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FINITE-DIMENSIONAL ODD HAMILTONIAN SUPERALGEBRAS OVER A FIELD OF PRIME CHARACTERISTIC

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Abstract

Let $\mathscr{H}(m;\underline{t})$ be the finite-dimensional odd Hamiltonian superalgebra over a field of prime characteristic. By determining ad-nilpotent elements in the even part, the natural filtration of $\mathscr{H}(m;\underline{t})$ is proved to be invariant in the following sense: If $\varphi : \mathscr{H}(m;\underline{t}) \to \mathscr{H}(m';\underline{t}')$ is an isomorphism then $\varphi(\mathscr{H}(m;\underline{t})_i) = \mathscr{H}(m';\underline{t}')_i$ for all $i \geq -1$. Using the result, we complete the classification of odd Hamiltonian superalgebras. Finally, we determine the automorphism group of the restricted odd Hamiltonian superalgebra and give further properties.

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As is well known, filtration structures provide useful tools in the research of Lie algebras and Lie superalgebras. In particular, they play an important role in the classifications of finite-dimensional simple modular Lie algebras and finite-dimensional simple Lie superalgebras of characteristic zero respectively (see [2, 5, 7, 21, 17]). We know that Cartan-type Lie algebras and Lie superalgebras possess natural filtration structures. By means of invariance of filtrations one can characterize intrinsic properties of Cartan-type Lie algebras and Lie superalgebras and determine the automorphism groups (see [22, 16, 24, 26]). In the case of Cartan-type modular Lie algebras, it is proved in [10] that the filtration of $X(m : \underline{1})$ is invariant under $\operatorname{Aut} X(m : \underline{1})$, where $X = W, S, H$ or K, and the same conclusion is obtained in [6] for all Cartan-type Lie algebras; by means of ad-nilpotent elements, the natural filtrations of infinite-dimensional Cartan-type Lie algebras are proved to be invariant under the automorphism groups (see [4]). In the case of characteristic zero, the natural filtrations of infinite-dimensional Lie algebras $X(m)$ is invariant, where $X = W, S, H$ or K (see [14]). In [23] the author discussed the simplicity and restrictiveness of the

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four classes of finite-dimensional modular Cartan-type Lie superalgebras. In [24] and [25], the invariance of natural filtrations of Hamiltonian superalgebras, generalized Witt superalgebras and special superalgebras are determined by means of image-space dimensions and ad-nilpotent elements, respectively.

In this paper, we discuss the finite-dimensional odd Hamiltonian superalgebra $\mathscr{H}(m;\underline{t})$ over a field of positive characteristic. In the case of characteristic zero, the infinite-dimensional odd Hamiltonian superalgebra $\mathscr{H}(m,m)$, which is defined by odd Hamiltonian differential forms, is even transitive irreducible simple Lie superalgebra (see [8, Theorem 4.1]). This Lie superalgebra was interpreted as the Lie superalgebra of polyvector fields on an m-dimensional space (see [1]). It was introduced in [11] by Leites, and was later called Leites superalgebra (see [9]). Paper [12] gave a description of the outer derivations of this superalgebra.

We denote the natural filtration of $\mathscr{H}(m;\underline{t})$ by $\{\mathscr{H}(m;\underline{t}), i\geq -1\}$. An isomorphism between any two odd Hamiltonian superalgebras is called f-isomorphism. In Section 2, we determine the ad-nilpotent elements with certain properties in the even part of $\mathscr{H}(m;\underline{t})$. The results are used in Section 3 to prove that the filtration of $\mathscr{H}(m;\underline{t})$ is invariant under any f-isomorphisms; that is, if $\varphi:\mathscr{H}(m;\underline{t})\to\mathscr{H}(m';\underline{t}')$ is an isomorphism then $\varphi(\mathscr{H}(m;\underline{t})_i)=\mathscr{H}(m';\underline{t}')_i$ for all $i\geq -1$. As a result, we complete the classification of odd Hamiltonian superalgebras. In Section 4, we first prove the automorphism group of the restricted odd Hamiltonian superalgebra $\mathscr{H}$ is isomorphic to $\mathrm{Aut}(\mathscr{U}:\mathscr{H})$, the admissible automorphism group of the base superalgebra $\mathscr{U}$. Then it is proved that the so-called standard normal series of $\mathrm{Aut}\,\mathscr{H}$ is sent to the one of $\mathrm{Aut}(\mathscr{U}:\mathscr{H})$. More detailed properties of $\mathrm{Aut}\,\mathscr{H}$ are also discussed. The works in this section are motivated by the results and methods involved in Lie algebras (see [19, 20, 4]), and based on [25, Theorem 1].

1. Preliminaries

1.1. Notation and conventions The following notation and conventions are used throughout this paper:

- $\mathbb{F}$ denotes the underlying field of characteristic $p>2$, $\mathbb{Z}_2$ the ring of integers modulo 2; $\mathbb{N}$ and $\mathbb{N}_0$ the positive integer set and nonnegative integer set, respectively.
- Fix $m\in\mathbb{N}\setminus\{1,2\}$.
- $\mathrm{U}(m)$ denotes the divided power algebra over $\mathbb{F}$ with the $\mathbb{F}$-basis $\{x^{(\alpha)}\mid\alpha\in\mathbb{N}_0^m\}$.
- $\Lambda(m)$ denotes the Grassmann superalgebra in m variables $x_{m+1},x_{m+2},\ldots,x_{2m}$.
- Denote the tensor product by $\Lambda(m,m):=\mathrm{U}(m)\otimes_{\mathbb{F}}\Lambda(m)$.
- We abbreviate $g\otimes f$ to gf where $g\in\mathrm{U}(m)$, $f\in\Lambda(m)$, and $x^{(\varepsilon_i)}$ to x_i, where $\varepsilon_i:=(\delta_{i1},\delta_{i2},\ldots,\delta_{im})$.
- Set $Y_0:=\{1,2,\ldots,m\}$, $Y_1:=\{m+1,m+2,\ldots,2m\}$ and $Y:=Y_0\cap Y_1$.

- Set $B_k := \{(i_1, \ldots, i_k) \mid m+1 \le i_1 < i_2 < \cdots < i_k \le 2m\}$, $B(m) := \bigcup_{k=0}^{m} B_k$, where $B_0 := \emptyset$. For $u \in B_k$, put $|u| := k$, $\{u\} := \{i_1, \ldots, i_k\}$, $x^u := x_{i_1} x_{i_2} \cdots x_{i_k}$, $x^{\emptyset} := \mathbf{1}$.
- Obviously, $\{x^{(\alpha)} x^u \mid \alpha \in \mathbb{N}_0^m, u \in B(m)\}$ is an $\mathbb{F}$-basis of $\Lambda(m, m)$.
- Define $D_1, \ldots, D_{2m}$ to be linear transformations of $\Lambda(m, m)$ such that

$$D_i(x^{(\alpha)} x^u) = \begin{cases} x^{(\alpha-\varepsilon_i)} x^u & i \in Y_0; \\ x^{(\alpha)} \partial x^u / \partial x_i & i \in Y_1, \end{cases}$$

where $x^{\beta} := 0$ whenever $\beta \notin \mathbb{N}_0^m$.
- If $\deg(x)$ occurs in this paper, we always regard x as a $\mathbb{Z}_2$-homogeneous element and $\deg(x)$ the $\mathbb{Z}_2$-degree of x.
- Define

$$\mu(i) := \begin{cases} \overline{0} & i \in Y_0; \\ \overline{1} & i \in Y_1. \end{cases}$$

- For $\underline{t} = (t_1, \ldots, t_m) \in \mathbb{N}^m$, put $\pi := (\pi_1, \ldots, \pi_m)$ where $\pi_i := p^{t_i} - 1$, $i \in Y_0$, and $A(m; \underline{t}) := \{\alpha \in \mathbb{N}_0^m \mid \alpha_i \le \pi_i,\ i \in Y_0\}$.
- Set

$$i' = \begin{cases} i + m & i \in Y_0; \\ i - m & i \in Y_1. \end{cases}$$

- Let $\xi := |\pi| + m = \sum_{i \in Y_0} p^{t_i}$.

1.2. The construction processes We know that $\Lambda(m, m)$ is an associative superalgebra with a $\mathbb{Z}_2$-gradation induced by the trivial $\mathbb{Z}_2$-gradation of $\mathrm{U}(m)$ and the natural $\mathbb{Z}_2$-gradation of $\Lambda(m)$. The following formulae hold in $\Lambda(m, m)$:

$$\begin{aligned} x^{(\alpha)} x^{(\beta)} &= \binom{\alpha+\beta}{\alpha} x^{(\alpha+\beta)}, && \alpha, \beta \in \mathbb{N}_0^m; \\ x_i x_j &= -x_j x_i, && i, j \in Y_1; \\ x^{(\alpha)} x_j &= x_j x^{(\alpha)}, && \alpha \in \mathbb{N}_0^m,\ j \in Y_1. \end{aligned}$$

Clearly, $D_1, \ldots, D_{2m}$ are superderivations of $\Lambda(m, m)$. Let

$$W(m, m) = \left\{ \sum_{i \in Y} a_i D_i \;\middle|\; a_i \in \Lambda(m, m),\ i \in Y \right\}.$$

Then $W(m, m)$ is an infinite-dimensional Lie superalgebra (see [23]), which is a subalgebra of $\mathrm{Der}_{\mathbb{F}}(\Lambda(m, m))$. We note that $W(m, m)$ is free $\Lambda(m, m)$-module with a $\Lambda(m, m)$-basis $\{D_1, \ldots, D_{2m}\}$.

The following formula holds in $W(m,m)$:

(1) $$[aD, bE] = aD(b)E - (-1)^{\deg(aD)\deg(bE)} bE(a)D + (-1)^{\deg(D)\deg(b)} ab[D,E].$$

Consequently,

(1′) $$\left[aD_i, bD_j\right] = aD_i(b)D_j - (-1)^{\deg(aD_i)\deg(bD_j)} bD_j(a)D_i$$

where $a, b \in \Lambda(m,m)$, $D, E \in W(m,m)$, $i, j \in Y$.

From the definition of $A(m;\underline{t})$, we obtain that

$$\Lambda(m,m;\underline{t}) := \mathrm{span}_{\mathbb{F}}\{x^{(\alpha)}x^u \mid \alpha \in A(m;\underline{t}), u \in B(m)\}$$

is a finite-dimensional subalgebra of $\Lambda(m,m)$. Set

$$W(m,m;\underline{t}) = \left\{ \sum_{i\in Y} a_i D_i \,\middle|\, a_i \in \Lambda(m,m;\underline{t}),\ i \in Y \right\},$$

then $W(m,m;\underline{t})$ is a finite-dimensional subalgebra of $W(m,m)$ (see [23]).

Define $\mathrm{T_H}(a) = \sum_{i\in Y}(-1)^{\mu(i)\deg(a)} D_i(a)D_{i'}$, where $a \in \Lambda(m,m;\underline{t})$. Then $\mathrm{T_H}$ is an odd linear mapping from $\Lambda(m,m;\underline{t})$ to $W(m,m;\underline{t})$, that is, $\mathrm{T_H}(\Lambda(m,m;\underline{t})_\theta) \subset W(m,m;\underline{t})_{\theta+\bar{1}}$, for $\theta \in \mathbb{Z}_2$. Let $\mathscr{H}(m;\underline{t}) = \{\mathrm{T_H}(a) \mid a \in \Lambda(m,m;\underline{t})\}$. Then $\mathscr{H}(m;\underline{t})$ is a subalgebra of $W(m,m;\underline{t})$, which is called odd Hamiltonian superalgebra (see [8, page 27]). We have the following formula (see [8, page 28]):

(2) $$\left[\mathrm{T_H}(a),\ \mathrm{T_H}(b)\right] = \mathrm{T_H}(\mathrm{T_H}(a)(b)).$$

Recall the natural $\mathbb{Z}$-gradations of $\Lambda(m,m;\underline{t})$ and $W(m,m;\underline{t})$:

$$\begin{aligned}
\Lambda(m,m;\underline{t}) &= \bigoplus_{i=0}^{\xi} \Lambda(m,m;\underline{t})_{[i]}, \quad \text{where}\\
\Lambda(m,m;\underline{t})_{[i]} &= \mathrm{span}_{\mathbb{F}}\{x^{(\alpha)}x^u \mid |\alpha| + |u| = i, \alpha \in A(m;\underline{t}),\ u \in B(m)\};\\
W(m,m;\underline{t}) &= \bigoplus_{i=-1}^{\xi-1} W(m,m;\underline{t})_{[i]}, \quad \text{where}\\
W(m,m;\underline{t})_{[i]} &= \mathrm{span}_{\mathbb{F}}\{a_j D_j \mid a_j \in \Lambda(m,m;\underline{t})_{[i+1]},\ j \in Y\}.
\end{aligned}$$

It is easy to verify that $\mathscr{H}(m;\underline{t})$ is a $\mathbb{Z}$-graded subalgebra of $W(m,m;\underline{t})$

$$\begin{aligned}
\mathscr{H}(m;\underline{t}) &= \bigoplus_{i=-1}^{\xi-2} \mathscr{H}(m;\underline{t})_{[i]}, \quad \text{where}\\
\mathscr{H}(m;\underline{t})_{[i]} &= \mathscr{H}(m;\underline{t}) \cap W(m,m;\underline{t})_{[i]}\\
&= \{\mathrm{T_H}(a) \mid a \in \Lambda(m,m;\underline{t})_{[i+2]}\}.
\end{aligned}$$

Set $W(m,m;\underline{t})_i=\bigoplus_{j\geq i}W(m,m;\underline{t})_{[j]}$, $\mathscr{H}(m;\underline{t})_i=\bigoplus_{j\geq i}\mathscr{H}(m;\underline{t})_{[j]}$. Recall that $\{W(m,m;\underline{t})_i, i\geq -1\}$ and $\{\mathscr{H}(m;\underline{t})_i, i\geq -1\}$ are said to be the natural filtrations of $W(m,m;\underline{t})$ and $\mathscr{H}(m;\underline{t})$, respectively.

From now on, we frequently abbreviate $W(m,m;\underline{t})$ and $\mathscr{H}(m;\underline{t})$ to W and $\mathscr{H}$, respectively.

2. The ad-nilpotent elements in $\mathscr{H}_{\bar{0}}$

Let L be a Lie superalgebra and S a nonempty subset of L. Recall that an element x of S is called ad-nilpotent, if $\operatorname{ad}x$ is a nilpotent linear transformation of L. We denote by $\operatorname{nil}(S)$ the set of ad-nilpotent elements in S.

For $\mathscr{H}(m;\underline{t})$ where $m\in\mathbb{N}\setminus\{1,2\}$ and $\underline{t}\in\mathbb{N}^m$, define

$$\begin{aligned}\Omega &:= \{E\in\operatorname{nil}(\mathscr{H}_{\bar{0}})\mid(\operatorname{ad}E)(\mathscr{H})\subset\operatorname{nil}(\mathscr{H})\},\\ \Gamma &:= \{E\in\operatorname{nil}(\mathscr{H}_{\bar{0}})\mid(\operatorname{ad}E)(\Omega)\subset\Omega\},\\ \Phi &:= \{E\in\mathscr{H}\mid(\operatorname{ad}E)(\mathscr{H}_1\cap\mathscr{H}_{\bar{0}})\subset\operatorname{nil}(\mathscr{H})\}.\end{aligned}$$

Let $m'\in\mathbb{N}\setminus\{1,2\}$, $\underline{t}'\in\mathbb{N}^{m'}$. For $\mathscr{H}(m';\underline{t}')$, the corresponding sets are denoted by Ω', Γ' and Φ', respectively.

Proceeding analogously to [18, Theorem 1.3.1] or [3, Theorem 2.1], we may prove the following lemma.

Lemma 2.1. *Let L be a finite-dimensional Lie superalgebra, and S a Lie subset of L, that is, S is closed under the multiplication of L. If $S\subset\operatorname{nil}(L)$, then $\operatorname{span}_{\mathbb{F}}S\subset\operatorname{nil}(L)$.*

For $\mathbb{Z}$-graded Lie superalgebras we have the following lemma.

Lemma 2.2. *Let L be a $\mathbb{Z}$-graded Lie superalgebra. Suppose that $x\in\operatorname{nil}(L)$. Then $\mathrm{m}_{\mathbb{Z}}(x)\in\operatorname{nil}(L)$, where $\mathrm{m}_{\mathbb{Z}}(x)$ is the nonzero $\mathbb{Z}$-component of x possessing the minimal $\mathbb{Z}$-degree.*

Proof. See [25, Lemma 2]. □

Now we return to the case of $\mathscr{H}(m;\underline{t})$.

Lemma 2.3. *Suppose that $a\in\Lambda(m,m;\underline{t})$. Then $\mathrm{T_H}(a)\in\operatorname{nil}(\mathscr{H})$ if and only if $\mathrm{T_H}(a)$ is a nilpotent transformation of $\Lambda(m,m;\underline{t})$.*

Proof. Let $b\in\Lambda(m,m;\underline{t})$. Applying (2) we obtain by induction on k that

$$\big(\operatorname{ad}\mathrm{T_H}(a)\big)^k\big(\mathrm{T_H}(b)\big)=\mathrm{T_H}\Big(\big(\mathrm{T_H}(a)\big)^k(b)\Big)\quad\text{for all }\ k\in\mathbb{N}.$$

Combining this with the fact $\operatorname{Ker}\mathrm{T_H}=\mathbb{F}\cdot 1$, we obtain the desired result. □

Since $\mathscr{H}$ is finite-dimensional, it is clear that $\mathscr{H}_{[-1]} \cup \mathscr{H}_1 \subset \mathrm{nil}(\mathscr{H})$. For the ad-nilpotent elements of $\mathscr{H}_{[0]}$, we have the following result.

LEMMA 2.4. *Let* $i, j \in Y$. *Then* $\mathrm{T_H}(x_i x_j) \in \mathrm{nil}(\mathscr{H})$ *if and only if* $i' \neq j$.

PROOF. By the definition of $\mathrm{T_H}$, we have

$$\mathrm{T_H}(x_i x_j) = (-1)^{\mu(i)+\mu(i)\mu(j)} x_j D_{i'} + (-1)^{\mu(j)} x_i D_{j'}. \tag{3}$$

Clearly, $x_i^p = x_j^p = 0$. Suppose that $i' \neq j$. It is easy to see that $(x_j D_{i'})^p = (x_i D_{j'})^p = 0$. From (1′), we have $[x_j D_{i'}, x_i D_{j'}] = 0$. In combination with (3), we have $(\mathrm{T_H}(x_i x_j))^{2p} = 0$. By virtue of Lemma 2.3, we obtain that $\mathrm{T_H}(x_i x_j) \in \mathrm{nil}(\mathscr{H})$, as desired.

Conversely, assume that $\mathrm{T_H}(x_i x_j) \in \mathrm{nil}(\mathscr{H})$ with $i' = j$. Without loss of generality, we may assume that $i \in Y_0$. By (3), $\mathrm{T_H}(x_i x_{i'}) = x_{i'} D_{i'} - x_i D_i$. Note that

$$(\mathrm{T_H}(x_i x_j))^k (x_{i'}) = x_{i'} \quad \text{for all } \ k \in \mathbb{N}.$$

Therefore, $\mathrm{T_H}(x_i x_{i'})$ is not a nilpotent transformation of $\Lambda(m, m; \underline{t})$, which contradicts Lemma 2.3. □

LEMMA 2.5. *Suppose that* $E_{[0]} \in \mathrm{nil}(\mathscr{H}_{[0]})$ *and* $[E_{[0]}, E_{[0]}] = 0$. *Then* $E_{[0]} + E_1 \in \mathrm{nil}(\mathscr{H})$ *for all* $E_1 \in \mathscr{H}_1$.

PROOF. Clearly, $\{E_{[0]}\} \cup \mathscr{H}_1$ is a Lie subset of $\mathscr{H}$, in which all elements are ad-nilpotent. By Lemma 2.1, $\mathrm{span}_{\mathbb{F}}(\{E_{[0]}\} \cup \mathscr{H}_1) \subset \mathrm{nil}(\mathscr{H})$. In particular, $E_{[0]} + E_1 \in \mathrm{nil}(\mathscr{H})$ for all $E_1 \in \mathscr{H}_1$. □

We shall prove that $\Omega \subset \mathscr{H}_1$. First we make the following preparatory remarks.

Consider $\mathscr{H}_{[0]}$-module $\mathscr{H}_{[-1]}$, and denote by ρ the corresponding representation, that is, $\rho(E) = (\mathrm{ad}\, E)\mid_{\mathscr{H}_{[-1]}}$, $E \in \mathscr{H}_{[0]}$. Fix the $\mathbb{F}$-basis $\{D_1, \ldots, D_{2m}\}$ of $\mathscr{H}_{[-1]}$. For $E \in \mathscr{H}_{[0]}$, we identify $\rho(E)$ with its matrix with respect to the fixed basis. Let $\mathrm{pl}(m, m)$ denote the general linear Lie superalgebra of $2m \times 2m$ matrices over $\mathbb{F}$ (see [15]). Let

$$\tilde{\mathrm{p}}(m) = \left\{ \begin{bmatrix} A & B \\ C & -A^{\mathrm{T}} \end{bmatrix} \in \mathrm{pl}(m, m) \;\middle|\; B = B^{\mathrm{T}}, C = -C^{\mathrm{T}} \right\}.$$

Then $\tilde{\mathrm{p}}(m)$ is a subalgebra of $\mathrm{pl}(m, m)$ (see [8, page 16]).

In the following e_{ij} denotes the $2m \times 2m$ matrix having 1 in (i, j) position and 0's elsewhere. The following lemma only needs straightforward verifications, which are omitted.

LEMMA 2.6. *The following statements hold*:

(i) $\mathrm{T_H}(x_{i'}x_j) = (-1)^{\mu(i')+\mu(i')\mu(j)}x_j D_i + (-1)^{\mu(j)}x_{i'}D_{j'}$, $i, j \in Y$.

(ii) $\rho(\mathrm{T_H}(x_{i'}x_j)) = (-1)^{\mu(i)}e_{ij} - (-1)^{\mu(i)\mu(j)}e_{j'i'}$, $i, j \in Y$.

(iii) ρ *is faithful.*

(iv) $\mathrm{Im}(\rho) = \tilde{\mathrm{p}}\,(m)$.

(v) *If* $E \in \mathrm{nil}(\mathscr{H}_{[0]})$ *then* $\rho(E)$ *is a nilpotent matrix.*

THEOREM 2.7. *Suppose that* $E \in \mathrm{nil}(\mathscr{H}_{\bar{0}})$ *and* $\mathrm{ad}\,E(\mathscr{H}) \subset \mathrm{nil}(\mathscr{H})$. *Then* $E \in \mathscr{H}_1$, *that is,* $\Omega \subset \mathscr{H}_1 \cap \mathscr{H}_{\bar{0}}$.

PROOF. Decompose $E = E_{[-1]} + E_0$, where $E_{[-1]} \in \mathscr{H}_{[-1]} \cap \mathscr{H}_{\bar{0}}$, $E_0 \in \mathscr{H}_0$. Let $E_{[-1]} = \sum_{i\in Y_0} c_i\,\mathrm{T_H}(x_{i'})$, $c_i \in \mathbb{F}$. Assume that $E_{[-1]} \neq 0$. Without loss of generality we may assume that $c_1 = 1$. Applying (2), we obtain

$$\left[E_{[-1]}, \mathrm{T_H}(x^{(2\varepsilon_1)}x_{1'})\right] = -\,\mathrm{T_H}(x_1x_{1'}).$$

By virtue of Lemma 2.4 and the equation above, we get $[E_{[-1]}, \mathrm{T_H}(x^{(2\varepsilon_1)}x_{1'})] \notin \mathrm{nil}(\mathscr{H})$. Now Lemma 2.2 shows $[E, \mathrm{T_H}(x^{(2\varepsilon_1)}x_{1'})] \notin \mathrm{nil}(\mathscr{H})$, contradicting the assumption. Hence $E_{[-1]} = 0$, $E = E_0 \in \mathscr{H}_0$.

Assume that $E = E_{[0]} + E_1$, where $E_{[0]} \in \mathscr{H}_{[0]} \cap \mathscr{H}_{\bar{0}}$, $E_1 \in \mathscr{H}_1 \cap \mathscr{H}_{\bar{0}}$. By Lemma 2.6 (iv), $\rho(E_{[0]}) \in \tilde{\mathrm{p}}\,(m)_{\bar{0}}$. Thus we may suppose that $\rho(E_{[0]}) = \left[\begin{smallmatrix} A & \\ & -A^{\mathrm{T}}\end{smallmatrix}\right]$.

Assume that $E_{[0]} \neq 0$. According to Lemma 2.6 (iii), A is a nonzero matrix. Put $A = (c_{ij})_{m\times m}$. Suppose that the l-th row is the leading nonzero row and the t-th column is the leading nonzero column.

We treat two cases separately.

Case (i): $l \le t$.

Let $k = \max\{j \in Y_0 \mid c_{lj} \neq 0\}$. Then $l \le t \le k$.

Assume that $l = k$. Then $l = t = k$ and $c_{ll} \neq 0$. Obviously, A is of the following block form $A = \left[\begin{smallmatrix} A_{ll} & 0 \\ * & * \end{smallmatrix}\right]$, where A_{ll} is an $l \times l$ matrix with (l, l)-entry $c_{ll} \neq 0$ and 0 elsewhere. So the matrix $\rho(E_{[0]})$ is not nilpotent. By Lemma 2.6 (v), $E_{[0]}$ is not ad-nilpotent. Then by Lemma 2.2, E is not ad-nilpotent. This contradicts the assumption that $E \in \Omega \subset \mathrm{nil}(\mathscr{H})$. Thus $l < k$.

Obviously,

$$\rho(E_{[0]}) = \sum_{j=t}^{k} c_{lj}e_{lj} + \sum_{i=l+1}^{m}\sum_{j=t}^{m} c_{ij}e_{ij} - \sum_{j=t}^{k} c_{lj}e_{j'l'} - \sum_{i=l+1}^{m}\sum_{j=t}^{m} c_{ij}c_{j'i'}.$$

Direct computation shows that

$$\begin{aligned}[\rho(E_{[0]}), e_{kl} - e_{l'k'}] \\ &= c_{lk}e_{ll} - \sum_{j=t}^{k} c_{lj}e_{kj} + \sum_{i=l+1}^{m} c_{ik}e_{il} - c_{lk}e_{l'l'} + \sum_{j=t}^{k} c_{lj}e_{j'k'} - \sum_{i=l+1}^{m} c_{ik}e_{l'i'}.\end{aligned}$$

This matrix possesses the block form $\left[\begin{smallmatrix} B_{ll} & 0 \\ * & * \end{smallmatrix}\right]$, where B_{ll} is an $l \times l$ matrix in which (l, l)-element is $c_{lk} \neq 0$ and the others are all 0. Therefore, the matrix $[\rho(E_{[0]}), e_{kl} - e_{l'k'}]$ is not nilpotent. By Lemma 2.6 (ii), $e_{kl} - e_{l'k'} = \rho(\mathrm{T_H}(x_{k'}x_l))$, and the matrix $\rho([(E_{[0]}, \mathrm{T_H}(x_{k'}x_l)])$ is not nilpotent. In combination with Lemma 2.6 (v), we see that $[E_{[0]}, \mathrm{T_H}(x_{k'}x_l)]$ is not ad-nilpotent. Now Lemma 2.2 ensures that $[E, \mathrm{T_H}(x_{k'}x_l)] \notin \mathrm{nil}(\mathscr{H})$. This contradicts the assumption that $E \in \Omega$.

Case (ii): $l > t$.

Let $k = \max\{i \in Y_0 \mid c_{it} \neq 0\}$. Then $k \geq l > t$, $a_{kt} \neq 0$ and

$$\rho(E_{[0]}) = \sum_{i=l}^{k} c_{it}e_{it} + \sum_{j=t+1}^{m}\sum_{i=l}^{m} c_{ij}e_{ij} - \sum_{i=l}^{k} c_{it}e_{t'i'} - \sum_{j=t+1}^{m}\sum_{i=l}^{m} c_{ij}e_{j'i'}.$$

By Lemma 2.6 (ii), $\rho(\mathrm{T_H}(x_{t'}x_k)) = e_{tk} - e_{k't'}$. Thus

$$\begin{aligned}&\left[\rho(E_{[0]}), \rho(\mathrm{T_H}(x_{t'}x_k))\right] \\ &\quad = \sum_{i=l}^{k} c_{it}e_{ik} - c_{kt}e_{tt} - \sum_{j=t+1}^{m} c_{kj}e_{tj} - \sum_{i=l}^{k} c_{it}e_{k'i'} + c_{kt}e_{t't'} + \sum_{j=t+1}^{m} c_{kj}e_{j't'}.\end{aligned}$$

This matrix is of the following form $\left[\begin{smallmatrix} A_{tt} & * \\ 0 & * \end{smallmatrix}\right]$, where A_{tt} is a $t \times t$ matrix whose (t, t)-entry is $-c_{kt} \neq 0$ and remaining entries are 0. Proceeding analogously to Case (i), we may prove that $[E, \mathrm{T_H}(x_{t'}x_k)]$ is not ad-nilpotent, contradicting the assumption that $E \in \Omega$.

We conclude that $E_{[0]} = 0$, $E = E_1 \in \mathscr{H}_1$. $\square$

3. Natural filtration and classification

For the sake of simplicity, an isomorphism between two odd Hamiltonian superalgebras will be called an f-isomorphism. In this section, we shall prove that the natural filtration of $\mathscr{H}$ is invariant under f-isomorphisms, that is, if $\varphi : \mathscr{H}(m; \underline{t}) \to \mathscr{H}(m'; \underline{t}')$ is an isomorphism of Lie superalgebras, then $\varphi(\mathscr{H}(m; \underline{t})_i) = \mathscr{H}(m'; \underline{t}')_i$ for all $i \geq -1$, where $m, m' \in \mathbb{N} \setminus \{1, 2\}$, $\underline{t} \in \mathbb{N}^m$, $\underline{t}' \in \mathbb{N}^{m'}$.

LEMMA 3.1. *Let $k, l \in Y_0$. Then $\mathrm{T_H}(x^{(2\varepsilon_k)}x_{l'}) \in \Omega$ if and only if $k \neq l$.*

PROOF. Assume that $k = l$. By (2), $[\mathrm{T_H}(x_{k'}), \mathrm{T_H}(x^{(2\varepsilon_k)}x_{k'})] = -\mathrm{T_H}(x_kx_{k'})$. By Lemma 2.4, we have $\mathrm{T_H}(x_kx_{k'}) \in \mathrm{nil}(\mathscr{H})$. Therefore, $\mathrm{T_H}(x^{(2\varepsilon_k)}x_{k'}) \notin \Omega$.

Conversely, let $k \neq l$. Let $E = E_{[-1]} + E_0$ be an element of $\mathscr{H}$, where $E_{[-1]} \in \mathscr{H}_{[-1]}$, $E_0 \in \mathscr{H}_0$. Assume that $E_{[-1]} = \sum_{i \in Y} c_i \mathrm{T_H}(x_i)$, where $c_i \in \mathbb{F}$. Put $D := [E_{[-1]}, \mathrm{T_H}(x^{(2\varepsilon_k)}x_{l'})]$. Then

$$D = \left[c_{k'}\mathrm{T_H}(x_{k'}) + c_l \mathrm{T_H}(x_l), \mathrm{T_H}(x^{(2\varepsilon_k)}x_{l'})\right] = -c_{k'}\mathrm{T_H}(x_kx_{l'}) + c_l \mathrm{T_H}(x^{(2\varepsilon_k)}). \tag{4}$$

By Lemma 2.4, $\mathrm{T_H}(x_k x_{l'})$ and $\mathrm{T_H}(x^{(2\varepsilon_k)})$ are all ad-nilpotent elements. Applying (2), we obtain that $[\mathrm{T_H}(x_k x_{l'}), \mathrm{T_H}(x^{(2\varepsilon_k)})] = 0$. So $S := \{0, \mathrm{T_H}(x_k x_{l'}), \mathrm{T_H}(x^{(2\varepsilon_k)})\}$ is a Lie subset of $\mathscr{H}$. By Lemma 2.1 and (4), we have $D \in \mathrm{nil}(\mathscr{H})$. Obviously,

$$[E, \mathrm{T_H}(x^{(2\varepsilon_k)} x_{l'})] = D + [E_0, \mathrm{T_H}(x^{(2\varepsilon_k)} x_{l'})], \tag{5}$$

where $[E_0, \mathrm{T_H}(x^{(2\varepsilon_k)} x_{l'})] \in \mathscr{H}_1$. Note that $k \neq l$. It is easy to verify that $[D, D] = 0$. By virtue of Lemma 2.5 and (5), we get $[E, \mathrm{T_H}(x^{(2\varepsilon_k)} x_{l'})] \in \mathrm{nil}(\mathscr{H})$. Hence $\mathrm{T_H}(x^{(2\varepsilon_k)} x_{l'}) \in \Omega$. $\square$

PROPOSITION 3.2. $\mathscr{H}_1 \cap \mathscr{H}_{\bar{0}} = \Gamma$.

PROOF. It is clear that $\mathscr{H}_1 \cap \mathscr{H}_{\bar{0}} \subset \mathrm{nil}(\mathscr{H}_{\bar{0}})$. By Theorem 2.7, $\Omega \subset \mathscr{H}_1 \cap \mathscr{H}_{\bar{0}}$ and therefore, $[\mathscr{H}_1 \cap \mathscr{H}_{\bar{0}}, \Omega] \subset [\mathscr{H}_1 \cap \mathscr{H}_{\bar{0}}, \mathscr{H}_1 \cap \mathscr{H}_{\bar{0}}] \subset \mathscr{H}_2 \cap \mathscr{H}_{\bar{0}} \subset \Omega$. Thus $\mathscr{H}_1 \cap \mathscr{H}_{\bar{0}} \subset \Gamma$.

To prove the converse inclusion, we suppose that $E \in \Gamma$ and decompose $E = E_{[-1]} + E_0$, where $E_{[-1]} \in \mathscr{H}_{[-1]}$, $E_0 \in \mathscr{H}_0$. Assume that $E_{[-1]} \neq 0$. Since $E_{[-1]} \in \mathscr{H}_{\bar{0}}$, without loss of generality, we may suppose that $E_{[-1]} = D_1 + \sum_{j=2}^{m} c_j D_j$, where $c_j \in \mathbb{F}$. Direct computation and application of Theorem 2.7 show that

$$\left[E, \mathrm{T_H}(x^{(2\varepsilon_1)} x_{2'})\right] = \mathrm{T_H}(x_1 x_{2'}) + \left[E_0, \mathrm{T_H}(x^{(2\varepsilon_1)} x_{2'})\right] \notin \Omega. \tag{6}$$

By Lemma 3.1, $\mathrm{T_H}(x^{(2\varepsilon_1)} x_{2'}) \in \Omega$. Moreover, (6) implies that $E \notin \Gamma$, which is a contradiction. So $E_{[-1]} = 0$, $E = E_0 \in \mathscr{H}_0$.

We next decompose $E_0 = E = E_{[0]} + E_1$, where $E_{[0]} \in \mathscr{H}_{[0]}$, $E_1 \in \mathscr{H}_1$. Assume that $E_{[0]} \neq 0$. Since $E_{[0]} \in \mathscr{H}_{\bar{0}}$, we may assume that $E_{[0]} = \sum_{i,j \in Y_0} c_{ij} \mathrm{T_H}(x_i x_{j'})$, where $c_{ij} \in \mathbb{F}$. Put

$$l := \min\{i \in Y_0 \mid c_{i j_0} \neq 0 \ \text{ for some } \ j_0 \in Y\},$$
$$t := \min\{j \in Y_0 \mid c_{i_0 j} \neq 0 \ \text{ for some } \ i_0 \in Y\}.$$

Case (i): $l \leq t$.

Let $k := \max\{j \in Y_0 \mid c_{lj} \neq 0\}$. Then $l \leq t \leq k$ and $c_{lk} \neq 0$.

If $l = k$, proceeding similarly as in the proof of Theorem 2.7, we may prove that E is not ad-nilpotent, which gives a contradiction.

If $l < k$, then

$$E_{[0]} = \sum_{j=t}^{k} c_{lj} \mathrm{T_H}(x_l x_{j'}) + \sum_{j=l+1}^{m} \sum_{j=t}^{m} c_{ij} \mathrm{T_H}(x_i x_{j'}).$$

Let $D := [\mathrm{T_H}(x^{(2\varepsilon_k)}x_{l'}), E_{[0]}]$. Then

$$\begin{aligned} D &= [x_k x_{l'} D_{k'} - x^{(2\varepsilon_k)} D_l, E_{[0]}] \\ &= c_{lk}\,\mathrm{T_H}(x_k x_{l'} x_l) - \sum_{j=t}^{k} c_{lj}\,\mathrm{T_H}(x^{(2\varepsilon_k)} x_{j'}) + \sum_{j=l+1}^{m} c_{ik}\,\mathrm{T_H}(x_k x_{l'} x_i). \end{aligned}$$

Therefore,

$$[\mathrm{T_H}(x_{k'}), D] = -c_{lk}\,\mathrm{T_H}(x_{l'}x_l) + \sum_{j=t}^{k} c_{lj}\,\mathrm{T_H}(x_k x_{j'}) - \sum_{j=l+1}^{m} c_{ik}\,\mathrm{T_H}(x_{l'}x_i).$$

By Lemma 2.6 (ii), we have

$$\rho([\mathrm{T_H}(x_{k'}), D]) = -c_{lk}(e_{ll} - e_{l'l'}) + \sum_{j=t}^{k} c_{lj}(e_{jk} - e_{k'j'}) - \sum_{j=l+1}^{m} c_{ik}(e_{li} - e_{i'l'}).$$

This matrix is of the following block form $\left[\begin{smallmatrix} A_{ll} & * \\ 0 & * \end{smallmatrix}\right]$, where A_{ll} is an $l \times l$ matrix whose (l, l)-entry is $-c_{lk} \neq 0$, but other entries are 0. Consequently, the matrix $\rho([\mathrm{T_H}(x_{k'}), D])$ is not nilpotent. This and Lemma 2.6 (v) show that $[\mathrm{T_H}(x_{k'}), D]$ is not ad-nilpotent. By Lemma 2.2, $[\mathrm{T_H}(x_{k'}), [\mathrm{T_H}(x^{(2\varepsilon_k)}x_{l'}), E]]$ is not ad-nilpotent. Furthermore, we obtain that

$$\big[\mathrm{T_H}(x^{(2\varepsilon_k)}x_{l'}), E\big] \notin \Omega. \tag{7}$$

On the other hand, by Lemma 3.1, $\mathrm{T_H}(x^{(2\varepsilon_k)}x_{l'}) \in \Omega$. Hence (7) implies that $E \notin \Gamma$, which is a contradiction.

Case (ii): $l > t$.

Let $k := \max\{i \in Y_0 \mid c_{it} \neq 0\}$. Then $k \geq l > t$, $c_{kt} \neq 0$ and

$$E_{[0]} = \sum_{i=l}^{k} c_{it}\,\mathrm{T_H}(x_i x_{t'}) + \sum_{i=l}^{m} \sum_{j=t+1}^{m} c_{ij}\,\mathrm{T_H}(x_i x_{j'}).$$

Put $G := [\mathrm{T_H}(x^{(2\varepsilon_t)}x_{k'}), E_{[0]}]$. Using (2) we compute

$$G = \sum_{i=l}^{k} c_{it}\mathrm{T_H}(x_t x_{k'} x_i) - c_{kt}\mathrm{T_H}(x^{(2\varepsilon_t)} x_{t'}) - \sum_{j=t+1}^{m} c_{kj}\,\mathrm{T_H}(x^{(2\varepsilon_t)} x_{j'}).$$

Therefore,

$$[\mathrm{T_H}(x_{t'}), G] = c_{kt}\,\mathrm{T_H}(x_t x_{t'}) - \sum_{i=l}^{k} c_{it}\,\mathrm{T_H}(x_{k'} x_i) + \sum_{j=t+1}^{m} c_{kj}\,\mathrm{T_H}(x_t x_{j'}).$$

By Lemma 2.6 (ii),

$$\rho([\mathrm{T_H}(x_{t'}), G]) = c_{kt}(e_{tt} - e_{t't'}) - \sum_{i=l}^{k} c_{it}(e_{ki} - e_{i'k'}) + \sum_{j=t+1}^{m} c_{kj}(e_{jt} - e_{t'j'}).$$

This matrix is of the form $\left[\begin{smallmatrix} B_{ll} & 0 \\ * & * \end{smallmatrix}\right]$, where B_{ll} is an $l \times l$ matrix whose (l, l)-entry is $c_{kt} \neq 0$, but other entries are 0. Similar to (i), we obtain that $[\mathrm{T_H}(x^{(2\varepsilon_t)} x_{k'}), E] \notin \Omega$. By Lemma 3.1, $\mathrm{T_H}(x^{(2\varepsilon_t)} x_{k'}) \in \Omega$ and therefore $E \notin \Gamma$, a contradiction.

Combining (i) and (ii), we conclude that $E_{[0]} = 0$ and $E = E_1 \in \mathscr{H}_1$. This proves that $\Gamma \subset \mathscr{H}_1 \cap \mathscr{H}_{\bar{0}}$. □

PROPOSITION 3.3. $\mathscr{H}_0 = \Phi$.

PROOF. The inclusion $\mathscr{H}_0 \subset \Phi$ is clear. So, we need only to prove the converse inclusion. Assume that $E = E_{[-1]} + E_0 \in \Phi$, where $E_{[-1]} \in \mathscr{H}_{[-1]}$, $E_0 \in \mathscr{H}_0$. Let $E_{[-1]} = \sum_{i \in Y} c_i \, \mathrm{T_H}(x_i)$, $c_i \in \mathbb{F}$. Assume that $E_{[-1]} \neq 0$. Then there exists some $k \in Y$ such that $c_k \neq 0$. If $k \in Y_1$, we may let $k = 1'$. Put $D := [E_{[-1]}, \mathrm{T_H}(x^{(2\varepsilon_1)} x_{1'})]$. Then we have

$$\begin{aligned} D &= [c_1 \, \mathrm{T_H}(x_1) + c_{1'} \, \mathrm{T_H}(x_{1'}), \mathrm{T_H}(x^{(2\varepsilon_1)} x_{1'})] \\ &= c_1 \, \mathrm{T_H}(x^{(2\varepsilon_1)}) - c_{1'} \, \mathrm{T_H}(x_1 x_{1'}) \\ &= c_1 x_1 D_{1'} - c_{1'}(x_{1'} D_{1'} - x_1 D_1). \end{aligned}$$

Therefore, $D^l(x_1) = c_{1'}^l x_1$ for all $l \in \mathbb{N}$. Thus D is not nilpotent as a linear transformation. By Lemma 2.3, D is not ad-nilpotent. Now Lemma 2.2 shows that $[E, \mathrm{T_H}(x^{(2\varepsilon_1)} x_{1'})]$ is not ad-nilpotent. Observe that $\mathrm{T_H}(x^{(2\varepsilon_1)} x_{1'}) \in \mathscr{H}_1 \cap \mathscr{H}_{\bar{0}}$. This contradicts the assumption that $E \in \Phi$. Hence $E_{[-1]} = \sum_{i \in Y_0} c_i \mathrm{T_H}(x_i)$. Without loss of generality, we may suppose that $c_1 \neq 0$. Let $G := \mathrm{T_H}(x_{1'} x_2 x_3 + x_{1'} x_{2'} x_{3'})$. Then

$$[E_{[-1]}, G] = c_1 \, \mathrm{T_H}(x_2 x_3 + x_{2'} x_{3'}) - c_2 \, \mathrm{T_H}(x_{1'} x_{3'}) + c_3 \, \mathrm{T_H}(x_{1'} x_{2'}).$$

Therefore,

$$(\mathrm{ad}[E_{[-1]}, G])^{4t}(\mathrm{T_H}(x_2 + x_3)) = c_1^{4t} \, \mathrm{T_H}(x_2 + x_3) \quad \text{for all } t \in \mathbb{N}.$$

By Lemma 2.2, $[E, G] \notin \mathrm{nil}(\mathscr{H})$. Notice that $G \in \mathscr{H}_1 \cap \mathscr{H}_{\bar{0}}$. This contradicts the assumption that $E \in \Phi$. Hence $E_{[-1]} = 0$, $E \in \mathscr{H}_0$. So $\Phi \subset \mathscr{H}_0$, as required. □

Before proving the following main theorem we recall the notation introduced in the beginning of Section 2.

THEOREM 3.4. *The natural filtrations of finite-dimensional odd Hamiltonian superalgebras are invariant under f-isomorphisms.*

PROOF. Let $m, m' \in \mathbb{N} \setminus \{1, 2\}$, $\underline{t} \in \mathbb{N}^m$, $\underline{t}' \in \mathbb{N}^{m'}$ and $\varphi : \mathscr{H}(m; \underline{t}) \to \mathscr{H}(m'; \underline{t}')$ be an f-isomorphism. Observe that φ preserves $\mathbb{Z}_2$-gradations. By the definition of Ω, it is clear that $\varphi(\Omega) = \Omega'$; furthermore, $\varphi(\Gamma) = \Gamma'$. By Proposition 3.2 and the definition of Φ, $\varphi(\Phi) = \Phi'$. This and Proposition 3.3 ensure that $\varphi(\mathscr{H}(m; \underline{t})_0) = \mathscr{H}(m'; \underline{t}')_0$.

As

$$\mathscr{H}_i = \{E \in \mathscr{H}_{i-1} \mid \operatorname{ad} E(\mathscr{H}) \subset \mathscr{H}_{i-1}\}, \quad i \geq 1,$$

we may prove, by induction on i, that $\varphi(\mathscr{H}(m; \underline{t})_i) = \mathscr{H}(m'; \underline{t}')_i$ for all $i \geq -1$. □

COROLLARY 3.5. *The filtration of finite-dimensional odd Hamiltonian superalgebra* $\mathscr{H}$ *is invariant under* $\operatorname{Aut} \mathscr{H}$.

PROOF. This is a direct consequence of Theorem 3.4. □

As a direct application of Theorem 3.4, we establish the following property of isomorphisms of odd Hamiltonian superalgebras.

By Theorem 3.4, we may easily prove the following

COROLLARY 3.6. *Let* ϕ *and* φ *be f-isomorphisms of* $\mathscr{H}(m; \underline{t})$ *to* $\mathscr{H}(m'; \underline{t}')$. *Then* $\phi = \varphi$ *if and only if* $\phi|_{\mathscr{H}_{[-1]}} = \varphi|_{\mathscr{H}_{[-1]}}$.

Employing Theorem 3.4, we may prove that m and $\underline{t}$ are intrinsic for the odd Hamiltonian superalgebra $\mathscr{H}(m; \underline{t})$, that is, we may give a classification of odd Hamiltonian superalgebras. For $\underline{t}, \underline{t}' \in \mathbb{N}^m$, $\underline{t}, \underline{t}'$ are said to be equivalent and denoted by $\underline{t} \sim \underline{t}'$ if there exists a permutation $\sigma \in S_m$ such that $t_{\sigma(i)} = t'_i$ for all $i \in Y_0$.

THEOREM 3.7. *Suppose that* $m, m' \in \mathbb{N} \setminus \{1, 2\}$, $\underline{t} \in \mathbb{N}^m$, $\underline{t}' \in \mathbb{N}^{m'}$. *Then* $\mathscr{H}(m; \underline{t}) \cong \mathscr{H}(m'; \underline{t}')$ *if and only if* $m = m'$ *and* $\underline{t} \sim \underline{t}'$.

PROOF. Assume that $\phi : \mathscr{H}(m; \underline{t}) \to \mathscr{H}(m'; \underline{t}')$ is an isomorphism of Lie superalgebras. Then Theorem 3.4 ensures that ϕ induces canonically an isomorphism of quotient spaces: $\mathscr{H}(m; \underline{t})/\mathscr{H}(m; \underline{t})_0 \to \mathscr{H}(m'; \underline{t}')/\mathscr{H}(m'; \underline{t}')_0$. Note that

$$\dim(\mathscr{H}(m; \underline{t})/\mathscr{H}(m; \underline{t})_0) = \dim \mathscr{H}(m; \underline{t})_{[-1]} = 2m.$$

It follows that $m = m'$.

Without loss of generality, we may suppose that $t_1 \geq \cdots \geq t_m$ and $t'_1 \geq \cdots \geq t'_m$. Assume on the contrary that $\underline{t} \neq \underline{t}'$. Then we may suppose that for some $k \in Y_0$,

$$t_k > t'_k \quad \text{but} \quad t_j = t'_j \quad \text{for} \quad k < j \leq m \quad (\text{maybe } k = m). \tag{8}$$

We assert that $\mathscr{H}(m;\underline{t})_{[p^{t'_k}-2]} \supsetneqq \mathscr{H}(m;\underline{t}')_{[p^{t'_k}-2]}$. According to (8) and the definition of $\mathscr{H}(m;\underline{t})$, the implication '$\supset$' is clear. Notice that

$$\mathrm{T_H}\left(x^{(p^{t'_k}\varepsilon_k)}\right) \in \mathscr{H}(m;\underline{t})_{[p^{t'_k}-2]} \quad \text{but} \quad \mathrm{T_H}\left(x^{(p^{t'_k}\varepsilon_k)}\right) \notin \mathscr{H}(m;\underline{t}')_{[p^{t'_k}-2]}.$$

So our assertion holds and therefore, $\dim \mathscr{H}(m;\underline{t})_{[p^{t'_k}-2]} > \dim \mathscr{H}(m;\underline{t}')_{[p^{t'_k}-2]}$. On the other hand, Theorem 3.4 implies that

$$\phi(\mathscr{H}(m;\underline{t})_i) = \mathscr{H}(m;\underline{t}')_i \quad \text{for all } i \geq -1. \tag{9}$$

From this we see easily that $\dim \mathscr{H}(m;\underline{t})_{[i]} = \dim \mathscr{H}(m;\underline{t}')_{[i]}$ for all $i \geq -1$. In particular, $\dim \mathscr{H}(m;\underline{t})_{[p^{t'_k}-2]} = \dim \mathscr{H}(m;\underline{t}')_{[p^{t'_k}-2]}$, contradicting to (9).

The converse implication is automatic. The proof is completed. $\square$

4. The automorphism group of $\mathscr{H}(m,m;\underline{1})$

Recall that a Lie superalgebra $L = L_{\bar{0}} \oplus L_{\bar{1}}$ over $\mathbb{F}$ is called *restricted*, if the Lie algebra $L_{\bar{0}}$ is restricted and the $L_{\bar{0}}$-module $L_{\bar{1}}$ is restricted (see [13]). The proof of Lemma 4.1 is analogous to [18, Theorem 4.4.5 (2)] or [23, Theorem 5].

LEMMA 4.1. *$\mathscr{H}(m;\underline{t})$ is restricted if and only if $\underline{t} = \underline{1}$.*

Let $\mathscr{A}$ be a finite-dimensional superalgebra over $\mathbb{F}$. Denote by $\operatorname{Aut}\mathscr{A}$ the (even) automorphism group of $\mathscr{A}$. If $\sigma \in \operatorname{Aut}\mathscr{A}$ and $D \in \operatorname{Der}\mathscr{A}$, then $D^\sigma := \sigma D\sigma^{-1}$ is again a superderivation of $\mathscr{A}$. It is easy to see that $\tilde{\sigma} : D \to D^\sigma$ is an automorphism of $\operatorname{Der}\mathscr{A}$. Suppose that $\mathscr{Q}$ is a Lie subsuperalgebra of $\operatorname{Der}\mathscr{A}$. We call $\sigma \in \operatorname{Aut}\mathscr{A}$ *admissible* to $\mathscr{Q}$ if $\tilde{\sigma}(\mathscr{Q}) \subset \mathscr{Q}$. Put $\operatorname{Aut}(\mathscr{A} : \mathscr{Q}) := \{\sigma \in \operatorname{Aut}\mathscr{A} \mid \tilde{\sigma}(\mathscr{Q}) \subset \mathscr{Q}\}$. Then $\operatorname{Aut}(\mathscr{A} : \mathscr{Q})$ is a subgroup of $\operatorname{Aut}\mathscr{A}$, and is referred to as the *admissible automorphism group* of $\mathscr{A}$ (to $\mathscr{Q}$). Obviously, $\Phi : \operatorname{Aut}(\mathscr{A} : \mathscr{Q}) \to \operatorname{Aut}\mathscr{Q}, \sigma \mapsto \tilde{\sigma}|_{\mathscr{Q}}$ is a homomorphism of groups. In this section, we only deal with the restricted odd Hamiltonian superalgebra $\mathscr{H}(m;\underline{1})$, and therefore adopt the convention $\mathscr{U} := \Lambda(m,m;\underline{1})$, $\mathscr{H} := \mathscr{H}(m;\underline{1})$ and $W := W(m,m;\underline{1})$.

The main result of this section is the following theorem.

THEOREM 4.2. *Let $\Phi : \operatorname{Aut}(\mathscr{U} : \mathscr{H}) \to \operatorname{Aut}\mathscr{H}$, $\sigma \mapsto \tilde{\sigma}|_{\mathscr{H}}$. Then Φ is an isomorphism of groups.*

To prove it, we need the following lemmas. First we introduce some notation. Let $\mathrm{M}_{2m}(\mathscr{U})$ denote the $\mathbb{F}$-algebra consisting of all $2m \times 2m$ matrices over $\mathscr{U}$, $\mathrm{pr}_{[0]}$ and pr_1 be the projections of $\mathscr{U}$ onto $\mathscr{U}_{[0]} = \mathbb{F}$ and $\mathscr{U}_1$, respectively. For $A = (a_{ij}) \in \mathrm{M}_{2m}(\mathscr{U})$, set $\mathrm{pr}_{[0]} A := (\mathrm{pr}_{[0]}(a_{ij}))$ and $\mathrm{pr}_1 A := (\mathrm{pr}_1(a_{ij}))$.

LEMMA 4.3. *The following statements hold*:

(i) *Let* $A \in \mathrm{M}_{2m}(\mathscr{U})$. *Then* A *is invertible if and only if* $\mathrm{pr}_{[0]} A$ *is invertible matrix over* $\mathbb{F}$.

(ii) *Suppose that* $\{E_1, \ldots, E_{2m}\}$ *is a* $\mathscr{U}$*-basis of* W. *Then* $\{\mathrm{pr}_{[-1]}(E_1), \ldots, \mathrm{pr}_{[-1]}(E_{2m})\}$ *is an* $\mathbb{F}$*-basis of* $W_{[-1]}$, *where* $\mathrm{pr}_{[-1]}$ *is the projection of* W *onto* $W_{[-1]}$.

(iii) *Suppose that* $\phi \in \mathrm{Aut}\,\mathscr{H}$ *and* $\{G_i \mid i \in Y\} \subset \mathscr{H}$ *is a* $\mathscr{U}$*-basis of* W. *Then* $\{\phi(G_i) \mid i \in Y\}$ *is also a* $\mathscr{U}$*-basis of* W.

PROOF. (i) Clearly, $A = \mathrm{pr}_{[0]} A + \mathrm{pr}_1 A$. Since every element of $\mathscr{U}_1$ is nilpotent, so is every $2m \times 2m$ matrix over $\mathscr{U}_1$. From these facts one may easily prove (i).

(ii) Suppose that $(D_1, \ldots, D_{2m})^{\mathrm{T}} = A(E_1, \ldots, E_{2m})^{\mathrm{T}}$, $A \in \mathrm{M}_{2m}(\mathscr{U})$. Then $(D_1, \ldots, D_{2m})^{\mathrm{T}} = (\mathrm{pr}_{[0]} A)(\mathrm{pr}_{[-1]}(E_1), \ldots, \mathrm{pr}_{[-1]}(E_{2m}))^{\mathrm{T}}$. Since $\{D_1, \ldots, D_{2m}\}$ is an $\mathbb{F}$-basis of $W_{[-1]}$, so is $\{\mathrm{pr}_{[-1]}(E_1), \ldots, \mathrm{pr}_{[-1]}(E_{2m})\}$.

(iii) By Corollary 3.5, the natural filtration $\{\mathscr{H}_i\}$ is invariant under ϕ. Thus ϕ induces canonically $\overline{\phi} \in \mathrm{GL}(\mathscr{H}/\mathscr{H}_0)$. Denote by $\overline{G}_i$ the image of G_i under the canonical map $\mathscr{H} \to \mathscr{H}/\mathscr{H}_0$. Then $\{\overline{G}_i \mid i \in Y\}$ is an $\mathbb{F}$-basis of $\mathscr{H}/\mathscr{H}_0$. Assume that

$$(\phi(G_1), \ldots, \phi(G_{2m}))^{\mathrm{T}} = A(D_1, \ldots, D_{2m})^{\mathrm{T}}, \quad A \in \mathrm{M}_{2m}(\mathscr{U}).$$

Decompose $A = \mathrm{pr}_{[0]} A + \mathrm{pr}_1 A$. We obtain that

$$(\overline{\phi}(\overline{G}_1), \ldots, \overline{\phi}(\overline{G}_{2m}))^{\mathrm{T}} = (\overline{\phi(G_1)}, \ldots, \overline{\phi(G_{2m})})^{\mathrm{T}} = (\mathrm{pr}_{[0]} A)(\overline{D}_1, \ldots, \overline{D}_{2m})^{\mathrm{T}}.$$

This implies that $\mathrm{pr}_{[0]} A$ is invertible. By (i), A is invertible and therefore $\{\phi(G_i) \mid i \in Y\}$ is a $\mathscr{U}$-basis of W. $\square$

LEMMA 4.4. *Suppose that* $\phi \in \mathrm{Aut}\,\mathscr{H}$. *Then there exist* $y_j \in \mathscr{U}_1$ *with* $\deg(y_j) = \mu(j)$ *such that* $(\phi(D_i))(y_j) = \delta_{ij} + \delta_{j1}\delta_{i1}$ *for* $i, j \in Y$. *In particular, the matrix* $((\phi(D_i))(y_j))_{i,j\in Y}$ *is invertible.*

PROOF. Let $j \in Y$. By Lemma 4.3 (iii), $\{\phi(D_1), \ldots, \phi(D_{2m})\}$ is a $\mathscr{U}$-basis of W. Thus we may suppose that $\phi(\mathrm{T}_{\mathrm{H}}(x_1x_j)) = \sum_{l=1}^{2m} a_{jl}\phi(D_l)$, where $a_{jl} \in \mathscr{U}$. From Lemma 4.3 (ii), we see easily that $a_{jl} \in \mathscr{U}_1$. Using (1), we obtain that

$$\phi([D_i, \mathrm{T}_{\mathrm{H}}(x_1x_j)]) = \left[\phi(D_i), \sum_{l=1}^{2m} a_{jl}\phi(D_l)\right] = \sum_{l=1}^{2m}(\phi(D_i)(a_{jl}))\phi(D_l). \tag{10}$$

On the other hand, by Lemma 2.6 (i), $\mathrm{T}_{\mathrm{H}}(x_1x_j) = x_jD_{1'} + (-1)^{\mu(j)}x_1D_{j'}$ and therefore,

$$\phi([D_i, \mathrm{T}_{\mathrm{H}}(x_1x_j)]) = \delta_{ij}\phi(D_{1'}) + (-1)^{\mu(j)}\delta_{i1}\phi(D_{j'}). \tag{11}$$

Comparing (10) and (11), one gets $\phi(D_i)(a_{j1'}) = \delta_{ij} + \delta_{j1}\delta_{i1}$. Put $y_j := a_{j1'}$ for $j \in Y$. We see that $\phi(D_i)(y_j) = \delta_{ij} + \delta_{j1}\delta_{i1}$, $y_j \in \mathscr{U}_1$ and $\deg(y_j) = \deg(a_{j1'}) = \mu(j') + \mu(1') = \mu(j)$, as desired. □

PROOF OF THEOREM 4.2. Let $\sigma \in \mathrm{Aut}(\mathscr{U} : \mathscr{H})$. Assume that $\tilde{\sigma}|_{\mathscr{H}} = 1|_{\mathscr{H}}$. We proceed by induction on $|\alpha| + |u|$ to show that $\sigma(x^{(\alpha)}x^u) = x^{(\alpha)}x^u$. Note that $W_{[-1]} = \mathscr{H}_{[-1]}$. We obtain that

$$D_j x_i = \delta_{ij} = \sigma(\delta_{ij}) = \sigma(D_j x_i) = D_j^{\sigma}(\sigma(x_i)) = D_j(\sigma(x_i)), \quad i, j \in Y.$$

This implies that $x_i - \sigma(x_i) \in \mathbb{F}$. Since $\sigma(\mathscr{U}_1) \subset \mathscr{U}_1$, it follows that $\sigma(x_i) = x_i$, $i \in Y$. Suppose that $|\alpha| + |u| > 1$. Then by induction hypothesis, we obtain

$$D_i(\sigma(x^{(\alpha)}x^u) - x^{(\alpha)}x^u) = \sigma(D_i(x^{(\alpha)}x^u)) - D_i(x^{(\alpha)}x^u) = 0 \quad \text{for all } i \in Y,$$

and therefore $\sigma(x^{(\alpha)}x^u) - x^{(\alpha)}x^u \in \mathbb{F}$. Thus $\sigma(x^{(\alpha)}x^u) = x^{(\alpha)}x^u$. Consequently, $\sigma = 1$ and Φ is injective.

We next prove that Φ is surjective. Let $\phi \in \mathrm{Aut}\,\mathscr{H}$. By Lemma 4.4 there exists $y_j \in \mathscr{U}$, with $\deg(y_j) = \mu(j)$ such that $(\phi(D_i))(y_j) = \delta_{ij} + \delta_{j1}\delta_{i1}$. Assume that $\phi(D_i) = \sum_{j=1}^{2m} a_{ij} D_j$, $a_{ij} \in \mathscr{U}$. Then we have the matrix equation $(\phi(D_i)(y_j)) = (a_{ij})(D_i y_j)$ and therefore,

$$(\delta_{ij} + \delta_{j1}\delta_{i1}) = (\phi(D_i)(y_j)) = \mathrm{pr}_{[0]}(\phi(D_i)(y_j)) = \mathrm{pr}_{[0]}(c_{ij})\,\mathrm{pr}_{[0]}(D_i y_j).$$

Thus $\mathrm{pr}_{[0]}(D_i y_j)$ is invertible. Define the endomorphism σ of $\mathscr{U}$ such that

$$\sigma(x_i) = y_j \quad \text{for all } i \in Y. \tag{12}$$

Then σ is even. We claim that $\sigma \in \mathrm{Aut}\,\mathscr{U}$. From (12) it is easy to see that σ leaves the natural filtration of $\mathscr{U}$ invariant, that is, $\sigma(\mathscr{U}_i) \subset \mathscr{U}_i$ for all $i \geq 0$. Therefore, it induces linear transformations σ_i of $\mathscr{U}_i/\mathscr{U}_{i+1}$, $i \geq 0$. Note that the matrix of σ_1 relative to $\mathbb{F}$-basis $\{x_1 + \mathscr{U}_2, \ldots, x_{2m} + \mathscr{U}_2\}$ is just $(\mathrm{pr}_{[0]}(D_i y_j))$. It follows that σ_1 is bijective. Proceeding by induction on $i \geq 1$, one may prove that σ_i is bijective. Now our claim follows.

Note that $\tilde{\sigma}(D_i)(y_j) = (\sigma D_i \sigma^{-1})(y_j) = \sigma(D_i x_j) = \delta_{ij} = \phi(D_i)(y_j)$ for all $i, j \in Y$. Since $\{y_j \mid j \in Y\}$ generates $\mathscr{U}$, we conclude that $\tilde{\sigma}(D_i) = \phi(D_i)$, $i \in Y$. By induction on k, we may prove that $\tilde{\sigma}|_{\mathscr{H}_{[k]}} = \phi|_{\mathscr{H}_{[k]}}$, $k \geq -1$, that is, $\tilde{\sigma}|_{\mathscr{H}} = \phi$. The proof is complete. □

To prove the next theorem, we establish the following lemma.

LEMMA 4.5. *The natural filtration of $\mathscr{U}$ is invariant under automorphisms of $\mathscr{U}$.*

PROOF. Since $\operatorname{Der}\mathscr{U} = W$, we have $\operatorname{Aut}\mathscr{U} = \operatorname{Aut}(\mathscr{U} : W)$. By [25, Theorem 1], the natural filtration of W is invariant under $\operatorname{Aut} W$. Note that $\tilde{\sigma}(aD_i) = \sigma(a)\tilde{\sigma}$, $\sigma \in \operatorname{Aut}\mathscr{U}$, $a \in \mathscr{U}$, $i \in Y$, which implies the desired result. □

Following [20], we introduce some notations. For $X = \mathscr{U}$ or $\mathscr{H}$, put

$$\operatorname{Aut}^* X = \{\sigma \in \operatorname{Aut} X \mid \sigma(X_{[j]}) \subset X_{[j]},\ j \in \mathbb{Z}\};$$
$$\operatorname{Aut}_i X = \{\sigma \in \operatorname{Aut} X \mid (\sigma - 1)(X_j) \subset X_{i+j},\ j \in \mathbb{Z}\}, \quad i \geq 0.$$

According to Lemma 4.5 and Corollary 3.5, the natural filtration of X is invariant under $\operatorname{Aut} X$. Thus $\operatorname{Aut}^* X < \operatorname{Aut} X$ and $\operatorname{Aut}_i X \lhd \operatorname{Aut} X$, $i \geq 0$. We call $\operatorname{Aut}_0 X > \operatorname{Aut}_1 X > \operatorname{Aut}_2 X > \cdots$ the standard normal series of $\operatorname{Aut} X$.

Set $\operatorname{Aut}^*(\mathscr{U} : \mathscr{H}) = \operatorname{Aut}^*\mathscr{U} \cap \operatorname{Aut}(\mathscr{U} : \mathscr{H})$ and $\operatorname{Aut}_i(\mathscr{U} : \mathscr{H}) = \operatorname{Aut}_i\mathscr{U} \cap \operatorname{Aut}(\mathscr{U} : \mathscr{H})$. We call $\operatorname{Aut}^*(\mathscr{U} : \mathscr{H})$ the homogeneous admissible automorphism group of $\mathscr{U}$, and $\operatorname{Aut}_0(\mathscr{U} : \mathscr{H}) > \operatorname{Aut}_1(\mathscr{U} : \mathscr{H}) > \cdots$ the standard normal series of $\operatorname{Aut}(\mathscr{U} : \mathscr{H})$.

THEOREM 4.6. *Suppose that Φ is defined as in Theorem 4.2. Then*

(i) $\Phi(\operatorname{Aut}_i(\mathscr{U} : \mathscr{H})) = \operatorname{Aut}_i\mathscr{H}$, $i \geq 0$;
(ii) $\Phi(\operatorname{Aut}^*(\mathscr{U} : \mathscr{H})) = \operatorname{Aut}^*\mathscr{H}$;
(iii) $\operatorname{Aut}_1\mathscr{H}$ *is a solvable normal subgroup of* $\operatorname{Aut}\mathscr{H}$;
(iv) $\operatorname{Aut}\mathscr{H} = \operatorname{Aut}_1\mathscr{H} \rtimes \operatorname{Aut}^*\mathscr{H}$.

PROOF. (i) We first prove the inclusion '$\subset$'. Let $\sigma \in \operatorname{Aut}_i(\mathscr{U} : \mathscr{H})$. Then $\sigma^{-1} \in \operatorname{Aut}_i(\mathscr{U} : \mathscr{H})$. For $k \in \mathbb{N}_0$ and $f \in \mathscr{U}_k$, we may suppose that $\sigma^{-1} f = f + f'$, $f' \in \mathscr{U}_{i+k}$, $\sigma(D_j f) = D_j f + f''$, $f'' \in \mathscr{U}_{i+k-1}$. By Lemma 4.5, $\sigma(D_j f') \in \mathscr{U}_{i+k-1}$. Note that

$$\begin{aligned}\tilde{\sigma}(D_j)(f) &= \sigma D_j \sigma^{-1}(f) = \sigma D_j(f + f')\\ &= \sigma(D_j f + D_j f') = D_j f + f'' + \sigma(D_j f').\end{aligned}$$

We obtain that $\tilde{\sigma}(D_j) f \equiv D_j f \pmod{\mathscr{U}_{i+k-1}}$. This implies that $\tilde{\sigma}(D_j) \equiv D_j \pmod{W_{i-1}}$, $j \in Y$. Notice that $\tilde{\sigma}(aD_j) = \sigma(a)\tilde{\sigma}(D_j)$, $j \in Y$, $a \in \mathscr{U}_l$. We may obtain that $\tilde{\sigma}(aD_j) \equiv aD_j \pmod{W_{i+l-1}}$. Therefore $\tilde{\sigma} \in \operatorname{Aut}_i W$. Thus $\tilde{\sigma} \in \operatorname{Aut}_i W \cap \operatorname{Aut}\mathscr{H} \subset \operatorname{Aut}_i\mathscr{H}$, and $\Phi(\operatorname{Aut}_i(\mathscr{U} : W)) \subset \operatorname{Aut}_i W$.

To prove the converse inclusion, suppose that $\varphi \in \operatorname{Aut}_i\mathscr{H}$, $i \geq 0$ and set $\sigma := \Phi^{-1}(\varphi)$. Given $j \in Y$, pick $k \in Y \setminus j'$. By Lemma 2.6 (i), $\mathrm{T}_{\mathrm{H}}(x_{k'}x_j) = (-1)^{\mu(k')+\mu(k')\mu(j)} x_j D_k + (-1)^{\mu(j)} x_{k'} D_{j'}$. Then

$$\begin{aligned}(13)\qquad &(-1)^{\mu(k')+\mu(k')\mu(j)}\sigma(x_j)(\varphi D_k) + (-1)^{\mu(j)}\sigma(x_{k'})(\varphi D_{j'})\\ &= \varphi(\mathrm{T}_{\mathrm{H}}(x_{k'}x_j))\\ &\equiv (-1)^{\mu(k')+\mu(k')\mu(j)} x_j D_k + (-1)^{\mu(j)} x_{k'} D_{j'} \pmod{\mathscr{H}_i}.\end{aligned}$$

Noticing that $\varphi \in \mathrm{Aut}_i\,\mathscr{H}$ and $W_{[-1]} = \mathscr{H}_{[-1]}$, we have

$$\text{(14)}\qquad \varphi(D_k) = D_k + E_1, \quad \varphi(D_{j'}) = D_{j'} + E_2, \quad \text{where } E_1, E_2 \in \mathscr{H}_{i-1}.$$

By Lemma 4.5, it is easy to see that $\sigma(x_j)E_1, \sigma(x_{k'})E_2 \in W_i$. Thus we obtain from (13) and (14),

$$(-1)^{\mu(k')+\mu(k')\mu(j)}(\sigma(x_j) - x_j)D_k + (-1)^{\mu(j)}(\sigma(x_{k'}) - x_{k'})D_{j'} \equiv 0 \pmod{W_i}.$$

Since $k \neq j'$, we obtain $\sigma(x_j) \equiv x_j \pmod{\mathscr{U}_{i+1}}$. Now using induction on $|\alpha| + |u|$, one may prove that $\sigma(x^{(\alpha)}x^u) \equiv x^{(\alpha)}x^u \pmod{\mathscr{U}_{|\alpha|+|u|+i}}$. This means $\sigma \in \mathrm{Aut}_i\,\mathscr{U}$ and therefore $\sigma \in \mathrm{Aut}_i(\mathscr{U} : \mathscr{H})$. Hence $\Phi(\mathrm{Aut}_i(\mathscr{U} : \mathscr{H})) \supset \mathrm{Aut}_i\,\mathscr{H}$.

(ii) The proof is completely analogous to (i), therefore is omitted.

(iii) Using the invariance of the natural filtration (see Corollary 3.5), one may verify directly that $[\mathrm{Aut}_i\,\mathscr{H}, \mathrm{Aut}_j\,\mathscr{H}] \subset \mathrm{Aut}_{i+j}\,\mathscr{H}$, $i, j \geq 0$ (see [19, page 210]). From this we see that the normal series $\mathrm{Aut}_1\,\mathscr{H} > \mathrm{Aut}_2\,\mathscr{H} > \cdots$ is abelian (that is, $\mathrm{Aut}_i\,\mathscr{H} / \mathrm{Aut}_{i+1}\,\mathscr{H}$ are abelian groups, for all $i \geq 1$), and reaches 0. Therefore $\mathrm{Aut}_1\,\mathscr{H}$ is solvable.

(iv) Let $\varphi \in \mathrm{Aut}\,\mathscr{H}$. Then there exists $\varphi_0, \varphi_1 \in \mathrm{Hom}_{\mathbb{F}}(\mathscr{H}, \mathscr{H})$ such that $\varphi = \varphi_0 + \varphi_1$ and $\varphi_0(\mathscr{H}_{[j]}) \subset \mathscr{H}_{[j]}$, $\varphi_1(\mathscr{H}_j) \subset \mathscr{H}_{j+1}$, $j \geq -1$. As the filtration of $\mathscr{H}$ is invariant under $\mathrm{Aut}\,\mathscr{H}$, we have $\varphi_0 \in \mathrm{Aut}^*\,\mathscr{H}$. Therefore, $\varphi_0^{-1}\phi = 1 + \varphi_0^{-1}\varphi_1 \in \mathrm{Aut}_1\,\mathscr{H}$. Hence (iv) holds. □

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