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ABOUT DIRECTED UNIONS OF ARTINIAN SUBRINGS OF A VON NEUMANN REGULAR RING

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Abstract

This work is concerned with the question of when a von Neumann regular ring is expressible as a directed union of Artinian subrings.

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1. Introduction

Given a commutative ring R, we recall that R is von Neumann regular if for each element r in R there exists an element r' such that $r = r^2r'$, or equivalently, R is reduced (0 is the only nilpotent element) and zero-dimensional (all prime ideals are maximal). Several papers in the literature have dealt with various aspects of von Neumann regular rings in commutative ring theory (see, for example, [5, 6, 8]). These works are source of motivation for this paper.

The purpose of this paper is to pursue the study of the problem of whether a von Neumann regular ring R is expressible as a directed union of Artinian subrings, raised by Gilmer and Heinzer in 1992 [1, Problem 42].

This fact leads us to consider the family of residue fields of R denoted $\mathscr{F}(R)$. We focus on the behaviour of the family of the residue fields of R. It is well known that if a zero-dimensional ring R with $\mathrm{Idem}(R)$ is finite, then R is a directed union of Artinian subrings [6, Theorem 5.4]. This leads us to examine the case where $\mathrm{Idem}(R)$ is infinite.

Let $\{R_\alpha\}_{\alpha\in A}$ be a nonempty family of rings and $\prod_{\alpha\in A} R_\alpha$ their product. We frequently consider $\prod_{\alpha\in A} R_\alpha$ as the set of all functions $f : A \to \bigcup_{\alpha\in A} R_\alpha$, such that $f(\alpha) \in R_\alpha$ for each $\alpha \in A$, with addition and multiplication defined pointwise: $(f+g)(\alpha) = f(\alpha) + g(\alpha)$ and $(fg)(\alpha) = f(\alpha)g(\alpha)$. In this perspective, the direct sum ideal of $\prod_{\alpha\in A} R_\alpha$, denoted by $\bigoplus_{\alpha\in A} R_\alpha$, is the set of functions $f \in \prod_{\alpha\in A} R_\alpha$ that are finitely nonzero (that is, $\{\alpha \in A : f(\alpha) \neq 0 \text{ in } R_\alpha\}$ is finite).

All rings considered in this paper are assumed to be commutative with a unity element. If S is a subring of a ring R, we assume that R and S have the same unity element. We denote by $\mathscr{A}(R)$ and $\mathscr{C}(R)$, respectively, the set of Artinian subrings of R, and the set $\{\text{char}(R/M) : M \text{ is a maximal ideal of } R\}$.

2. Directed unions of Artinian subrings

Let us fix notation for much of Section 2. The data will consist of a directed system (R_j, f_{jk}) of rings indexed by a directed set $(I, \leq)$. Let $R = \bigcup_{j\in I} R_j$, together with the canonical maps $f_j : R_j \to R$. The ring R is said to be a directed union of the R_j's if the f_{jk}'s are inclusion maps. Thus, directed unions can be treated by assuming all f_{jk} to be monomorphisms. Notice that if R_j is a ring for each $j \in I$, then R is also a ring. However, R needs not be Artinian even if each R_j is. If $R = \bigcup_{j\in I} R_j$ is a directed union of Artinian subrings, then we regard each R_i as a subrings of R, that is, contains the same unity. Turning to the Artinian case, two key properties of an Artinian ring R that come into play are that $\text{Spec}(R)$ is finite and that R has only finitely many idempotents.

We begin this section with two lemmas that we use throughout this paper.

LEMMA 2.1. *Let R be a ring.*

(1) (a) *If R is a directed union of Artinian subrings then $\mathscr{C}(R)$ is finite.*
(b) *If $\mathscr{C}(R)$ is infinite then $\mathscr{A}(R) = \emptyset$.*

(2) *If R is a zero-dimensional ring with finite spectrum, then R is expressible as a finite product of zero-dimensional quasi-local subrings.*

(3) *If R is a von Neumann regular ring, then R is Artinian if and only if R is a finite product of fields if and only if R is Noetherian.*

PROOF. (1) (a) Suppose that R is a directed union of Artinian subrings. Then $\mathscr{A}(R) \neq \emptyset$ and hence $\mathscr{C}(R)$ is finite [4, Proposition 1].

(b) Assume that $\mathscr{A}(R) \neq \emptyset$, let $S \in \mathscr{A}(R)$, then $\mathscr{C}(R) \subseteq \mathscr{C}(S)$ and hence $\mathscr{C}(S)$ is infinite, a contradiction with S is Artinian ring ($\mathscr{C}(R)$ is finite).

(2) Let $\text{Spec}(R) = \{M_i\}_{i=1}^n$ be the spectrum of R. Let $S_{M_i}(0)$ denote $\text{Ker}\,\varphi_i$ for each $i = 1, \ldots, n$, where $\varphi_i : R \to R_{M_i}$ and $\varphi_i(r) = r/1$, is the canonical homomorphism.

Since $\mathrm{Rad}(S_{M_i}(0)) = M_i$, $S_{M_i}(0)$ is a primary ideal. Note that $\bigcap_{i=1}^n S_{M_i}(0) = (0)$ and $S_{M_i}(0) + S_{M_j}(0) = R$ for each $i \neq j$ in $\{1, \dots, n\}$. Therefore, $R \simeq R/\bigcap_{i=1}^n S_{M_i}(0)$. By the Chinese Remainder Theorem, $R \simeq \prod_{i=1}^n R/S_{M_i}(0)$, where $R/S_{M_i}(0)$ is quasi-local and zero-dimensional, for $i = 1, \dots, n$.

(3) This follows from the fact that a von Neumann regular ring with only finitely many idempotent elements is a finite product of fields, and then it is Artinian. □

Lemma 2.1 leads us to state that if R is a von Neumann regular ring, then R is a directed union of Artinian subrings if and only if R is a directed union of zero-dimensional semi-quasilocal subrings. This follows from the fact that a von Neumann regular ring with only finitely many idempotent elements is a finite product of fields and then it is Artinian.

LEMMA 2.2. *Let R be a ring and S a multiplicatively closet subset of R. Then*:

(a) *If R is a directed union of Artinian subrings, then so is $S^{-1}R$.*

(b) *R is a directed union of zero-dimensional semi-quasilocal subrings if and only if $S^{-1}R$ is a directed union of zero-dimensional quasilocal subrings.*

(c) *If R is reduced, then R is a directed union of Artinian subrings if and only if $S^{-1}R$ is a directed union of Artinian subrings.*

PROOF. (a) If $R = \bigcup_{i\in I} R_i$ is a directed union of Artinian subrings, then $S^{-1}R = \bigcup_{i\in I} S_i^{-1}R_i$, where $S_i = S \cap R_i$ for each $i \in I$. By [11, (6.17)], $S_i^{-1}R_i$ is Noetherian for each $i \in I$ since each R_i is Noetherian. Each $S^{-1}R_i$ is zero-dimensional as each R_i is (see, for example, [2, Proposition 1.21]). By [3, Theorem 8.5], $S^{-1}R_i$ is an Artinian ring for each $i \in I$. The family $\{S^{-1}R_i\}_{i\in I}$ is directed because so is the family $\{R_i\}_{i\in I}$. Thus $S^{-1}R$ is a directed union of Artinian subrings.

(b) Suppose that $R = \bigcup_{i\in I} R_i$ is a directed union of zero-dimensional semi-quasilocal subrings, we have in (a) $S^{-1}R = \bigcup_{i\in I} S_i^{-1}R_i$. Let $\mathrm{Spec}(S^{-1}R) = \{S^{-1}\mathfrak{m} : \mathfrak{m} \in \mathrm{Spec}(R) \text{ and } \mathfrak{m} \cap S = \emptyset\}$, then $S_i^{-1}R_i$ is semi-quasilocal. From (a) we conclude that $S^{-1}R$ is a directed union of the required type. Conversely, suppose that $S^{-1}R = \bigcup_{i\in I} T_i$ is a directed union of zero-dimensional semi-quasilocal subrings. Let $R_i = T_i \cap R$, then R_i is zero-dimensional (see, for example, [7, Theorem 2.4]). Also $|\,\mathrm{Idem}(R_i)| \le |\,\mathrm{Idem}(T_i)|$. Any zero-dimensional ring with only finitely many idempotent elements is a semi-quasilocal ring. Therefore, $R = \bigcup_{i\in I} R_i$ is a directed union of zero-dimensional semi-quasilocal subrings.

(c) First, we claim that if R is reduced, then so is $S^{-1}R$. Let $r/s \in N(S^{-1}R)$, where $N(S^{-1}R)$ is the nilradical of $S^{-1}R$. Then there exists $n_o \in \mathbb{N}^*$ such that $(r/s)^{n_o} = 0$; so there exists $u \in U$ such that $(ru)^{n_o} = 0$. Since R is reduced, we have $ru = 0$ and hence $r/s = 0$. In other words, $N(S^{-1}R) = (0)$. Thus, $S^{-1}R$ is reduced. Now, assume that $S^{-1}R = \bigcup_{i\in I} T_i$ is a directed union of Artinian subrings. By [10, Corollary 4], R is also a directed union of Artinian subrings. □

EXAMPLE 2.3. Let $R = \mathbb{Q}^{\omega_o}$ be a countable direct product of copies of $\mathbb{Q}$, where $\mathbb{Q}$ denotes the field of rational numbers. We consider

$$\mathbb{Q}^{(i)} = \left\{ \{x_j\}_{j=1}^{\infty} \in \mathbb{Q}^{w_o} : x_{i-1} = x_i = \cdots \right\}$$

a subring of R, and $\mathbb{Q}^{(i)} \simeq \mathbb{Q}^i$, the product of i copies of $\mathbb{Q}$, an Artinian von Neumann regular ring. It follows that $\mathscr{A}(R) \neq \emptyset$. We can easily see that $\mathbb{Q}^{(i)}$, $i \geq 1$, are the only Artinian subrings of R. However, R is not a directed union of Artinian subrings. Indeed, consider $y = (y_i)_{i \in \mathbb{N}^*} \in R$ such that $y_i \neq y_j$ for $i \neq j$. Then, for each $i \in \mathbb{N}^*$, $y \notin \mathbb{Q}^{(i)}$. It follows that $\varinjlim \mathbb{Q}^{(i)} \subsetneq \mathbb{Q}^{\omega_o}$. However $\mathscr{S} = \varinjlim \mathbb{Q}^{(i)}$ is the biggest subring of R which expressible as a directed union of Artinian subrings.

Hence $\mathscr{A}(R) \neq \emptyset$ does not imply that R is a directed union of Artinian subrings.

Let R be a von Neumann regular ring and $\{M_\lambda\}_{\lambda \in \Lambda}$ its spectrum. Since R is a reduced ring, we have $\bigcap_{\lambda \in \Lambda} M_\lambda = (0)$. This allows us to regard R as a subring of $S = \prod_{\lambda \in \Lambda} R/M_\lambda$. It is known that if there exists $k \in \mathbb{Z} \backslash \{0\}$ such that the set $\{\lambda \in \Lambda : |R/M_\lambda| > k\}$ is finite, then $\prod_{\lambda \in \Lambda} R/M_\lambda$ is a directed union of Artinian subrings [6, Theorem 6.7]. By [10, Corollary 4], R is also a directed union of Artinian subrings. Hence, we further assume that for each $k \in \mathbb{Z}^*$ the set $\{\lambda \in \Lambda : |R/M_\lambda| > k\}$ is infinite. Under these assumptions, it would be interesting to know whether $\mathscr{A}\left(R, \prod_{\lambda \in \Lambda} R/M_\lambda\right) \neq \emptyset$, where $\mathscr{A}\left(R, \prod_{\lambda \in \Lambda} R/M_\lambda\right)$ is the family of Artinian subrings of S containing R. The following result provides an answer to this problem.

PROPOSITION 2.4. *Let R be a von Neumann regular ring with* $\operatorname{Spec}(R) = \{M_\lambda : \lambda \in \Lambda\}$ *. The following statements are equivalent*:

(i) $\mathscr{A}\left(R, \prod_{\lambda \in \Lambda} R/M_\lambda\right) \neq \emptyset$.
(ii) *There exist $\lambda_1, \ldots, \lambda_n$ in Λ such that $\bigcap_{i=1}^n M_{\lambda_i} = (0)$.*
(iii) $\operatorname{Idem}(R)$ *is a finite set.*
(iv) *R is Artinian.*

PROOF. (i) $\Rightarrow$ (ii). If $\mathscr{A}\left(R, \prod_{\lambda \in \Lambda} R/M_\lambda\right) \neq \emptyset$, then there exists an Artinian ring A such that $R \subseteq A \subset \prod_{\lambda \in \Lambda} R/M_\lambda$. Since $\prod_{\lambda \in \Lambda} R/M_\lambda$ is reduced, so is the ring A. Therefore, $A \simeq \prod_{i=1}^n K_i$, where K_i is a field for each $i = 1, \ldots, n$. Now, let $Q_i = \prod\{K_j : j = 1, \ldots, n \text{ and } j \neq i\}$, where $i = 1, \ldots, n$. Then $\prod_{j=1}^n K_j/Q_i \simeq K_i$ and hence $Q_i \in \operatorname{Spec}(A)$. Thus, for each $i = 1, \ldots, n$, $M_{\lambda_i} = Q_i \cap R \in \operatorname{Spec}(R)$ and $\bigcap_{i=1}^n M_{\lambda_i} = (0)$.

(ii) $\Rightarrow$ (iii). Since $\bigcap_{i=1}^n M_{\lambda_i} = (0)$, then R is imbeddable in $\prod_{i=1}^n R/M_{\lambda_i}$. Hence R has only finitely many idempotent elements.

(iii) $\Rightarrow$ (iv). It follows form [10, Lemma 1].

(iv) $\Rightarrow$ (i). Obvious. □

Let $S \subseteq T$ be an extension of rings with S an Artinian and von Neumann regular, and let F be a finite set of idempotents of T then $S[F]$ is Artinian. Indeed, $S[F]$ is an integral extension of S since each idempotent element e of F satisfies $e^2 = e$. Therefore, $\dim S[F] = 0$. Furthermore, $S[F]$ is finitely generated over the Noetherian ring S. Hence $S[F]$ is Noetherian and then Artinian.

LEMMA 2.5. *Let R be a von Neumann regular subring of a ring T and* $\mathrm{Idem}(T) = \{e_\lambda\}_{\lambda\in\Lambda}$ *the set of idempotent elements of T. Then R is a directed union of Artinian subrings if and only if $R[\{e_\lambda : \lambda \in \Lambda\}]$ is a directed union of Artinian subrings.*

PROOF. Assume that $R = \bigcup_{i\in I} R_i$ is a directed union of Artinian subrings. We write $\Lambda = \bigcup_{j\in J} \Lambda_j$, where Λ_j ranges over all finite subsets of Λ, thus

$$\{e_\lambda\}_{\lambda\in\Lambda} = \bigcup_{j\in J} \{e_\lambda\}_{\lambda\in\Lambda_j}, \quad \text{and} \quad R[\{e_\lambda : \lambda \in \Lambda\}] = \bigcup_{(i,j)\in I\times J} R_i[\Lambda_j],$$

where $R_i[\Lambda_j] = R_i[\{e_k : k \in \Lambda_j\}]$. Now, endowed with the lexicographic order, the set $I \times J$ is directed and clearly the family $\{R_i[\Lambda_j] : (i, j) \in I \times J\}$ is also directed. It follows that $R[\{e_\lambda : \lambda \in \Lambda\}]$ is a directed union of Artinian subrings. The converse follows from [10, Lemma 1 and Corollary 4]. □

PROPOSITION 2.6. *Let $\{K_i\}_{i\in I}$ be a family of fields. Then the following statements are equivalent*:

(i) $\mathscr{A}\left(\prod_{i\in I} K_i\right) \neq \emptyset$.
(ii) $\{\mathrm{char}(K_i) : i \in I\}$ *is finite.*

PROOF. (i) ⇒ (ii). Let A be an Artinian subring of $\prod_{i\in I} K_i$. By [4, Proposition 1] $\mathscr{C}\left(\prod_{i\in I} K_i\right) \subseteq \mathscr{C}(A)$. Since A is Artinian, $\mathrm{Max}(A)$ is finite and hence $\mathscr{C}(A)$ is finite. Thus $\{\mathrm{char}(K_i) : i \in I\}$ is finite since $\{\mathrm{char}(K_i) : i \in I\} \subseteq \mathscr{C}\left(\prod_{i\in I} K_i\right)$.

(ii) ⇒ (i). If $\mathrm{char}\left(\prod_{i\in I} K_i\right) \neq 0$, then $\mathscr{A}\left(\prod_{i\in I} K_i\right) \neq \emptyset$, since the prime subring of $\prod_{i\in I} K_i$ is finite, and hence it is Artinian. So suppose that $\mathrm{char}\left(\prod_{i\in I} K_i\right) = 0$. Since $\{\mathrm{char}(K_i) : i \in I\} = \{p_1, \ldots, p_k\}$ is finite, we may set $I = A_1 \cup \cdots \cup A_k$, where $A_s = \{i \in A : \mathrm{char}(K_i) = p_s\}$ for $s = 1, \ldots, k$. Since $\mathrm{char}(R) = 0$, we may suppose that $p_1 = 0$ and write $R = R_1 \times \cdots \times R_k$, where $R_i = \prod_{j\in A_i} K_j$, for each $i = 1, \ldots, k$. Hence $\mathbb{Q}^* \times \pi_2^* \times \cdots \times \pi_k^* \in \mathscr{A}(R)$, where π_i is the prime subring of R_i for each $i = 2, \ldots, k$ and $\mathbb{Q}^*$ (respectively, π_i^*) denotes the diagonal imbedding of $\mathbb{Q}$ (respectively, π_i) into $\prod_{j\in A_1} K_j$ (respectively, $\prod_{j\in A_i} K_j$, for $i = 2, \ldots, k$). □

Let $\mathscr{B}\left(R, \prod_{\alpha\in A} R/M_\alpha\right)$ denote the set of all the intermediate directed unions of Artinian subrings between R and $\prod_{\alpha\in A} R/M_\alpha$. By Example 2.3, it may happen that $\mathscr{B}\left(R, \prod_{\alpha\in A} R/M_\alpha\right) \neq \emptyset$ even if $\prod_{\alpha\in A} R/M_\alpha$ is not a directed union of Artinian subrings. In this case R must be a directed union of Artinian subrings. By [5, Corollary 1.2], if $\mathscr{C}(R)$ is infinite, then R admits no Artinian subring. So we restrict our study to the case when $\mathscr{C}(R)$ is a finite set.

If $x \in N(R)$, we denote by $\eta(x)$ the index of nilpotency of x, that is, $\eta(x) = k$ if $x^k = 0$ but $x^{k-1} \neq 0$. We define $\eta(R)$ to be $\operatorname{Sup}\{\eta(x) : x \in N(R)\}$; if the set $\{\eta(x) : x \in N(R)\}$ is unbounded, then we write $\eta(R) = \infty$. From [6, Theorem 3.4], we have $\dim \prod_{\alpha\in A} T_\alpha = 0$, for any family $\{T_\alpha\}_{\alpha\in A}$ of zero-dimensional rings, equivalent to $\{\alpha \in A : \eta(T_\alpha) > k\}$ is finite for some $k \in \mathbb{Z}^+$.

It is worth to mention that $\prod_{\alpha\in A} R_\alpha$ needs not be a directed union of Artinian subrings even if $\dim\left(\prod_{\alpha\in A} R_\alpha\right) = 0$, where $\{R_\alpha\}_{\alpha\in A}$ is family of zero-dimensional rings. For instance, let K be an infinite field, X be an indeterminate over K, and $R_i = K[X]/(X^3)$ for each $i \in \mathbb{Z}^+$. We have $N(R_i) = (X)/(X^3)$, $i \in \mathbb{Z}^+$. Since $\eta(R_i) = 3$ for each $i \in \mathbb{Z}^+$, by [6, Theorem 3.4], $\dim \prod_{i=1}^{\infty} R_i = 0$ and from [6, Theorem 6.7], $\prod_{i=1}^{\infty} R_i$ is not a directed union of Artinian subrings.

Let R be a von Neumann regular ring and $\operatorname{Max}(R) = \{M_\lambda\}_{\lambda\in A}$ the set of maximal ideals of R. Let $\varphi : R \hookrightarrow \prod_{\lambda\in A} L_\lambda$ be a monomorphism defined by $\varphi(x) = \{x_\lambda\}_{\lambda\in A}$ such that $x_\lambda \equiv \overline{x}$ modulo M_λ, where $R/M_\lambda \simeq L_\lambda$ for each $\lambda \in A$. We assume that $\mathscr{C}(R) = \{p_1, \ldots, p_n\}$ is finite, then we have $\prod_{\lambda\in A} L_\lambda = \bigoplus_{j=1}^{n} R_j$, where $R_j = \prod_{\lambda\in A_j} L_\lambda$ and $A_j = \{\lambda \in A : \operatorname{char}(L_\lambda) = p_j\}$. For each j, we suppose there exists a field Ω_j that contains all fields L_λ, $\lambda \in A_j$. We now regard R as a subring of $\prod_{j=1}^{n} S_j$, where $S_j = \Omega_j^{I_j}$. Let e_j be the primitive idempotent of $\prod_{j=1}^{n} S_j$ associated with $\{j\}$ for $j = 1, \ldots, n$. Then $R[e_1, \ldots, e_n] = Re_1 \oplus \cdots \oplus Re_n$. Therefore, R is a directed union of Artinian subrings if and only if $R[e_1, \ldots, e_n]$ is a directed union of Artinian subrings if and only if every Re_i is a directed union of Artinian subrings. Also $Re_j \subseteq \Omega_j^{I_j}$. We denote by Ω_j^* the diagonal imbedding of Ω_j into $\Omega_j^{I_j}$. Without loss of generality we assume that $Re_j = R$, In other words, $\mathscr{C}(R) = \{p\}$.

THEOREM 2.7. *Let R be a von Neumann regular ring with $\mathscr{F}(R) = \{L_\alpha\}_{\alpha\in A}$, $\mathscr{C}(R) = \{p\}$, and $\mathscr{S} = \big\{\{r_\alpha\}_{\alpha\in A} \in \prod_{\alpha\in A} L_\alpha : \{r_\alpha\}_{\alpha\in A}$ has only finitely many distinct coordinates$\big\}$. Assume that each L_α is absolutely algebraic and there exists a field Ω that contains all but finitely many L_α's. Then $\mathscr{S}$ is a directed union of Artinian subrings.*

PROOF. To show that $\mathscr{S}$ is a directed union of Artinian subrings, it suffices to prove that $\mathscr{S}$ is covered by a directed union of finite products of fields. Let $f \in \mathscr{S}$, then $\{f(\alpha) : \alpha \in A\} = \{f_1, \ldots, f_t\}$ a finite set. Let $A_i = \{\alpha \in A : f(\alpha) = f_i\}$ and denote $f_i^* = (f_i, f_i, \ldots, f_i, \ldots) \in \prod_{\alpha\in A_i} L_\alpha$. Then $\{f(\alpha)\}_{\alpha\in A} = (f_1^*, \ldots, f_t^*)$. Since $A = \bigcup_{i=1}^{t} A_i$, and all the fields L_α, $\alpha \in A_i$, have the same characteristic, then, up to isomorphism, $\bigcap_{\alpha\in A_i} L_\alpha = K_i$ is a field with $f_i^* \in K_i^*$, the diagonal imbedding of K_i in $\prod_{\alpha\in A_i} L_\alpha$. It follows that $f \in K_1^* \times \cdots \times K_t^* \simeq K_1 \times \cdots \times K_t$. Therefore,

$\mathscr{S}$ is covered by a directed union of Artinian subrings. □

EXAMPLE 2.8. Let p be a positive prime integer and $\{q_i\}_{i\in\mathbb{N}^*}$ be an infinite family of distinct prime integers. Let $\mathscr{F} = \{\mathbb{Q}\} \cup \{\mathbb{Q}(\zeta_i)\}_{i=1}^{\infty}$ be an infinite family of fields, where ζ_i is a p^{q_i}-primitive root of unity. We denote by φ_i the imbedding of $\mathbb{Q}$ into $\mathbb{Q}(\zeta_i)$ for each $i \in \mathbb{Z}^+$. Let $\varphi = \prod_{i=1}^{\infty} \varphi_i$, $T = \prod_{i=1}^{\infty} \mathbb{Q}(\zeta_i)$ and let $I = \bigoplus_{i=1}^{\infty} \mathbb{Q}(\zeta_i)$ be the direct sum ideal of T. We denote by $\mathbb{Q}^* = \varphi(\mathbb{Q}) = R_o \simeq \mathbb{Q}$ the diagonal imbedding of $\mathbb{Q}$ in T. Let $S = R_o + I$, since S is a subring of T and dim(S)=0 [9, Proposition 2.7], the ring S is a Von Neumann regular ring. Let

$$S_1 = \mathbb{Q}(\zeta_1) \times \mathbb{Q}^* \simeq \mathbb{Q}(\zeta_1) \times \mathbb{Q}, \quad \ldots,$$
$$S_i = \mathbb{Q}(\zeta_1) \times \cdots \times \mathbb{Q}(\zeta_i) \times \mathbb{Q}^* \simeq \mathbb{Q}(\zeta_1) \times \cdots \times \mathbb{Q}(\zeta_i) \times \mathbb{Q}.$$

We have that $S_j \subset S_{j+1}$, for each positive integer $j \in \mathbb{Z}^+$. Therefore, $S = \bigcup_{i\in\mathbb{N}^*} S_i$ is an increasing union of Artinian subrings.

THEOREM 2.9. *Let R be a von Neumann regular ring and $\mathscr{F}(R) = \{L_\alpha\}_{\alpha\in A}$ the set of residue fields of R. Suppose that each L_α is absolutely algebraic. Then R is a directed union of Artinian subrings if and only if for each $x = \{x_\lambda\}_{\lambda\in A} \in R$, x has only finitely many distinct coordinates.*

PROOF. Assume that $R = \bigcup_{i\in I} R_i$ is a directed union of Artinian subrings. Since R is imbeddable in $\prod_{\lambda\in A} R/M_\lambda$, we regard R as a subring of $\prod_{\lambda\in A} R/M_\lambda$, and hence each R_i is a subring of $\prod_{\lambda\in A} R/M_\lambda$. Let $x = \{x_\lambda\}_{\lambda\in A} \in R$ then $x \in R_{i_o}$ for some $i_o \in I$. Since each R_i is isomorphic to a finite product of fields, say $\prod_{i=1}^{n} K_i$, which in turn is isomorphic to $\simeq \prod_{i=1}^{n} K_i^*$, where K_i^* is the diagonal imbedding of K_i into $\prod_{\lambda\in A_i} R/M_\lambda$ and $A_i = \{\lambda \in A : x_\lambda = x_i\}$. It follows that $\{x_\lambda\}_{\lambda\in A} = (x_1^*, \ldots, x_n^*)$ with $x_i \in K_i$, $i \in \{1, \ldots, n\}$. The converse follows from the fact that the $R \subseteq \mathscr{S}$ (see Theorem 2.7 and [10, Corollary 4]). □

EXAMPLE 2.10. Let F be an absolutely algebraic infinite field, α an infinite cardinal, and S the direct product of α copies of F. Let A be a set of cardinality α and consider S as the set of all functions $f : A \to F$ under pointwise addition and multiplication. Let $R = \{f \in S : f(A) \text{ is finite}\}$, let $\mathscr{P} = \{A_1, \ldots, A_n\}$ be any finite partition of A, where A_i is a subset of A for each i. Moreover, $R_{\mathscr{P}} = \{f \in S : f \text{ is constant on each } A_i\}$ is a subring of R containing π, the prime subring of S. Clearly, $R_{\mathscr{P}} \simeq F^n$. We use F^* to denote the diagonal imbedding of F in S. From the proof of [6, Proposition 5.2], the family $\{R_{\mathscr{P}} : \mathscr{P} \text{ is a finite partition of } A\}$ is directed. Each $R_{\mathscr{P}}$ is Artinian and von Neumann regular and $R = \cup R_{\mathscr{P}} \simeq F^* + I$, where $I = \bigoplus_{\beta\in A} F$ is the direct sum ideal of S, is the biggest subring of S which expressible as a directed union of Artinian subrings. On the other hand, if α is countable, then for each $i \geq 0$, let $F^{(i)} = \{\{x_j\}_{j=1}^{\infty} \in S : x_{i-1} = x_i = \cdots\}$ be a subring of S isomorphic to F^i. According to [6, Proposition 5.1], $R = \bigcup_{i=1}^{\infty} F^{(i)}$ is a directed union of Artinian von Neumann regular subrings.

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