

THE HIGHER ORDER COMMUTATORS OF THE FRACTIONAL INTEGRALS ON HARDY SPACES

SHUNCHAO LONG[✉] and JIAN WANG

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Abstract

In this paper we investigate the boundedness on Hardy spaces for the higher order commutator $T_{b,m}^\tau$ generated by the *BMO* function b and fractional integral type operator T^τ , and establish the boundedness theorems for $T_{b,m}^\tau$ from $H_{b,m}^{p_1, q_1, s}$ to L^{p_2} and to H^{p_2} ($0 < p_1 \leq 1$), and from $H\dot{K}_{q_1, b, m}^{\alpha, p_1, s}$ to $\dot{K}_{q_2}^{\alpha, p_2}$ and to $H\dot{K}_{q_2}^{\alpha, p_2}$, respectively, for certain ranges of α , p_1 , q_1 , p_2 , q_2 and s .

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1. Introduction

Let T be a linear operator and b a *BMO* function. The higher order commutator operators are defined as

$$T_{b,m}f(x) = T((b(\cdot) - b(x))^m f(\cdot))(x), \quad m = 0, 1, 2, \dots$$

Obviously, $T_{b,0} = T$, $T_{b,1} = [b, T]$ which is the commutator in [6], and

$$T_{b,m} = [b, T_{b,m-1}], \quad m = 1, 2, \dots$$

Coifman, Rochberg and Weiss [6] stated that if T is a Calderón-Zygmund singular integral operator, then $T_{b,1} = [b, T]$ is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. Chanillo [4] extended this result to the fractional integral. Subsequently, many authors have

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studied the L^p -boundedness of the commutator $T_{b,1} = [b, T]$ (see, for example, [3, 22, 2, 10]) and the higher order commutators $T_{b,m}$, $m = 0, 1, \dots$, (see [23, 15, 12, 8]). The case $0 < p \leq 1$ was also considered by many authors. When T is a Calderón-Zygmund singular integral operator, Perez [21], Pluszyski [20] and Alvarez [1] showed that $[b, T]$ does not map $H^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$. However, Perez [21] proved that $T_{b,m}$ maps the modified spaces $H_{b,m}^{1,\infty,0}(\mathbb{R}^n)$ to $L^1(\mathbb{R}^n)$, Alvarez [1] obtained that $[b, T]$ maps $H_{b,1}^{p,\infty,0}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$, and Long and Wang [15] proved that $T_{b,m}$ maps $H_{b,m}^{p,q,s}(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ and to $H^p(\mathbb{R}^n)$. In this paper, we will extend these results to the higher order commutators $T_{b,m}^\tau$ of fractional integral type operators T^τ and consider their boundedness from $H_{b,m}^{p_1,q_1,s}(\mathbb{R}^n)$ to $L^{p_2}(\mathbb{R}^n)$ and to $H^{p_2}(\mathbb{R}^n)$ for $0 < p_1 \leq 1$.

On the other hand, Herz type Hardy spaces were recently studied by many authors (see [5, 9, 17, 19, 11, 12, 18, 16]). The boundedness of some operators on Herz spaces and Herz type Hardy spaces can be found in [17, 19, 11, 12, 16, 15, 14]. If T is a standard Calderón-Zygmund operator, $[b, T]$ is bounded on Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ (or $K_q^{\alpha,p}(\mathbb{R}^n)$) for $-n/q \leq \alpha < n(1 - 1/q)$, but not for $\alpha \geq n(1 - 1/q)$ (see [11, 19]). It is not bounded even from $H\dot{K}_q^{\alpha,p,0}(\mathbb{R}^n)$ (or $HK_q^{\alpha,p,0}(\mathbb{R}^n)$) into $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ (or $K_q^{\alpha,p}(\mathbb{R}^n)$) for $\alpha \geq n(1 - 1/q)$. However, $[b, T]$ is bounded from $H\dot{K}_{q,b}^{\alpha,p,0}(\mathbb{R}^n)$ (or $HK_{q,b}^{\alpha,p,0}(\mathbb{R}^n)$) into $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ (or $K_q^{\alpha,p}(\mathbb{R}^n)$) for $n(1 - 1/q) \leq \alpha < n(1 - 1/q) + \gamma$ (see [19]), just as the cases involving the standard Hardy space $H^1(\mathbb{R}^n)$ and the Lebesgue space $L^1(\mathbb{R}^n)$. Long and Wang [15] obtained the boundedness of the higher order commutators $T_{b,m}$ of the Calderón-Zygmund singular integral operators T from $H\dot{K}_{q,b,m}^{\alpha,p,s}(\mathbb{R}^n)$ (or $HK_{q,b,m}^{\alpha,p,s}(\mathbb{R}^n)$) into $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ (or $K_q^{\alpha,p}(\mathbb{R}^n)$), and from $H\dot{K}_{q,b,m}^{\alpha,p,s}(\mathbb{R}^n)$ (or $HK_{q,b,m}^{\alpha,p,s}(\mathbb{R}^n)$) into $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ (or $HK_q^{\alpha,p}(\mathbb{R}^n)$) for some ranges of p, q, s and α . The boundedness of higher order commutators of fractional integrals on Herz spaces was obtained for a range of α in [12]. Here, we will also investigate the boundedness for the higher order commutators $T_{b,m}^\tau$ of the fractional integral type operators T^τ from the Herz type Hardy spaces $H\dot{K}_{q_1,b,m}^{\alpha,p_1,s}(\mathbb{R}^n)$ (or $HK_{q_1,b,m}^{\alpha,p_1,s}(\mathbb{R}^n)$) to Herz spaces $\dot{K}_{q_2}^{\alpha,p_2}(\mathbb{R}^n)$ (or $K_{q_2}^{\alpha,p_2}(\mathbb{R}^n)$) and from $H\dot{K}_{q_1,b,m}^{\alpha,p_1,s}(\mathbb{R}^n)$ (or $HK_{q_1,b,m}^{\alpha,p_1,s}(\mathbb{R}^n)$) to $H\dot{K}_{q_2}^{\alpha,p_2}(\mathbb{R}^n)$ (or $HK_{q_2}^{\alpha,p_2}(\mathbb{R}^n)$) for certain ranges of $\alpha, p_1, q_1, p_2, q_2$ and s .

Let us introduce some definitions below.

DEFINITION 1. Let $0 \leq \tau < n$, $0 < \gamma \leq 1$, $s \in \mathbb{N} \cup \{0\}$, $1 < q_1 \leq q_2 < \infty$ be such that $1/q_1 - 1/q_2 = \tau/n$. T^τ is said to be a $(q_1, \tau; s, \gamma)$ -fractional integral type operator if T^τ is a bounded singular integral operator from $L^{q_1}(\mathbb{R}^n)$ into $L^{q_2}(\mathbb{R}^n)$ with kernel $K(x, y)$, which is C^∞ away from the origin and satisfies the following conditions:

- (i) $T^\tau f(x) = \int_{\mathbb{R}^n} K(x, y) f(y) dy$, if $x \neq y$;
- (ii) $\left| \frac{\partial^\zeta K(x, y)}{\partial y^\zeta} - \frac{\partial^\zeta K(x, y')}{\partial y^\zeta} \right| \leq C_\zeta \frac{|y - y'|^\gamma}{|x - y|^{n-\tau+s+\gamma}}$, if $|x - y| \geq 2|y - y'|$, where

$\zeta = (\zeta_1, \dots, \zeta_n)$ is any multi-index and $s = |\zeta| = \zeta_1 + \dots + \zeta_n$.

Denote by $[r]$ the integer part of the real number r . For $\beta = (\beta_1, \dots, \beta_n) \in (\mathbb{N} \cup \{0\})^n$, $x^\beta = x_1^{\beta_1} \dots x_n^{\beta_n}$ and $|\beta| = \beta_1 + \dots + \beta_n$. Let

$$\|f\|_{L^q(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |f(x)|^q dx \right)^{1/q}.$$

Denote by T^* the conjugate operator of T .

DEFINITION 2. Let $0 < p \leq 1 \leq q \leq \infty$, $p < q$, $[n(1/p - 1)] \leq s < \infty$. A function $a(x)$ is said to be a $(p, q, s; b)$ -atom of order m if there exists a ball B for which

- (i) $\text{supp } a \subseteq B = B(x_0, r) = \{x : |x - x_0| < r\}$;
- (ii) $\|a\|_{L^q(\mathbb{R}^n)} \leq |B|^{1/q-1/p}$;
- (iii) $\int_{\mathbb{R}^n} a(x)b^i(x)x^\beta dx = 0$; $|\beta| \leq s$, $i = 0, 1, \dots, m$.

DEFINITION 3. Let $0 < p \leq 1 \leq q \leq \infty$, $p < q$, $[n(1/p - 1)] \leq s < \infty$. We define $f \in H_{b,m}^{p,q,s}(\mathbb{R}^n)$ if and only if $f(x) = \sum_{k \in \mathbb{N}} \lambda_k a_k(x)$, where each a_k is a $(p, q, s; b)$ -atom of order m , $\sum_{k \in \mathbb{N}} |\lambda_k|^p < +\infty$, and $\|f\|_{H_{b,m}^{p,q,s}(\mathbb{R}^n)} \sim \left(\sum_{k \in \mathbb{N}} |\lambda_k|^p \right)^{1/p}$.

Obviously, $H_{b,m}^{1,\infty,0}(\mathbb{R}^n)$ are the spaces $H_{b,m}^1(\mathbb{R}^n)$ which were introduced by Perez in [21]. By the atomic decomposition theory of Coifman and Weiss [7, 13, 25], if $0 < p \leq 1 \leq q \leq \infty$, $p < q$ and $[n(1/p - 1)] \leq s < \infty$, it is easy to see that $H_{b,0}^{p,q,s}(\mathbb{R}^n) = H^p(\mathbb{R}^n)$, the classical Hardy spaces.

THEOREM 1.1. Let $0 < p_1 \leq 1$, $1/p_2 = 1/p_1 - \tau/n$, $0 < \gamma \leq 1$, $s > [n(1/p_1 - 1)]$, $1 < q_1 \leq \infty$, and let T^τ be a $(q_1, \tau; s, \gamma)$ -fractional integral type operator (as in Definition 1) and $b \in BMO$. If $n/(n - \tau + s + \gamma) < p_2 < +\infty$ and $0 \leq \tau < n$, then $T_{b,m}^\tau$ maps $H_{b,m}^{p_1,q_1,s}(\mathbb{R}^n)$ into $L^{p_2}(\mathbb{R}^n)$.

REMARK 1. Theorem 1.1 is equivalent to [21, Theorem 1.9] when $\tau = 0$, $p_1 = p_2 = 1$, $s = 0$, $q_1 = \infty$; and to [1, Theorem 1.5] when $\tau = 0$, $m = 1$, $s = 0$.

THEOREM 1.2. Let $0 < p_1 \leq 1$, $1/p_2 = 1/p_1 - \tau/n$, $0 < \gamma \leq 1$, $s > [n(1/p_1 - 1)]$, $1 < q_1 \leq \infty$, and let T^τ be a $(q_1, \tau; s, \gamma)$ -fractional integral type operator and $b \in BMO$. Assume that $(T^\tau)^*(g_{i,\beta}) = C$ (a constant), $g_{i,\beta}(x) = b^i(x)x^\beta$, $|\beta| \leq s$, $i = 0, 1, \dots, m$. If $n/(n - \tau + s + \gamma) < p_2 \leq 1$ and $0 \leq \tau < \gamma$, then $T_{b,m}^\tau$ maps $H_{b,m}^{p_1,q_1,s}(\mathbb{R}^n)$ into $H^{p_2}(\mathbb{R}^n)$.

Let $B_k = \{x \in \mathbb{R}^n : |x| \leq 2^k\}$, $C_k = B_k \setminus B_{k-1}$, and $\chi_k = \chi_{C_k}$ for $k \in \mathbb{Z}$, where χ_{C_k} is the characteristic function of set C_k .

DEFINITION 4. Let $0 < \alpha < \infty$, $0 < p < \infty$, $1 \leq q < \infty$. The Herz spaces are defined by

(a) $\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \{f \in L_{loc}^q(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} < +\infty\}$ (homogeneous space), where

$$\|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)} = \left(\sum_{k \in \mathbb{Z}} |B_k|^{\alpha p/n} \|f \chi_k\|_{L^q(\mathbb{R}^n)}^p \right)^{1/p},$$

(b) $K_q^{\alpha,p}(\mathbb{R}^n) = L^q(\mathbb{R}^n) \cap \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ (non-homogeneous space), and $\|f\|_{K_q^{\alpha,p}(\mathbb{R}^n)} = \|f\|_{L^q(\mathbb{R}^n)} + \|f\|_{\dot{K}_q^{\alpha,p}(\mathbb{R}^n)}$.

Obviously, $\dot{K}_p^{0,p}(\mathbb{R}^n) = L^p(\mathbb{R}^n) = K_p^{0,p}(\mathbb{R}^n)$ for all $0 < p \leq \infty$.

DEFINITION 5. Let $-\infty < \alpha < \infty$, $0 < p < \infty$, $1 \leq q < \infty$ and $s \in \mathbb{N} \cup \{0\}$. A function $a(x)$ is said to be an m th order central $H\dot{K} - (\alpha, q, ; b)_s$ -atom if $a(x)$ satisfies

- (i) $\text{supp } a \subseteq B(0, r) = \{x \in \mathbb{R}^n : |x| < r, r > 0\}$;
- (ii) $\|a\|_{L^q(\mathbb{R}^n)} \leq |B(0, r)|^{-\alpha/n}$;
- (iii) $\int_{\mathbb{R}^n} a(x) b^i(x) x^\beta dx = 0, |\beta| \leq s, i = 0, 1, \dots, m$.

A function $a(x)$ is said to be an m th order central $HK - (\alpha, q, ; b)_s$ -atom if $a(x)$ satisfies (ii), (iii) and

- (i') $\text{supp } a \subseteq \overline{B(0, r)} = \{x \in \mathbb{R}^n : |x| < r, r > 1\}$.

DEFINITION 6. Let $0 < p < \infty$, $1 < q < \infty$, $n(1-1/q) \leq \alpha < \infty$ and $s \in \mathbb{N} \cup \{0\}$. We define $f \in H\dot{K}_{q,b,m}^{\alpha,p,s}(\mathbb{R}^n)$ (or $HK_{q,b,m}^{\alpha,p,s}(\mathbb{R}^n)$) if and only if $f(x) = \sum_{k \in \mathbb{Z}} \lambda_k a_k(x)$ (or $f(x) = \sum_{k \geq 0} \lambda_k a_k(x)$), where each a_k is an m order central $H\dot{K} -$ (or $HK -$) $(\alpha, q, ; b)_s$ -atom with the support B_k , $\sum_{k \in \mathbb{Z}} |\lambda_k|^p < +\infty$ (or $\sum_{k \geq 0} |\lambda_k|^p < +\infty$), and

$$\|f\|_{H\dot{K}_{q,b,m}^{\alpha,p,s}(\mathbb{R}^n)} \sim \left(\sum_{k \in \mathbb{Z}} |\lambda_k|^p \right)^{1/p} \quad \left(\text{or } \|f\|_{HK_{q,b,m}^{\alpha,p,s}(\mathbb{R}^n)} \sim \left(\sum_{k \geq 0} |\lambda_k|^p \right)^{1/p} \right).$$

If $0 < p < \infty$, $1 < q < \infty$, $n(1-1/q) \leq \alpha < \infty$ and $s \geq [\alpha - n(1-1/q)]$, it is easy to see that $H\dot{K}_{q,b,0}^{\alpha,p,s}(\mathbb{R}^n) = H\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$, $HK_{q,b,0}^{\alpha,p,s}(\mathbb{R}^n) = HK_q^{\alpha,p}(\mathbb{R}^n)$ (see [17, 11] or [19, 18]). For $0 < p < \infty$, $H\dot{K}_p^{0,p}(\mathbb{R}^n) = HK_p^{0,p}(\mathbb{R}^n)$ which are the usual Hardy spaces $H^p(\mathbb{R}^n)$. In particular, $H^p(\mathbb{R}^n) = L^p(\mathbb{R}^n)$ when $p > 1$. When $1 < q < \infty$, $-n/q < \alpha < n(1-1/q)$ and $0 < p \leq \infty$, $H\dot{K}_q^{\alpha,p}(\mathbb{R}^n) = \dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ and $HK_q^{\alpha,p}(\mathbb{R}^n) = K_q^{\alpha,p}(\mathbb{R}^n)$ (see [11]).

THEOREM 1.3. Let $0 \leq \tau < n$, $s \in \mathbb{N} \cup \{0\}$, $0 < \gamma \leq 1$, $0 < p_1 \leq p_2 < \infty$, $1 < q_1 < \infty$, $1/q_2 = 1/q_1 - \tau/n$, $n(1-1/q_1) \leq \alpha < s + \gamma + n(1-1/q_1)$, and let

$b \in BMO$ and T^τ be a $(q_1, \tau; s, \gamma)$ -fractional integral type operator. Then $T_{b,m}^\tau$ maps $H\dot{K}_{q_1,b,m}^{\alpha,p_1,s}(\mathbb{R}^n)$ into $\dot{K}_{q_2}^{\alpha,p_2}(\mathbb{R}^n)$ and $HK_{q_1,b,m}^{\alpha,p_1,s}(\mathbb{R}^n)$ into $K_{q_2}^{\alpha,p_2}(\mathbb{R}^n)$, respectively.

REMARK 2. Theorem 3 is [19, Theorem 3.1] when $\tau = 0, s = 0, m = 1$.

THEOREM 1.4. Let $p_1, p_2, q_1, q_2, \tau, \alpha, s, \gamma, T^\tau$ and b be as in Theorem 1.3. Assume that $(T^\tau)^*(g_{i,\beta}) = C, g_{i,\beta}(x) = b^i(x)x^\beta, |\beta| \leq s, i = 0, 1, 2, \dots, m$. Then $T_{b,m}^\tau(\mathbb{R}^n)$ maps $H\dot{K}_{q_1,b,m}^{\alpha,p_1,s}(\mathbb{R}^n)$ into $H\dot{K}_{q_2}^{\alpha,p_2}(\mathbb{R}^n)$ and $HK_{q_1,b,m}^{\alpha,p_1,s}(\mathbb{R}^n)$ into $HK_{q_2}^{\alpha,p_2}(\mathbb{R}^n)$, respectively.

Denote by BMO the space of measurable functions b such that

$$\int_B |b(x) - b_B| dx < C|B|$$

holds for all balls B , where $b_B = |B|^{-1} \int_B b(x) dx$ with a constant C independent of B .

Throughout this paper, C always means a constant independent of the main parameters involved, but which may be different from line to line. For any power exponent p with $1 \leq p \leq \infty$, we denote the conjugate exponent $p/(p-1)$ by p' .

2. Proofs of the theorems

First we prove two lemmas.

LEMMA 2.1. Let $1 < q \leq \infty, T^\tau$ be the $(q_1, \tau; s, \gamma)$ -fractional integral type operator defined as above and $b(x) \in BMO$. If for $i = 0, 1, \dots, m, a(x)b^i(x)$ satisfy s order vanishing moments with $\text{supp } a \subset B$ with the center at $x_0 = 0$ and $x \in (2B)^c$, then

$$|T_{b,m}^\tau a(x)| \leq C|B|^{(s+\gamma)/n+1/q'} \|a\|_{L^q(\mathbb{R}^n)} \left(\frac{|b(x) - b_B|^m}{|x|^{n-\tau+s+\gamma}} + \frac{\|b\|_{BMO}^m}{|x|^{n-\tau+s+\gamma}} \right).$$

PROOF. Using the s th order vanishing moments of $a(x)b^i(x), i = 0, 1, \dots, m$, we have

$$\begin{aligned} T_{b,m}^\tau a(x) &= \int_{\mathbb{R}^n} (b(x) - b(y))^m K(x, y) a(y) dy \\ &= \int_{\mathbb{R}^n} (b(x) - b(y))^m (K(x, y) - P(x, y)) a(y) dy, \end{aligned}$$

where $P(x, y)$ is the $(s-1)$ th order Taylor's expansion for $K(x, y)$ as a function of y at $y = 0$. Again, using the s th order vanishing moments of $a(x)b^i(x)$, $i = 0, 1, \dots, m$, we have

$$\begin{aligned} T_{b,m}^\tau a(x) &= \int_{\mathbb{R}^n} \sum_{|\zeta|=s} C_{s,\zeta} \frac{\partial^\zeta K(x, y_0)}{\partial y^\zeta} y^\zeta (b(x) - b(y))^m a(y) dy \\ &= \int_{\mathbb{R}^n} \sum_{|\zeta|=s} C_{s,\zeta} \left(\frac{\partial^\zeta K(x, y_0)}{\partial y^\zeta} - \frac{\partial^\zeta K(x, 0)}{\partial y^\zeta} \right) y^\zeta (b(x) - b(y))^m a(y) dy, \end{aligned}$$

where y_0 is a point on the line segment connecting y and 0 . Thus, by (ii) in Definition 1, and since $y_0 \in B$, $y \in B$, $|\zeta| = s$, $\|a\|_{L^1(\mathbb{R}^n)} \leq |B|^{1/q'} \|a\|_{L^q(\mathbb{R}^n)}$ and

$$\begin{aligned} \int_{\mathbb{R}^n} |b(y) - b_B|^m |a(y)| dy &\leq \|a\|_{L^q(\mathbb{R}^n)} \left(\int_B |b(x) - b_B|^{mq'} dy \right)^{1/q'} \\ &\leq C \|b\|_{BMO}^m |B|^{1/q'} \|a\|_{L^q(\mathbb{R}^n)}, \end{aligned}$$

we have

$$\begin{aligned} |T_{b,m}^\tau a(x)| &\leq C \frac{|B|^{(s+\gamma)/n}}{|x|^{n-\tau+s+\gamma}} \int_{\mathbb{R}^n} |b(x) - b(y)|^m |a(y)| dy \\ &\leq C \frac{|B|^{(s+\gamma)/n}}{|x|^{n-\tau+s+\gamma}} \left(|b(x) - b_B|^m \int_{\mathbb{R}^n} |a(y)| dy \right. \\ &\quad \left. + \int_{\mathbb{R}^n} |b(y) - b_B|^m |a(y)| dy \right) \\ &\leq C |B|^{(s+\gamma)/n+1/q'} \|a\|_{L^q(\mathbb{R}^n)} \left(\frac{|b(x) - b_B|^m}{|x|^{n-\tau+s+\gamma}} + \frac{\|b\|_{BMO}^m}{|x|^{n-\tau+s+\gamma}} \right). \quad \square \end{aligned}$$

LEMMA 2.2. *Let $q > 0$, $m \in \mathbb{N} \cup \{0\}$, $B = B(0, r)$. Then*

$$\begin{aligned} \Pi &= \int_{\mathbb{R}^n \setminus 4B} \frac{|b(x) - b_B|^{mq}}{|x|^{(n-\tau+s+\gamma)q}} |x|^\delta dx \\ &\leq C \|b\|_{BMO}^{mq} |B|^{-((n-\tau+s+\gamma)q-\delta-n)/n} \sum_{j=2}^{\infty} (j+1)^{mq} 2^{-j((n-\tau+s+\gamma)q-\delta-n)}. \end{aligned}$$

PROOF.

$$\begin{aligned} \Pi &= \sum_{j=2}^{\infty} \int_{2^j r < |x| < 2^{j+1} r} \frac{|b(x) - b_B|^{mq}}{|x|^{(n-\tau+s+\gamma)q-\delta}} dx \\ &\leq \sum_{j=2}^{\infty} \frac{1}{|2^j r|^{(n-\tau+s+\gamma)q-\delta}} \int_{2^{j+1} B} |b(x) - b_B|^{mq} dx \\ &= C \sum_{j=2}^{\infty} \frac{1}{(2^j r)^{(n-\tau+s+\gamma)q-\delta-n}} \frac{1}{(2^{j+1} r)^n} \int_{2^{j+1} B} |b(x) - b_B|^{mq} dx. \end{aligned}$$

When $mq \geq 1$, we have

$$\begin{aligned}
& \frac{1}{(2^{j+1}r)^n} \int_{2^{j+1}B} |b(x) - b_B|^{mq} dx \\
& \leq C \frac{1}{(2^{j+1}r)^n} \int_{2^{j+1}B} |b(x) - b_{2^{j+1}B}|^{mq} dx + C \left(\sum_{i=0}^j |b_{2^i B} - b_{2^{i+1}B}| \right)^{mq} \\
& \leq C \|b\|_{BMO}^{mq} + C \left(\sum_{i=0}^j \|b\|_{BMO} \right)^{mq} \\
& = C(j+1)^{mq} \|b\|_{BMO}^{mq}.
\end{aligned}$$

When $0 < mq < 1$, by Hölder's inequality, we have

$$\begin{aligned}
& \frac{1}{(2^{j+1}r)^n} \int_{2^{j+1}B} |b(x) - b_B|^{mq} dx \\
& \leq \frac{1}{(2^{j+1}r)^n} \left(\int_{2^{j+1}B} |b(x) - b_B| dx \right)^{mq} |2^{j+1}B|^{1-mq} \\
& \leq \left(\frac{1}{(2^{j+1}r)^n} \int_{2^{j+1}B} |b(x) - b_{2^{j+1}B}| dx + \sum_{i=0}^j |b_{2^i B} - b_{2^{i+1}B}| \right)^{mq} \\
& \leq C(j+1)^{mq} \|b\|_{BMO}^{mq}.
\end{aligned}$$

When $mq = 0$,

$$\frac{1}{(2^{j+1}r)^n} \int_{2^{j+1}B} |b(x) - b_B|^{mq} dx = C.$$

Thus, we have finished the proof of Lemma 2.2. \square

PROOF OF THEOREM 1.1. We need to prove that there exists a constant C such that for each function f in $H_{b,m}^{p_1, q_1, s}(\mathbb{R}^n)$,

$$\|T_{b,m}^\tau f\|_{L^{p_2}(\mathbb{R}^n)} \leq C \|f\|_{H_{b,m}^{p_1, q_1, s}(\mathbb{R}^n)},$$

where q_1, q_2 are as in Definition 1, that is, $1 < q_1 \leq q_2 \leq \infty$ such that $1/q_1 - 1/q_2 = \tau/n (= 1/p_1 - 1/p_2)$. By a standard argument, it is enough to show that there exists a constant C independent of a such that $\|T_{b,m}^\tau a\|_{L^{p_2}(\mathbb{R}^n)} \leq C$ for each $(p_1, q_1, s; b)$ -atom a of order m .

To prove this, without loss of generality, we may suppose that $\text{supp } a \subset B$ with center at the origin. Then

$$\begin{aligned}
\int_{\mathbb{R}^n} |T_{b,m}^\tau a(x)|^{p_2} dx &= \left(\int_{4B} + \int_{\mathbb{R}^n \setminus 4B} \right) |T_{b,m}^\tau a(x)|^{p_2} dx \\
&= \text{I} + \text{II}.
\end{aligned}$$

By the boundedness of $T_{b,m}^\tau$ from $L^{q_1}(\mathbb{R}^n)$ into $L^{q_2}(\mathbb{R}^n)$ and Hölder's inequality, we have

$$\begin{aligned} \text{I} &\leq C \left(|B|^{1/p_2-1/q_2} \left(\int_{4B} |T_{b,m}^\tau a(x)|^{q_2} dx \right)^{1/q_2} \right)^{p_2} \\ &\leq C \left(\|b\|_{BMO}^m |B|^{1/p_1-1/q_1} \|a\|_{L^{q_1}(\mathbb{R}^n)} \right)^{p_2} \\ &\leq C \left(\|b\|_{BMO}^m |B|^{1/p_1-1/q_1} |B|^{1/q_1-1/p_1} \right)^{p_2} \\ &= C \|b\|_{BMO}^{mp_2}. \end{aligned}$$

For II, using Lemma 2.1 ($q = q_1$), Lemma 2.2 ($\delta = 0$, $q = p_2$) and $\|a\|_{L^{q_1}(\mathbb{R}^n)} \leq |B|^{1/q_1-1/p_1}$, we have

$$\begin{aligned} \text{II} &\leq C |B|^{(s+\gamma)p_2/n+p_2/q_1'} \|a\|_{L^{q_1}(\mathbb{R}^n)}^{p_2} \\ &\quad \times \left(\int_{\mathbb{R}^n \setminus 4B} \frac{|b(x) - b_B|^{mp_2}}{|x|^{(n-\tau+s+\gamma)p_2}} dx + \int_{\mathbb{R}^n \setminus 4B} \frac{\|b\|_{BMO}^{mp_2}}{|x|^{(n-\tau+s+\gamma)p_2}} dx \right) \\ &\leq C |B|^{(s+\gamma)p_2/n+p_2/q_1'} |B|^{(1/q_1-1/p_1)p_2} \\ &\quad \times \|b\|_{BMO}^{mp_2} |B|^{-(n-\tau+s+\gamma)p_2/n+1} \sum_{j=2}^{\infty} (j+1)^{mp_2} 2^{-j((n-\tau+s+\gamma)p_2-n)} \\ &= C \|b\|_{BMO}^{mp_2}, \end{aligned}$$

since $n/(n-\tau+s+\gamma) < p_2$. This concludes the proof of Theorem 1.1. \square

PROOF OF THEOREM 1.2. By the atom-molecule theory of Coifman and Weiss (see [7, 13, 25]) we need only to prove that there exists a constant C such that for each $(p_1, q_1, s; b)$ -atom a of order m , $T_{b,m}^\tau a$ is a $(p_2, q_2, s, \varepsilon)$ -molecule, that is,

- (i) $|x|^{nd} T_{b,m}^\tau a(x) \in L^{q_2}(\mathbb{R}^n)$;
- (ii) $N_{q_2}(T_{b,m}^\tau a) = \|T_{b,m}^\tau a\|_{L^{q_2}(\mathbb{R}^n)}^{c/d} \| |x|^{nd} T_{b,m}^\tau a(x) \|_{L^{q_2}(\mathbb{R}^n)}^{1-c/d} = C < \infty$;
- (iii) $\int_{\mathbb{R}^n} T_{b,m}^\tau a(x) x^\beta dx = 0$, $|\beta| \leq s$,

where $s \geq s_0 = [n(1/p_2 - 1)]$,

$$\frac{s-\tau+\gamma}{n} > \varepsilon > \max \left\{ \frac{s}{n}, \frac{1}{p_2} - 1 \right\}$$

(($s-\tau+\gamma)/n > 1/p_2 - 1$ since $p_2 > n/(n-\tau+s+\gamma)$), $c = 1 - 1/p_2 + \varepsilon$, $d = 1 - 1/q_2 + \varepsilon$, $0 < p_1 \leq 1$, $0 < \gamma \leq 1$, $s \geq 0$, $1/q_1 - 1/q_2 = \tau/n$.

Let $\text{supp } a \subset B$ with the center at the origin. By the boundedness of $T_{b,m}^\tau$ from $L^{q_1}(\mathbb{R}^n)$ into $L^{q_2}(\mathbb{R}^n)$, we have $\|T_{b,m}^\tau a\|_{L^{q_2}(\mathbb{R}^n)} \leq C \|b\|_{BMO}^m \|a\|_{L^{q_1}(\mathbb{R}^n)}$. Let

$$\begin{aligned} \int_{\mathbb{R}^n} |x|^{q_2 nd} |T_{b,m}^\tau a(x)|^{q_2} dx &= \left(\int_{4B} + \int_{\mathbb{R}^n \setminus 4B} \right) |x|^{q_2 nd} |T_{b,m}^\tau a(x)|^{q_2} dx \\ &= \text{I} + \text{II}. \end{aligned}$$

For I, by the $L^q(\mathbb{R}^n)$ -boundedness of $T_{b,m}^\tau$ and $\|a\|_{L^{q_1}(\mathbb{R}^n)} \leq |B|^{1/q_1-1/p_1}$, we have

$$I \leq C|B|^{q_2d} \|T_{b,m}^\tau a\|_{L^{q_2}(\mathbb{R}^n)}^{q_2} \leq C|B|^{q_2d} \|b\|_{BMO}^{q_2m} \|a\|_{L^{q_1}(\mathbb{R}^n)}^{q_2}.$$

For II, by Lemma 2.1 and Lemma 2.2 ($q = q_2$, $\delta = q_2nd$), we have

$$\begin{aligned} II &\leq C|B|^{q_2(s+\gamma)/n+q_2/q_1'} \|a\|_{L^{q_1}(\mathbb{R}^n)}^{q_2} \\ &\quad \times \left(\int_{\mathbb{R}^n \setminus 4B} \frac{|b(x) - b_B|^{q_2m}}{|x|^{q_2(n-\tau+s+\gamma)-q_2nd}} dx + \int_{\mathbb{R}^n \setminus 4B} \frac{\|b\|_{BMO}^{q_2m}}{|x|^{q_2(n-\tau+s+\gamma)-q_2nd}} dx \right) \\ &\leq C|B|^{q_2(s+\gamma)/n+q_2/q_1'} \|a\|_{L^{q_1}(\mathbb{R}^n)}^{q_2} \\ &\quad \times \|b\|_{BMO}^{q_2m} |B|^{-q_2(n-\tau+s+\gamma)/n+q_2d+1} \sum_{j=2}^{\infty} (j+1)^{q_2m} 2^{-j(q_2(n-\tau+s+\gamma)-nd)-n} \\ &= C\|b\|_{BMO}^{q_2m} |B|^{q_2d} \|a\|_{L^{q_1}(\mathbb{R}^n)}^{q_2}, \end{aligned}$$

since $q_2(n-\tau+s+\gamma) - q_2nd - n = q_2(s+\gamma-\tau-n\varepsilon) > 0$.

Thus,

$$\begin{aligned} N_{q_2}(T_{b,m}^\tau a) &\leq C\|b\|_{BMO}^{mc/d} \|a\|_{L^{q_1}(\mathbb{R}^n)}^{c/d} \|b\|_{BMO}^{m(1-c/d)} |B|^{d(1-c/d)} \|a\|_{L^{q_1}(\mathbb{R}^n)}^{1-c/d} \\ &= C\|b\|_{BMO}^m \|a\|_{L^{q_1}(\mathbb{R}^n)} |B|^{1/p_2-1/q_2} \leq C. \end{aligned}$$

We have proved (i) and (ii). By $(T^\tau)^*(g_{i,\beta}) = C$, $|\beta| \leq s$, $i = 0, 1, 2, \dots, m$, and the vanishing moments of a , (iii) is obvious.

This concludes the proof of Theorem 1.2. \square

PROOF OF THEOREM 1.3. Let $f \in H\dot{K}_{q_1, b, m}^{\alpha, p_1, s}(\mathbb{R}^n)$, that is, $f(x) = \sum_{j=-\infty}^{\infty} \lambda_j a_j(x)$, where each a_j is an m -order dyadic central $H\dot{K}(\alpha, q_1; b)_s$ -atom with support B_j , and $\|f\|_{H\dot{K}_{q_1, b, m}^{\alpha, p_1, s}(\mathbb{R}^n)} \sim (\sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1})^{1/p_1}$. Then,

$$\begin{aligned} \|T_{b,m}^\tau f\|_{\dot{K}_{q_2}^{\alpha, p_2}(\mathbb{R}^n)}^{p_1} &\leq C \left(\sum_{k=-\infty}^{\infty} 2^{k\alpha p_2} \|(T_{b,m}^\tau f)\chi_k\|_{L^{q_2}(\mathbb{R}^n)}^{p_2} \right)^{p_1/p_2} \\ &\leq C \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \|(T_{b,m}^\tau f)\chi_k\|_{L^{q_2}(\mathbb{R}^n)}^{p_1} \\ &\leq C \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| \|(T_{b,m}^\tau a_j)\chi_k\|_{L^{q_2}(\mathbb{R}^n)} \right)^{p_1} \\ &\quad + C \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left(\sum_{j=k-1}^{\infty} |\lambda_j| \|(T_{b,m}^\tau a_j)\chi_k\|_{L^{q_2}(\mathbb{R}^n)} \right)^{p_1} \\ &= I_1 + I_2. \end{aligned}$$

For I_2 , by the $L^q(\mathbb{R}^n)$ -boundedness of $T_{b,m}^\tau$ and Hölder's inequality, we have

$$\begin{aligned}
I_2 &\leq C \sum_{k=-\infty}^{\infty} 2^{k\alpha p_1} \left(\sum_{j=k-1}^{\infty} \|b\|_{BMO}^m |\lambda_j| \|a_j\|_{L^{q_1}(\mathbb{R}^n)} \right)^{p_1} \\
&\leq C \|b\|_{BMO}^{p_1 m} \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-1}^{\infty} 2^{(k-j)\alpha} |\lambda_j| \right)^{p_1} \\
&\leq \begin{cases} C \|b\|_{BMO}^{p_1 m} \sum_{k=-\infty}^{\infty} \sum_{j=k-1}^{\infty} 2^{(k-j)p_1 \alpha} |\lambda_j|^{p_1} & \text{if } 0 < p_1 \leq 1 \\ C \|b\|_{BMO}^{p_1 m} \sum_{k=-\infty}^{\infty} \left(\sum_{j=k-1}^{\infty} 2^{(k-j)p_1 \alpha/2} |\lambda_j|^{p_1} \right) \\ \quad \times \left(\sum_{j=k-1}^{\infty} 2^{(k-j)p_1' \alpha/2} \right)^{p_1/p_1'} & \text{if } 1 < p_1 < \infty \end{cases} \\
&\leq \begin{cases} C \|b\|_{BMO}^{p_1 m} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1} \sum_{k=-\infty}^{j+1} 2^{(k-j)p_1 \alpha} & \text{if } 0 < p_1 \leq 1 \\ C \|b\|_{BMO}^{p_1 m} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1} \sum_{k=-\infty}^{j+1} 2^{(k-j)p_1 \alpha/2} & \text{if } 1 < p_1 < \infty \end{cases} \\
&\leq C \|b\|_{BMO}^{p_1 m} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}.
\end{aligned}$$

That is, $I_2 \leq C \|b\|_{BMO}^{p_1 m} \|f\|_{HK_{q_1, b, m}^{\alpha, p_1, s}(\mathbb{R}^n)}^{p_1}$. For I_1 , since

$$\begin{aligned}
|b(x) - b_{B_j}|^{q_2 m} &\leq C |b(x) - b_{B_k}|^{q_2 m} + C \left(\sum_{i=j}^{k-1} |b_{B_i} - b_{B_{i+1}}| \right)^{q_2 m} \\
&\leq C |b(x) - b_{B_k}|^{q_2 m} + C (k-j)^{q_2 m} \|b\|_{BMO}^{q_2 m},
\end{aligned}$$

and $\int_{C_k} |b(x) - b_{B_k}|^{q_2 m} dx \leq |B_k| \|b\|_{BMO}^{q_2 m} \leq C 2^{kn} (k-j) \|b\|_{BMO}^{q_2 m}$, for $k > j$, we have

$$\begin{aligned}
&\|(T_{b,m}^\tau a_j) \chi_k\|_{L^2(\mathbb{R}^n)}^2 \\
&\leq C |B_j|^{(s+\gamma)q_2/n+q_2/q_1'} \|a_j\|_{L^{q_1}(\mathbb{R}^n)}^2 \\
&\quad \times \left(\int_{C_k} \frac{|b(x) - b_{B_j}|^{q_2 m}}{|x|^{q_2(n-\tau+s+\gamma)}} dx + \int_{C_k} \frac{\|b\|_{BMO}^{q_2 m}}{|x|^{q_2(n-\tau+s+\gamma)}} dx \right) \\
&\leq C |B_j|^{(s+\gamma)q_2/n+q_2/q_1'} \|a_j\|_{L^{q_1}(\mathbb{R}^n)}^2 \\
&\quad \times \left(\int_{C_k} \frac{|b(x) - b_{B_k}|^{q_2 m}}{|x|^{q_2(n-\tau+s+\gamma)}} dx + \int_{C_k} \frac{(k-j)^{q_2 m} \|b\|_{BMO}^{q_2 m}}{|x|^{q_2(n-\tau+s+\gamma)}} dx \right)
\end{aligned}$$

$$\begin{aligned}
&\leq C 2^{j((s+\gamma)q_2+nq_2-nq_2/q_1)-j\alpha q_2} 2^{-k(n-\tau+s+\gamma)q_2} \times 2^{kn} (k-j)^{q_2 m} \|b\|_{BMO}^{q_2 m} \\
&= C 2^{(j-k)((s+\gamma+n)q_2-nq_2/q_1)} 2^{-j\alpha q_2} (k-j)^{q_2 m} \|b\|_{BMO}^{q_2 m}.
\end{aligned}$$

Hence

$$\begin{aligned}
I_1 &\leq C \|b\|_{BMO}^{p_1 m} \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} (k-j)^m 2^{(j-k)(n+s+\gamma-n/q_1-\alpha)} |\lambda_j| \right)^{p_1} \\
&\leq \begin{cases} C \|b\|_{BMO}^{p_1 m} \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{k-2} (k-j)^{mp_1} 2^{(j-k)(n+s+\gamma-n/q_1-\alpha)p_1} |\lambda_j|^{p_1} & \text{if } 0 < p_1 \leq 1 \\ C \|b\|_{BMO}^{p_1 m} \sum_{k=-\infty}^{\infty} \left(\sum_{j=-\infty}^{k-2} 2^{(j-k)(n+s+\gamma-n/q_1-\alpha)p_1/2} |\lambda_j|^{p_1} \right) \\ \quad \times \left(\sum_{j=-\infty}^{k-2} (k-j)^{mp_1'} 2^{(j-k)(n+s+\gamma-n/q_1-\alpha)p_1'/2} \right)^{p_1/p_1'} & \text{if } 1 < p_1 < \infty \end{cases} \\
&\leq \begin{cases} C \|b\|_{BMO}^{p_1 m} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1} \sum_{k=j+2}^{\infty} (k-j)^{mp_1} 2^{(j-k)(n+s+\gamma-n/q_1-\alpha)p_1} & \text{if } 0 < p_1 \leq 1 \\ C \|b\|_{BMO}^{p_1 m} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1} \sum_{k=j+2}^{\infty} 2^{(j-k)(n+s+\gamma-n/q_1-\alpha)p_1/2} & \text{if } 1 < p_1 < \infty \end{cases} \\
&\leq C \|b\|_{BMO}^{p_1 m} \sum_{j=-\infty}^{\infty} |\lambda_j|^{p_1}
\end{aligned}$$

for $\gamma > \alpha + n/q_1 - n - s$. That is, $I_1 \leq C \|b\|_{BMO}^{p_1 m} \|f\|_{H\dot{K}_{q_1, b, m}^{\alpha, p_1, s}(\mathbb{R}^n)}^{p_1}$.

Thus, we have proved the case of homogeneous spaces of Theorem 1.3. The proof of the non-homogeneous case is similar to that of homogeneous case. This concludes the proof of Theorem 1.3. \square

PROOF OF THEOREM 1.4. By the atom-molecule theory of $H\dot{K}_q^{\alpha, p}(\mathbb{R}^n)$ (see [18, Theorem 2.5]), we need only to prove that there exists a constant C such that for each dyadic central m -order $H\dot{K}(\alpha, q_1; b)$ -atom a_k with support B_k , $T_{b, m}^{\tau} a_k$ is a dyadic central $(\alpha, q_2, s, \varepsilon)_k$ -molecule, that is,

- (i) $\|T_{b, m}^{\tau} a_k\|_{L^{q_2}(\mathbb{R}^n)} \leq 2^{-k\alpha}$;
- (ii) $N(T_{b, m}^{\tau} a_k) = \|T_{b, m}^{\tau} a_k\|_{L^{q_2}(\mathbb{R}^n)}^{c/d} \| |x|^{nd} T_{b, m}^{\tau} a_k(x) \|_{L^{q_2}(\mathbb{R}^n)}^{1-c/d} = C < \infty$;
- (iii) $\int_{\mathbb{R}^n} T_{b, m}^{\tau} a_k(x) x^{\beta} dx = 0$, $|\beta| \leq s$,

where $s \geq [\alpha + n(1/q_1 - 1)]$, $(s + \gamma - \tau)/n > \varepsilon > \max\{s/n, \alpha/n + 1/q_1 - 1\}$ (since $(s + \gamma - \tau)/n > \alpha/n + 1/q_1 - 1$), $c = 1 - 1/q_1 - \alpha/n + \varepsilon$, $d = 1 - 1/q_2 + \varepsilon$.

(i) By the $L^q(\mathbb{R}^n)$ -boundedness of $T_{b, m}^{\tau}$, we have

$$\|T_{b, m}^{\tau} a_k\|_{L^{q_2}(\mathbb{R}^n)} \leq C \|b\|_{BMO}^m \|a_k\|_{L^{q_1}(\mathbb{R}^n)} \leq C \|b\|_{BMO}^m 2^{-k\alpha}.$$

(ii)

$$\begin{aligned} \int_{\mathbb{R}^n} |x|^{q_2nd} |T_{b,m}^\tau a_k(x)|^{q_2} dx &= \left(\int_{B_{k+2}} + \int_{\mathbb{R}^n \setminus B_{k+2}} \right) |x|^{q_2nd} |T_{b,m}^\tau a_k(x)|^{q_2} dx \\ &= \text{I} + \text{II}. \end{aligned}$$

For I, by the $L^q(\mathbb{R}^n)$ -boundedness of $T_{b,m}^\tau$, we have

$$\text{I} \leq C 2^{kndq_2} \|T_{b,m}^\tau a_k\|_{L^{q_2}(\mathbb{R}^n)}^{q_2} \leq C 2^{kndq_2} \|b\|_{BMO}^{q_2m} \|a_k\|_{L^{q_1}(\mathbb{R}^n)}^{q_2} \leq C \|b\|_{BMO}^{q_2m} 2^{kq_2(nd-\alpha)}.$$

For II, by Lemma 2.1 and Lemma 2.2 ($\delta = q_2nd$), we have

$$\begin{aligned} \text{II} &\leq C |B_k|^{(s+\gamma)q_2/n+q_2/q_1'} \|a_k\|_{L^{q_1}(\mathbb{R}^n)}^{q_2} \\ &\quad \times \left(\int_{\mathbb{R}^n \setminus B_{k+2}} \frac{|b(x) - b_{B_k}|^{q_2m}}{|x|^{q_2(n-\tau+s+\gamma)-q_2nd}} dx + \int_{\mathbb{R}^n \setminus B_{k+2}} \frac{\|b\|_{BMO}^{q_2m}}{|x|^{q_2(n-\tau+s+\gamma)-q_2nd}} dx \right) \\ &\leq C 2^{nk((s+\gamma)q_2/n+q_2/q_1')} \|a_k\|_{L^{q_1}(\mathbb{R}^n)}^{q_2} \\ &\quad \times \|b\|_{BMO}^{q_2m} 2^{-nk((n-\tau+s+\gamma)q_2/n-q_2d-1)} \sum_{j=2}^{\infty} (j+1)^{q_2m} 2^{-j(q_2((n-\tau+s+\gamma)-nd)-n)} \\ &= C \|b\|_{BMO}^{q_2m} 2^{kq_2nd} \|a_k\|_{L^{q_1}(\mathbb{R}^n)}^{q_2} \sum_{j=2}^{\infty} (j+1)^{q_2m} 2^{-j(s+\gamma-\tau-\varepsilon n)q_2} \\ &= C \|b\|_{BMO}^{q_2m} 2^{kq_2(nd-\alpha)}, \end{aligned}$$

since $(s + \gamma - \tau - \varepsilon n)q_2 > 0$. Therefore,

$$N(T_{b,m}^\tau a_k) \leq C 2^{-k\alpha c/d} 2^{k(nd-\alpha)(1-c/d)} = C.$$

(iii) By $(T^\tau)^*(g_{i,\beta}) = C$, $|\beta| \leq s$, $i = 0, 1, \dots, m$, and the vanishing moments of a_k , the vanishing moments of $T_{b,m}^\tau a_k$ are obvious.

Thus, we have proved the case of homogeneous spaces of Theorem 1.4. The proof of the non-homogeneous case is similar to that of homogeneous case. This concludes the proof of Theorem 1.4. \square

3. Fractional integrals

Let $0 \leq \tau < n$, we define the fractional integrals of T by

$$T^\tau f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\tau}} dx.$$

The proof of the boundedness of T^τ from $L^{q_1}(\mathbb{R}^n)$ into $L^{q_2}(\mathbb{R}^n)$ for $1 < q_1 < q_2 < \infty$ such that $1/q_1 - 1/q_2 = \tau/n$ can be found in [24]. Thus, the fractional integrals T^τ as above satisfy the condition in Definition 1 for $\gamma = 1$ and any $s \geq 0$. Hence, as the special case of Theorems 1.1–1.4, we have the following corollaries.

COROLLARY 3.1. *Let $0 < p_1 \leq 1$, $1/p_2 = 1/p_1 - \tau/n$, $s > [n(1/p_1 - 1)]$, $1 < q_1 \leq \infty$, and let T^τ be a fractional integral operator (as above) and $b \in BMO$. If $n/(n - \tau + s + 1) < p_2 < +\infty$ and $0 \leq \tau < n$, then $T_{b,m}^\tau$ maps $H_{b,m}^{p_1, q_1, s}(\mathbb{R}^n)$ into $L^{p_2}(\mathbb{R}^n)$.*

The case $p_1 = 1$ and $m = 0$ of Corollary 3.1 was proved by Stein and Weiss.

COROLLARY 3.2. *Let $0 < p_1 \leq 1$, $1/p_2 = 1/p_1 - \tau/n$, $s > [n(1/p_1 - 1)]$, $1 < q_1 \leq \infty$, and let T^τ be a fractional integral operator and $b \in BMO$. Assume that $(T^\tau)^*(b^i(x)x^\beta) = C$, $|\beta| \leq s$, $i = 0, 1, \dots, m$. If $n/(n - \tau + s + 1) < p_2 \leq 1$ and $0 \leq \tau < 1$, then $T_{b,m}^\tau$ maps $H_{b,m}^{p_1, q_1, s}(\mathbb{R}^n)$ into $H^{p_2}(\mathbb{R}^n)$.*

The case $0 < p_1 < p_2 \leq 1$ and $m = 0$ of Corollary 3.2 was proved by Taibleson and Weiss [25].

COROLLARY 3.3. *Let $0 \leq \tau < n$, $s \geq 0$, $0 < p_1 \leq p_2 < \infty$, $1 < q_1 < \infty$, $1/q_2 = 1/q_1 - \tau/n$, $n(1 - 1/q_1) \leq \alpha < s + 1 + n(1 - 1/q_1)$, and let T^τ be a fractional integral and $b \in BMO$. Then $T_{b,m}^\tau$ maps $H\dot{K}_{q_1, b, m}^{\alpha, p_1, s}(\mathbb{R}^n)$ into $\dot{K}_{q_2}^{\alpha, p_2}(\mathbb{R}^n)$ and $HK_{q_1, b, m}^{\alpha, p_1, s}(\mathbb{R}^n)$ into $K_{q_2}^{\alpha, p_2}(\mathbb{R}^n)$, respectively.*

COROLLARY 3.4. *Let $p_1, p_2, q_1, q_2, \tau, \alpha, s, T^\tau$ and b be as in Corollary 3.3. Assume that $(T^\tau)^*(b^i(x)x^\beta) = C$, $|\beta| \leq s$, $i = 0, 1, \dots, m$. Then $T_{b,m}^\tau$ maps $H\dot{K}_{q_1, b, m}^{\alpha, p_1, s}(\mathbb{R}^n)$ into $H\dot{K}_{q_2}^{\alpha, p_2}(\mathbb{R}^n)$ and $HK_{q_1, b, m}^{\alpha, p_1, s}(\mathbb{R}^n)$ into $HK_{q_2}^{\alpha, p_2}(\mathbb{R}^n)$, respectively.*

The case $m = 0$ of Corollary 3.3 and Corollary 3.4 can be found in [18, 14].

The case $m = 1$ of Corollary 3.3 can be found in [19].

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Mathematics Department

Xiangtan University

Xiangtan, 411105

P. R. China

e-mail: sclong@xtu.edu.cn, jwang@hnedu.com