$ can be obtained by n-times repeating the case $\dim N = 1$.

Let now $A \in E(S)$ and $A \subseteq N^\perp$. If $x \in N^\perp$, then $x = x_A + x_{A^\perp}$, where $x_A \in A$ and $x_{A^\perp} \in A^\perp$ so that $x - x_A = x_{A^\perp} \in A^{\perp_N}$ which gives $A + A^{\perp_N} = N^\perp$ and $A \in E(N^\perp)$.

Conversely, let $A \in E(N^\perp)$. Then $A + A^{\perp_N} = N^\perp$ and $A + A^{\perp_N} + N = N^\perp + N = S$. If $a \in A$ and $u \in A^{\perp_N}$, $v \in N$, then $(a, u + v) = 0$, that is, $A^{\perp_N} + N \subseteq A^\perp$. If now $x \in A^\perp$, then $x = x_A + x_{A^{\perp_N}} + x_N$ which gives $x_A = 0$, that is, $A^\perp \subseteq A^{\perp_N} + N$. ☐

Therefore, if $\dim N = n \geq 1$, $N \subseteq S$, then any charge m on $F(S)$ $(E(S))$ can be restricted by Proposition 3.1 to a charge $m_{N^\perp}$ on $F(N^\perp)$ $(E(N^\perp))$ by $m_{N^\perp}(M) = m(M)$ if $M \in F(N^\perp)$.

If $\dim S < \infty$, then it can happen that m is unbounded. In what follows, we show that if $\dim S = \infty$, then every sign-preserving charge on $F(S)$ is $P_1(S)$-bounded as well as $P(S)$-bounded. We will follow the basic ideas of Dorofeev-Sherstnev [1] (see also [2, Theorem 3.2.20]), who proved an analogical result for the frame-type functions.

Let us recall that if H is a Hilbert space, then by a self-adjoint operator on H we mean always an operator A defined on a subspace, S, of H which is dense in H.

Inspiring that, let us denote by $\mathrm{SPC}(H)$ the set of all $P_1(S)$-unbounded sign-preserving charges defined on $F(S)$, where S is an arbitrary dense subspace of H.

Our aim is to show that $\mathrm{SPC}(H) = \emptyset$.

LEMMA 3.2. *Let* $\mathrm{SPC}(H) \neq \emptyset$, $\dim H = \infty$. *There exist a dense subspace S of H and a charge $m \in \mathrm{SPC}(H)$ on $F(S)$ such that, for any one-dimensional subspace N of S with $|m(N)| > 1$, we have* $m_{N^\perp} \notin \mathrm{SPC}(N^{\perp_H})$.

PROOF. If $\dim N < \infty$, then $N^\perp$ is dense in $N^{\perp_H}$, where ${}^{\perp_H}$ denotes the orthocomplementation in H, and a sign-preserving charge on $F(S)$ is also a sign-preserving charge on $F(N^\perp)$.

Suppose that the assertion does not hold. Then, for any dense subspace S of H, for any charge $m \in \mathrm{SPC}(H)$ on $F(S)$, there exists a one-dimensional subspace N_1 of S with $|m(N_1)| > 1$ such that $m_{N_1^\perp} \in \mathrm{SPC}(N_1{}^{\perp_H})$.

Since H is an infinite-dimensional Hilbert space, it is isomorphic with its subspace $N_1^{\perp_H}$. Consequently, any charge from $\mathrm{SPC}(N_1^{\perp_H})$ also does not fulfil the hypothesis. In particular, for m_{N_1} and we can find a one-dimensional subspace N_2 of $N_1^\perp$ with $|m(N_2)| > 1$ such that $m_{(N_1 \vee N_2)^\perp} \in \mathrm{SPC}((N_1 \vee N_2)^{\perp_H})$.

Continuing this process by induction, we find a sequence of mutually orthogonal subspaces $\{N_n\}$ of S such that $|m(N_n)| > 1$ and $m_{(N_1 \vee \cdots \vee N_n)^\perp} \notin \mathrm{SPC}((N_1 \vee \cdots \vee N_n)^{\perp_H})$ for any $n \geq 1$.

There are infinitely many n's such that $m(N_n) > 1$ or $m(N_n) < -1$. Without loss of generality, we can assume that all $m(N_n)$ have the same sign.

Denote by $A = \bigvee_n N_n$. In the first case, for any integer $n \geq 1$, we have

$$
\begin{aligned}
m(S) = m(A^\perp) + m(A) &= m(A^\perp) + \sum_{i=1}^{n} m(N_i) + m\left(\bigvee_{i=n+1}^{\infty} N_i\right) \\
&\geq m(A^\perp) + \sum_{i=1}^{n} m(N_i) \geq m(A^\perp) + n,
\end{aligned}
$$

when we have used the sign preserving property of m, which gives a contradiction.

In a similar way we deal with the second case. □

LEMMA 3.3. *Let* $\mathrm{SPC}(H) \neq \emptyset$, $\dim H = \infty$ *There exists* $m \in \mathrm{SPC}(H)$ *and a one-dimensional subspace* X_0 *of* S, S *dense in* H, *such that*

$$
\max\left\{|m(X_0)|, \sup\{|m(Y)| : Y \in P_1(X_0^\perp)\}\right\} = 1. \tag{3.2}
$$

PROOF. Take m from Lemma 3.2 and multiplying m by some non-zero constant, if necessary, we obtain (3.2). □

Since the proofs of the following two lemmas are identical with those in [2, Lemma 3.2.18] and [2, Lemma 3.2.19], they are omitted.

LEMMA 3.4. *Let* $m \in \mathrm{SPC}(H)$, $\dim H = \infty$, *satisfy the condition of Lemma* 3.3. *Then there exist orthonormal vectors* $e_1, e_2, e_3 \in S$, S *being the dense subspace of* H, *such that* $|m(\mathrm{sp}(e_i))| > 1$ *for any* $i = 1, 2, 3$.

LEMMA 3.5. *Let* H *be a real four-dimensional Hilbert space. Let* $e_1, e_2, e_3, e \in \mathscr{S}(H)$ *such that* e_1, e_2, e_3 *are mutually orthogonal, and* $e \notin \{e_1\}^\perp \cup \{e_2\}^\perp \cup \{e_3\}^\perp$, *be given. Then there exist two non-zero vectors* x *and* y *in* H *such that*

(1) $e = x + y$;
(2) $(x, e_1) = (y, e_2) = (x, y) = (y - \|y\|^2 e, e_3) = 0$, $y - \|y\|^2 e \neq 0$.

We recall that a closed subset R of a complex or quaternion Hilbert space H which is a manifold with respect to the real field $\mathbb{R}$ is said to be *completely real* if the inner product $(\cdot, \cdot)$ from H takes real values on $R \times R$. Equivalently, if and only if there is an orthonormal set $\{e_j\}$ in R such that R is the closure of the real linear combinations of the e_j.

PROPOSITION 3.6. *Any sign-preserving charge on* $F(S)$, $\dim S = \infty$, *is* $P_1(S)$-*bounded.*

PROOF. Suppose the converse, that is, let $\mathrm{SPC}(H) \neq \emptyset$, and let $m \in \mathrm{SPC}(H)$ satisfy (3.2). Let us set $f(x) := m(\mathrm{sp}(x))$, $x \in \mathscr{S}(S)$. Select orthonormal vectors e_1, e_2, e_3 from Lemma 3.4 with $|f(e_i)| > 1$, $i = 1, 2, 3$, and define the constant

$$C = \max_{1 \le i \le 3} \left\{ |f(e_i)|, \sup\{|f(x)| : x \in \mathscr{S}(\{e_i\}^\perp)\} \right\}.$$

From the unboundedness of f it follows that there is a vector $h \in \mathscr{S}(S)$ such that $|f(h)| > 3C$. It is clear that $h \notin \bigcup_{i=1}^3 \{e_i\}^\perp$ and put $\lambda_i = (h, e_i)/|(h, e_i)|$, $i = 1, 2, 3$. Then $(h, \lambda_i e_i)$ is real for $i = 1, 2, 3$. Let M be a completely real subspace of dimension 4 containing h and all $\lambda_i e_i$'s.

Applying Lemma 3.5 to vectors $\lambda_i e_i$'s and h, we find two non-zero vectors x and y in M such that

$$(x, \lambda_2 e_2) = (y, \lambda_3 e_3) = (x, y) = (z, \lambda_1 e_1) = 0, \ \ h = x + y,$$

where $z = y - \|y\|^2 h$ is a non-zero vector. Since $\mathrm{sp}\{z, h\} = \mathrm{sp}\{x, y\} = \mathrm{sp}\{y, h\}$, we have $f(h) + f(z/\|z\|) = f(x/\|x\|) + f(y/\|y\|)$. From the construction we conclude that $z \in \{e_1\}^\perp$, so that $|f(z/\|z\|)| \le C$. Similarly, $|f(x/\|x\|)|$, $|f(y/\|y\|)| \le C$. Since $|f(h)| \le |f(h) + f(z/\|z\|)| + |f(z/\|z\|)|$, then

$$|f(h) + f(z/\|z\|)| \ge |f(h)| - |f(z/\|z\|)| > 3C - C = 2C,$$

we finally obtain from the last equality

$$2C \ge |f(x/\|x\|) + f(y/\|y\|)| = |f(h) + f(z/\|z\|)| > 2C,$$

which is a desired contradiction. □

THEOREM 3.7. *Any sign-preserving charge on* $F(S)$, $\dim S = \infty$, *is* $P(S)$-*bounded. Moreover, there is a unique Hermitian trace operator* T *on* H *such that*

$$m(\mathrm{sp}(x)) = (Tx, x), \quad x \in \mathscr{S}(S).$$

PROOF. In view of Proposition 3.6, $f(x) := m(\mathrm{sp}(x))$, $x \in \mathscr{S}(S)$, is bounded. Therefore, by (1) of Lemma 2.1, there is a Hermitian operator T on $\overline{S}$ such that $f(x) = (Tx, x)$, $x \in \mathscr{S}(S)$.

We now show that $T \in \mathrm{Tr}(H)$. If $T = 0$, the statement is evident. Let now $T \neq 0$ and suppose $T \notin \mathrm{Tr}(H)$. Then there is an ONS $\{f_1, \ldots, f_{n_1}\}$ in H such that $\sum_{k=1}^{n_1} |(Tf_k, f_k)| > 1$. Choose an $\epsilon > 0$ such that $\sum_{k=1}^{n_1} |(Tf_k, f_k)| > 1 + \epsilon$. It is easy to see that for $\{f_1, \dots, f_{n_1}\}$ we can find an ONS $\{h_1, \dots, h_{n_1}\}$ in S such that $\|h_k - f_k\| < \epsilon/(2n_1\|T\|)$, $k = 1, \dots, n_1$. Then

$$\begin{aligned} |f(h_k) - (Tf_k, f_k)| &\leq |(T(h_k - f_k), f_k)| + |(Th_k, h_k - f_k)| \\ &\leq 2\|T\|\|h_k - f_k\| < \epsilon/n_1, \end{aligned}$$

so that

$$\sum_{k=1}^{n_1} |f(h_k)| \geq \sum_{k=1}^{n_1} |(Tf_k, f_k)| - \sum_{k=1}^{n_1} |(Tf_k, f_k) - f(h_k)| > 1.$$

Put $H_1 = \{h_1, \dots, h_{n_1}\}^{\perp_H}$, then $S_1 = H_1$ is a dense subspace in H_1, so that, $m|F(S_1)$ is a sign-preserving charge on $F(S_1)$. Therefore, as in the beginning of the present proof, there is a Hermitian operator T_1 $(= P_{H_1}TP_{H_1})$ on H_1 such that $f(x) = (T_1x, x) = (Tx, x)$, $x \in \mathscr{S}(S_1)$. Here T_1 is not any trace operator since $T \notin \mathrm{Tr}(H)$.

Repeating the same reasonings as above, we find an ONS $\{f_{n_1+1}, \dots, f_{n_2}\}$ in H_1 such that $\sum_{k=n_1+1}^{n_2} |(Tf_k, f_k)| > 1$, and we find an ONS $\{h_{n_1+1}, \dots, h_{n_2}\}$ in S_1 with $\sum_{k=n_1+1}^{n_2} |f(h_k)| > 1$. Continuing this process, we find a countable family of orthonormal vectors $\{h_1, h_2, \dots\} \subset S$ and a sequence of integers, $\{n_i\}_{i=0}^{\infty}$, $n_0 = 0$, such that $\sum_{k=n_{i-1}+1}^{n_i} |f(h_k)| > 1$, for any $i \geq 1$, which gives $\sum_{k=1}^{\infty} |f(h_k)| = \infty$.

Without loss of generality, we can assume that all $f(h_n) > 0$ or $f(h_n) < 0$. Set $A = \bigvee_n \mathrm{sp}(h_n)$. In the first case, for any $k \geq 1$,

$$m(S) = m(A^{\perp}) + \sum_{i=1}^{k} \sum_{j=n_{i-1}+1}^{n_i} m(\mathrm{sp}(h_j)) + m\left(\bigvee_{i>n_k} \mathrm{sp}(h_i)\right) \geq m(A^{\perp}) + k,$$

which is a contradiction. In a similar way we deal with the second case. Therefore, $T \in \mathrm{Tr}(H)$, and this proves that m is $P(S)$-bounded. ☐

4. Sign-preserving regular charges and completeness criterion

In this section, we present a new completeness criterion showing that S is complete if and only if $F(S)$ admits at least one non-zero sign-preserving regular charge. This result extends measure-type completeness criteria given, for example, in [2, Section 4.3.2].

We say that a charge m on $F(S)$ $(E(S))$ is *regular* if, given $M \in F(S)$ $(M \in E(S))$ and given $\epsilon > 0$, there is a finite-dimensional subspace N of M such that

$$|m(M \cap N^{\perp})| < \epsilon.$$

THEOREM 4.1. *An inner product space S is complete if and only if $F(S)$ admits at least one non-zero sign-preserving regular charge.*

PROOF. The necessity is evident. Suppose, therefore, that S is an infinite-dimensional inner product space, and let m be a non-zero sign-preserving regular charge. According to Theorem 3.7, m is $P(S)$-bounded. Let T be a Hermitian operator from (2.3).

Let B be an arbitrary orthogonally closed subspace of S and let $\{e_i\}$ be any MONS in B and define $B_0 = \{e_i\}^{\perp\perp}$. Then $B_0 \subseteq B$. We claim that $B_0 = B$.

We see that

$$(\star) \qquad m(B_0) = m(B_0) + m(B \cap B_0^{\perp}) = m(B_0) + 1 - m(B^{\perp} \vee B_0) = m(B)$$

(which is true for any charge m on $F(S)$).

If we had $B_0 \neq B$, then $\overline{B}_0 \neq \overline{B}$, and we can find a unit vector $v \in \overline{B}$ which is orthogonal to $\overline{B}_0$. There exists a unit vector $e \in S$ such that $m(\mathrm{sp}(e)) \neq 0$. Indeed, there exists $M \in F(S)$ such that, say, $m(M) > 0$. Given M, we find a sequence $\{M_n\}$ in $P(S)$ of non-decreasing subspaces of M such that $m(M) = \lim_n m(M_n)$. Without loss of generality we can assume that $m(\mathrm{sp}(e)) > 0$. Applying Lemma 2.1 to $\epsilon = m(\mathrm{sp}(e))/3 > 0$ and to $v \in \overline{B}$, we can find a $\delta > 0$ such that, for any unit vector $w \in B$ with $\|w - v\| < \delta$ and any $A \perp v$, $\dim A < \infty$, we have (2.4) for every $P_1(S)$-bounded charge s on $F(S)$ for which $\|T_s\| = \|T\|$.

Define a unitary operator $U : S \to S$ such that $Ue = w$ and $Uf = f$ for any $f \perp e, w$. Then m_U defined via $m_U(M) = m(U^{-1}(M))$, $M \in F(S)$, is a $P_1(S)$-bounded, regular charge on $F(S)$ for which $\|T_{m_U}\| = \|T\|$.

Hence, for B there exists a sequence $\{B_n\}$ of finite-dimensional subspaces of B, $B_n \subseteq B_{n+1}$ for $n \geq 1$, such that $m_U(B) = \lim_n m_U(B_n)$.

We assert that $m_U(B) = \lim_n m_U(B_n \vee \mathrm{sp}(w))$.

Calculate,

$$|m_U(B_n \vee \mathrm{sp}(w)) - m_U(B)| \leq |m_U(B_n \vee \mathrm{sp}(w)) - m_U(B_n)| + |m_U(B_n) - m_U(B)|.$$

We now follow the ideas and symbols from the proof of (2) of Lemma 2.1 with norm $\|T\|$ less than a constant $K > 0$. Let $\epsilon > 0$ be given. Set

$$w_n' = (I - P_{B_n}(w))/\|(I - P_{B_n}(w)\|.$$

Then $\|w - w_n'\| < \epsilon/2K$, $B_n \vee \mathrm{sp}(w) = B_n \vee \mathrm{sp}(w_n')$, and $w_n' \perp B_n$. Hence,

$$\begin{aligned}
&|m_U(B_n \vee \mathrm{sp}(w)) - m_U(B_n)| \\
&\quad = |m_U(\mathrm{sp}(w_n'))| = |(T_{m_U} w_n', w_n')| \\
&\quad \leq |(T_{m_U} w_n', w_n') - (T_{m_U} w_n', w)| + |(T_{m_U} w_n', w) - (T_{m_U} w, w)| \\
&\quad \leq \|T_{m_U}\| \|w_n'\| \|w_n' - w\| + \|T_{m_U}\| \|w_n' - w\| \|w\| \leq \epsilon.
\end{aligned}$$

Consequently, $m_U(B) = \lim_n m_U(B_n \vee \mathrm{sp}(w))$, and by $(\star)$, $m_U(B) = m_U(B_0 \vee \mathrm{sp}(w))$.

Therefore, given $\epsilon > 0$ there is an integer n_0 such that for any $n > n_0$

$$m_U(B_n \vee \mathrm{sp}(w)) - \epsilon < m_U(B_0 \vee \mathrm{sp}(w)) < m_U(B_n \vee \mathrm{sp}(w)) + \epsilon$$

and

$$m_U(B_0) - \epsilon < m_U(B_n) < m_U(B_0) + \epsilon.$$

Using these inequalities and (2.4), we get

$$\begin{aligned} m_U(B_0) &= m_U(B_0 \vee \mathrm{sp}(w)) > m_U(B_n \vee \mathrm{sp}(w)) - \epsilon \\ &> m_U(B_n) + m_U(\mathrm{sp}(w)) - 2\epsilon > m_U(B_0) + m(\mathrm{sp}(e)) - 3\epsilon = m_U(B_0), \end{aligned}$$

which contradicts the beginning and the end of former inequalities, and this proves $B_0 = B$.

Due to the arbitrariness of $B \in F(S)$, we conclude that $F(S)$ is orthomodular. The criterion of Amemiya and Araki [2, Theorem 4.1.2], yields that S is complete, as claimed. □

THEOREM 4.2. *Any sign-preserving regular charge on $F(S)$ of an inner product space S, $\dim S = \infty$, is completely additive, and there is a trace operator T on $\overline{S}$ such that $m(M) = \mathrm{tr}(TP_M)$, $M \in F(S)$. In addition, the regular charge is always bounded.*

PROOF. If m is a zero function, the statement is trivially satisfied. Suppose that m is a non-zero sign-preserving regular charge.

According to Theorem 4.1, S is a Hilbert space, and due to (i) of Lemma 3.2, there is a Hermitian operator T on S such that $(Tx, x) = m(\mathrm{sp}(x))$ for any unit vector $x \in S$. Moreover, by Theorem 3.7, T is a trace operator on S.

Express $T = T^+ - T^-$, where T^+ and T^- are positive and negative parts of T. Let S^+, S^- and S_0 be the subspaces of S generated $\{x_i : \lambda_i > 0\}$, $\{x_i : \lambda_i < 0\}$, and $\{x_i : \lambda_i = 0\}$, respectively, where $T = \sum_i \lambda_i(\cdot, x_i)x_i$. Then, for any unit vector $x \in S^+$, $m(\mathrm{sp}(x)) > 0$ and, for any unit vector $y \in S^-$, $m(\mathrm{sp}(y)) < 0$. Therefore, $m(S^+) = \lim_n m(S_n)$, where $S_n \subseteq S_{n+1}$ are finite-dimensional subspaces of S^+. Hence, $m(S^+) \geq \sum_i m(\mathrm{sp}(x_i))$ for any ONB $\{x_i\}$ in S^+ which implies $m(S^+) = \mathrm{tr}(T^+)$. In a similar way, we have $m(S^-) = -\mathrm{tr}(T^-)$. Since $m(S_0) = 0$, we have $m(S) = \mathrm{tr}(T)$.

If now M is an arbitrary subspace of $F(S)$, then T_M is the restriction of $P_M T P_M$ onto M, where P_M is the orthogonal projector of S onto M, is a trace operator. We repeat the above reasoning for T_M. Hence, $m(M) = \mathrm{tr}(T_M) = \mathrm{tr}(TP_M)$, $M \in F(S)$.

It is easy to show that the mapping $M \mapsto \mathrm{tr}(TP_M)$, $M \in F(S)$, is a completely additive function on $F(S)$ and bounded. □

We recall that Theorem 4.1 does not hold for the case of $E(S)$. Indeed, let x be a unit vector in S. The mapping $m_x(M) = \|x_M\|^2$, $M \in E(S)$, where $x = x_M + x_{M^\perp}$ and $x_M \in M$, $x_{M^\perp} \in M^\perp$, is a regular charge on $E(S)$ for any complete or incomplete S.

We conclude the article with some comments.

(1) We recall that we do not know whether any regular charge on $F(S)$ is sign-preserving.

(2) If a regular charge is $P_1(S)$-bounded, then Theorem 4.1 holds for any $P_1(S)$-bounded regular charge.

(3) We do not know whether every regular charge on $F(S)$ with $\dim S = \infty$ is $P_1(S)$-bounded. This is unknown even if S is a Hilbert space.

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