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PARTIAL AUTOMORPHISMS OF STABLE C*-ALGEBRAS AND HILBERT C*-BIMODULES

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Abstract

Let A be a C*-algebra and $\mathbb{K}$ the C*-algebra of all compact operators on a countably infinite dimensional Hilbert space. In this note, we shall show that there is an isomorphism of a semigroup of equivalence classes of certain partial automorphisms of $A \otimes \mathbb{K}$ onto a semigroup of equivalence classes of certain countably generated A-A-Hilbert bimodules.

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0. Introduction

Let A be a C*-algebra and $\mathbb{K}$ the C*-algebra of all compact operators on a countably infinite dimensional Hilbert space. Brown, Green and Rieffel [4] defined $\mathrm{Pic}(A)$, the Picard group of A as the group of isomorphic classes of A-A-equivalence bimodules and showed that there is a homomorphism of $\mathrm{Aut}(A \otimes \mathbb{K})$, the group of all automorphisms of $A \otimes \mathbb{K}$ to $\mathrm{Pic}(A)$ and that its kernel is $\mathrm{Int}(A \otimes \mathbb{K})$, the normal subgroup of $\mathrm{Aut}(A \otimes \mathbb{K})$ of all generalized inner automorphisms of $A \otimes \mathbb{K}$. If A is σ-unital, the homomorphism of $\mathrm{Aut}(A \otimes \mathbb{K})$ to $\mathrm{Pic}(A)$ is surjective. Hence $\mathrm{Out}(A \otimes \mathbb{K})(= \mathrm{Aut}(A \otimes \mathbb{K})/\mathrm{Int}(A \otimes \mathbb{K}))$ is isomorphic to $\mathrm{Pic}(A)$.

In this note, we shall give a similar result to the above one for certain partial automorphisms of $A \otimes \mathbb{K}$ and certain countably generated A-A-Hilbert bimodules.

For a C*-algebra B, let $\mathrm{Aut}(B)$ be the group of all automorphisms of B and $M(B)$ its multiplier algebra.

1. Equivalence relations and a map

Let A and $\mathbb{K}$ be as above. Let $\mathrm{PAut}(A\otimes\mathbb{K})$ be the set of all partial automorphisms of $A\otimes\mathbb{K}$ defined by Exel [6]. We shall give an equivalence relation $\sim$ as follows; for $\Theta_j=(I_j, J_j, \theta_j)\in\mathrm{PAut}(A\otimes\mathbb{K})$ $(j=1,2)$, $\Theta_1\sim\Theta_2$ if $I_1=I_2$, $J_1=J_2$ and there is a unitary element $w\in M(I_1)$ such that $\theta_2=\theta_1\circ\mathrm{Ad}(w)$. We denote by $[\Theta]$ the equivalence class of a partial automorphism $\Theta\in\mathrm{PAut}(A\otimes\mathbb{K})$. We also denote by $\mathrm{PAut}(A\otimes\mathbb{K})/\sim$ the quotient set of $\mathrm{PAut}(A\otimes\mathbb{K})$ by the above equivalence relation. We shall define a product in $\mathrm{PAut}(A\otimes\mathbb{K})/\sim$ as follows. Let $\Theta_j=(I_j, J_j, \theta_j)\in\mathrm{PAut}(A\otimes\mathbb{K})$ $(j=1,2)$, and let $I_3=\theta_1^{-1}(J_1\cap I_2)$, $J_3=\theta_2(J_1\cap I_2)$, $\theta_3=\theta_2\circ\theta_1$. Finally, let $\Theta_3=(I_3, J_3, \theta_3)\in\mathrm{PAut}(A\otimes\mathbb{K})$. We define $[\Theta_1][\Theta_2]=[\Theta_3]$. By routine computations, we can see that $[\Theta_3]$ is independent of the choices of representatives of $[\Theta_1]$ and $[\Theta_2]$. By the above product, $\mathrm{PAut}(A\otimes\mathbb{K})/\sim$ is a semigroup. Let $\Theta_e=(A\otimes\mathbb{K}, A\otimes\mathbb{K}, id_{A\otimes\mathbb{K}})\in\mathrm{PAut}(A\otimes\mathbb{K})$. Then by easy computations $[\Theta_e]$ is the unit element in $\mathrm{PAut}(A\otimes\mathbb{K})/\sim$ and the group of all invertible elements in $\mathrm{PAut}(A\otimes\mathbb{K})/\sim$ is

$$\{[(A\otimes\mathbb{K}, A\otimes\mathbb{K}, \beta)] \mid \beta\in\mathrm{Aut}(A\otimes\mathbb{K})\}\cong\mathrm{Out}(A\otimes\mathbb{K}).$$

We identify it with $\mathrm{Out}(A\otimes\mathbb{K})$ and denote an element $[(A\otimes\mathbb{K}, A\otimes\mathbb{K}, \beta)]\in\mathrm{PAut}(A\otimes\mathbb{K})/\sim$ by $[\beta]$.

Let $\mathrm{HB}(A)$ be the set of all A-A-Hilbert bimodule isomorphic classes of A-A-Hilbert bimodules defined in Brown, Mingo and Shen [5] and Abadie, Eilers and Exel [1]. For any A-A-Hilbert bimodule X, we denote by $[X]$ the A-A-Hilbert bimodule isomorphic class of X. We define a product in $\mathrm{HB}(A)$ as the relative tensor product with respect to A.

In the same way as in Abadie, Eilers and Exel [1, Example 3.2], for any $\Theta=(I, J, \theta)\in\mathrm{PAut}(A\otimes\mathbb{K})$ we define an $A\otimes\mathbb{K}$-$A\otimes\mathbb{K}$-Hilbert bimodule X_Θ as follows. Let X_Θ be the vector space I and the obvious left action of $A\otimes\mathbb{K}$ and the obvious left inner product, but we define the right action of $A\otimes\mathbb{K}$ on X_Θ by $x\cdot a=\theta^{-1}(\theta(x)a)$ for any $a\in A\otimes\mathbb{K}$ and $x\in X_\Theta$ and the right $A\otimes\mathbb{K}$-valued inner product by $\langle x, y\rangle_{A\otimes\mathbb{K}}=\theta(x^*y)$ for any $x, y\in X_\Theta$.

LEMMA 1.1. *Let $\Theta_j=(I_j, J_j, \theta_j)\in\mathrm{PAut}(A\otimes\mathbb{K})$ $(j=1,2)$. If $\Theta_1\sim\Theta_2$, then $X_{\Theta_1}\cong X_{\Theta_2}$ as $A\otimes\mathbb{K}$-$A\otimes\mathbb{K}$-Hilbert bimodules.*

PROOF. We shall prove this lemma in the same way as in Brown, Green and Rieffel [4, Proposition 3.1]. Since $\Theta_1\sim\Theta_2$, $I_1=I_2$ and $J_1=J_2$. Put $I=I_1=I_2$ and $J=J_1=J_2$. Then there is a unitary element $w\in M(I)$ such that $\theta_2=\theta_1\circ\mathrm{Ad}(w)$. Let Φ be a map from X_{Θ_1} to X_{Θ_2} defined by $\Phi(x)=xw$ for any $x\in X_{\Theta_1}$. Then by direct computations, we can see that Φ is an $A\otimes\mathbb{K}$-$A\otimes\mathbb{K}$- Hilbert bimodule isomorphism of X_{Θ_1} onto X_{Θ_2} since $\theta_2=\theta_1\circ\mathrm{Ad}(w)$. Hence we obtain the conclusion. □

By Lemma 1.1, the map $[\Theta]\in\mathrm{PAut}(A\otimes\mathbb{K})/\sim\ \to[X_\Theta]\in\mathrm{HB}(A\otimes\mathbb{K})$ can be defined.

Lemma 1.2. *The map* $[\Theta]\in\mathrm{PAut}(A\otimes\mathbb{K})/\sim\ \to[X_\Theta]\in\mathrm{HB}(A\otimes\mathbb{K})$ *is a semigroup homomorphism.*

Proof. Let $\Theta_j=(I_j,J_j,\theta_j)\in\mathrm{PAut}(A\otimes\mathbb{K})$ $(j=1,2)$. Let $\Theta_3=(\theta_1^{-1}(J_1\cap I_2),\theta_2(J_1\cap I_2),\theta_2\circ\theta_1)\in\mathrm{PAut}(A\otimes\mathbb{K})$. Then by the definition of the product in $\mathrm{PAut}(A\otimes\mathbb{K})/\sim$, $[\Theta_3]=[\Theta_1][\Theta_2]$. Hence it suffices to show that $X_{\Theta_1}\otimes_{A\otimes\mathbb{K}}X_{\Theta_2}\cong X_{\Theta_3}$ as $A\otimes\mathbb{K}-A\otimes\mathbb{K}$-Hilbert bimodules. We note that $X_{\Theta_1}=I_1$, $X_{\Theta_2}=I_2$ and $X_{\Theta_3}=\theta_1^{-1}(J_1\cap I_2)$ as vector spaces. Let Φ be a map of $X_{\Theta_1}\otimes_{A\otimes\mathbb{K}}X_{\Theta_2}$ to X_{Θ_3} defined by $\Phi(x_1\otimes x_2)=\theta_1^{-1}(\theta_1(x_1)x_2)$ for any $x_1\in X_{\Theta_1}$ and $x_2\in X_{\Theta_2}$. For any $a\in A\otimes\mathbb{K}$ and $x_1\in X_{\Theta_1}$, $x_2\in X_{\Theta_2}$ with $x_1,x_2\geq 0$,

$$\Phi(x_1\otimes(a\cdot x_2))=\theta_1^{-1}(\theta_1(x_1)ax_2)=\Phi((x_1\cdot a)\otimes x_2).$$

Thus Φ is well defined. It is also clear that Φ is surjective. Furthermore, by routine computations, Φ preserves the both inner products. Hence, by the remark after Jensen and Thomsen [7, Definition 1.1.18], $X_{\Theta_1}\otimes_{A\otimes\mathbb{K}}X_{\Theta_2}\cong X_{\Theta_3}$ as $A\otimes\mathbb{K}$-$A\otimes\mathbb{K}$-Hilbert bimodules. Therefore, we obtain the conclusion. □

For any A-A-Hilbert bimodule, let $I_L(X)$ be the closure of linear span of $\{{}_A\langle x,y\rangle \mid x,y\in X\}$ and $I_R(X)$ the closure of linear span of $\{\langle x,y\rangle_A \mid x,y\in X\}$. By Brown, Mingo and Shen [5, Remark 1.9], $I_L(X)$ and $I_R(X)$ are closed two-sided ideals of A and by restriction we regard X as an $I_L(X)$-$I_R(X)$-equivalence bimodule.

Lemma 1.3. *The homomorphism* $[\Theta]\in\mathrm{PAut}(A\otimes\mathbb{K})/\sim\ \to[X_\Theta]\in\mathrm{HB}(A\otimes\mathbb{K})$ *is injective.*

Proof. Let $\Theta_j\in\mathrm{PAut}(A\otimes\mathbb{K})$ $(j=1,2)$. We suppose that $X_{\Theta_1}\cong X_{\Theta_2}$ as $A\otimes\mathbb{K}$-$A\otimes\mathbb{K}$-Hilbert bimodules. Thus there is an $A\otimes\mathbb{K}$-$A\otimes\mathbb{K}$-Hilbert bimodule isomorphism Φ of X_{Θ_1} onto X_{Θ_2}. Then since Φ preserves the both inner products, $I_L(X_{\Theta_1})=I_L(X_{\Theta_2})$ and $I_R(X_{\Theta_1})=I_R(X_{\Theta_2})$. On the other hand, $I_L(X_{\Theta_j})=I_j$, $I_R(X_{\Theta_j})=J_j$ by the definition of X_{Θ_j} for $j=1,2$. Thus $I_1=I_2$ and $J_1=J_2$. We regard X_{Θ_1} and X_{Θ_2} as I-J-equivalence bimodules where $I=I_1=I_2$ and $J=J_1=J_2$. Then $X_{\Theta_1}\cong X_{\Theta_2}$ as I-J-equivalence bimodules. Hence, by Brown, Green and Rieffel [4, Corollary 3.2], there is a unitary element $w\in M(I)$ such that $\theta_2=\theta_1\circ\mathrm{Ad}(w)$. Therefore, $\Theta_1\sim\Theta_2$. □

Let B be a C*-algebra. For each A-B-Hilbert bimodule X, let $\tilde{X}$ be its dual Hilbert C*-bimodule. Let $\mathbb{H}_A$ be the standard equivalence bimodule between $A \otimes \mathbb{K}$ and A, that is, $\mathbb{H}_A = (A \otimes \mathbb{K})(1 \otimes e)$, where 1 is the unit element in $M(A)$, e is a rank one projection in $\mathbb{K}$ and we identify A with $A \otimes e$. For any $\Theta \in \mathrm{PAut}(A \otimes \mathbb{K})$, let $Y_\Theta = \tilde{\mathbb{H}}_A \otimes_{A\otimes\mathbb{K}} X_\Theta \otimes_{A\otimes\mathbb{K}} \mathbb{H}_A$. Then the map $[X] \to [\tilde{\mathbb{H}}_A \otimes_{A\otimes\mathbb{K}} X \otimes_{A\otimes\mathbb{K}} \mathbb{H}_A]$ is an isomorphism of $\mathrm{HB}(A \otimes \mathbb{K})$ onto $\mathrm{HB}(A)$. Thus the map $[\Theta] \in \mathrm{PAut}(A \otimes \mathbb{K})/\!\sim\ \to [Y_\Theta] \in \mathrm{HB}(A)$ is injective.

COROLLARY 1.4. *The map* $[\Theta] \in \mathrm{PAut}(A\otimes\mathbb{K})/\!\sim\ \to [Y_\Theta] \in \mathrm{HB}(A)$ *is an injective homomorphism.*

2. A certain class of countably generated Hilbert bimodules and a certain class of partial automorphisms

Let Y be an A-A-Hilbert bimodule. We say that Y is countably generated as equivalence bimodules if Y is a countably generated left Hilbert $I_L(Y)$-module and Y is a countably generated right Hilbert $I_R(Y)$-module.

Let Y be an A-A-Hilbert bimodule countably generated as equivalence bimodules. By Blackadar [2, Corollary 13.6.3] $I_L(Y)$ and $I_R(Y)$ are σ-unital. Put $I = I_L(Y) \otimes \mathbb{K}$, $J = I_R(Y) \otimes \mathbb{K}$. Then I and J are also σ-unital by Brown, Green and Rieffel [4, Proposition 2.4]. Let $X = \mathbb{H}_A \otimes_A Y \otimes_A \tilde{\mathbb{H}}_A$. Then

$$\begin{aligned} I_L(X) &= \overline{{}_{A\otimes\mathbb{K}}\langle \mathbb{H}_A\ {}_A\langle Y\, {}_A\langle \tilde{\mathbb{H}}_A, \tilde{\mathbb{H}}_A\rangle, Y\rangle, \mathbb{H}_A\rangle} \\ &= \overline{{}_{A\otimes\mathbb{K}}\langle \mathbb{H}_A\ {}_A\langle YA, Y\rangle, \mathbb{H}_A\rangle} = \overline{{}_{A\otimes\mathbb{K}}\langle \mathbb{H}_A I_L(Y), \mathbb{H}_A\rangle} = I_L(Y) \otimes \mathbb{K} \end{aligned}$$

since $\overline{YA} = Y$ by Brown, Mingo and Shen [5, Proposition 1.7]. Similarly $I_R(X) = I_R(Y) \otimes \mathbb{K}$.

LEMMA 2.1. *Let Y be an A-A-Hilbert bimodule countably generated as equivalence bimodules. Then there is a partial automorphism Θ_Y of $A \otimes \mathbb{K}$ such that $Y_{\Theta_Y} \cong Y$ as A-A-Hilbert bimodules.*

PROOF. Let I and J be as above. Then I and J are σ-unital. Hence, by Brown, Green and Rieffel [4, Theorem 3.4], there is an isomorphism θ of I onto J such that $X_\theta \cong X$ as I-J-equivalence bimodules where X_θ is an I-J-equivalence bimodule induced by the isomorphism θ of I onto J which is defined in Brown, Green and Rieffel [4, Section 3] and $X = \mathbb{H}_A \otimes_A Y \otimes_A \tilde{\mathbb{H}}_A$. Let Θ_Y be a partial automorphism of $A \otimes \mathbb{K}$ defined by $\Theta_Y = (I, J, \theta)$. Then by the definition of X_{Θ_Y}, $X_{\Theta_Y} = X_\theta$ as I-J-equivalence bimodules. Hence $X_{\Theta_Y} \cong X$ as I-J-equivalence bimodules. By the remark after Jensen and Thomsen [7, Definition 1.1.18], $X_{\Theta_Y} \cong X$ as $A \otimes \mathbb{K}$-$A \otimes \mathbb{K}$-Hilbert bimodules. Therefore, $Y_{\Theta_Y} \cong Y$ as A-A-Hilbert bimodules. □

Let $\mathrm{PAut}_\sigma(A\otimes\mathbb{K})$ be the set of all partial automorphisms $\Theta=(I, J, \theta)$ of $A\otimes\mathbb{K}$ satisfying that I and J are σ-unital. Let $\mathrm{HB_c}(A)$ be the subsemigroup of all A-A-Hilbert bimodules countably generated as equivalence bimodules.

LEMMA 2.2. $\mathrm{HB_c}(A)$ *is the image of a subsemigroup* $\mathrm{PAut}_\sigma(A\otimes\mathbb{K})/\sim$ *by the homomorphism* $[\Theta]\in\mathrm{PAut}(A\otimes\mathbb{K})/\sim\ \to [Y_\Theta]\in\mathrm{HB}(A)$.

PROOF. By the proof of Lemma 2.1, we see that $\mathrm{HB_c}(A)$ is contained in the image of $\mathrm{PAut}_\sigma(A\otimes\mathbb{K})/\sim$ by the map $[\Theta]\in\mathrm{PAut}(A\otimes\mathbb{K})/\sim\ \to [Y_\Theta]\in\mathrm{HB}(A)$. Hence we have only to show the inverse inclusion. Let $\Theta=(I, J, \theta)\in\mathrm{PAut}_\sigma(A\otimes\mathbb{K})$. Then since $\overline{X_\Theta(A\otimes\mathbb{K})}=X_\Theta$ by Brown, Mingo and Shen [5, Proposition 1.7],

$$\begin{aligned}
I_L(Y_\Theta) &= \overline{{}_A\langle\tilde{\mathbb{H}}_A\ {}_{A\otimes\mathbb{K}}\langle X_\Theta\ {}_{A\otimes\mathbb{K}}\langle\mathbb{H}_A, \mathbb{H}_A\rangle, X_\Theta\rangle, \tilde{\mathbb{H}}_A\rangle} \\
&= \overline{{}_A\langle\tilde{\mathbb{H}}_A\ {}_{A\otimes\mathbb{K}}\langle X_\Theta, X_\Theta\rangle, \tilde{\mathbb{H}}_A\rangle} = \overline{\langle I\mathbb{H}_A, \mathbb{H}_A\rangle_A} \\
&= (1\otimes e)I(1\otimes e).
\end{aligned}$$

Let $(A\otimes\mathbb{K})''$ and I'' be the universal von Neumann algebras of $A\otimes\mathbb{K}$ and I, respectively. Then by Pedersen [8], there is a central open projection $p\in(A\otimes\mathbb{K})''$ with $I''=p(A\otimes\mathbb{K})''$ and $I=p(A\otimes\mathbb{K})''\cap(A\otimes\mathbb{K})$. Put $q=(1\otimes e)p$. Then q is a full projection in $M(I)$ and $I_L(Y_\Theta)=qIq$. Since I is σ-unital, by Brown [3, Corollary 2.6], $I_L(Y_\Theta)$ is stably isomorphic to I. Hence, by Brown, Green and Rieffel [4, Proposition 2.4] $I_L(Y_\Theta)$ is σ-unital since I is σ-unital. Similarly, $I_R(Y_\Theta)$ is also σ-unital. Thus, by Blackadar [2, Corollary 13.6.3], Y_Θ is countably generated as equivalence bimodules. Therefore, $[Y_\Theta]\in\mathrm{HB_c}(A)$. □

By Corollary 1.4 and Lemma 2.2, we obtain the following theorem.

THEOREM 2.3. *The map* $[\Theta]\to[Y_\Theta]$ *is an isomorphism of* $\mathrm{PAut}_\sigma(A\otimes\mathbb{K})/\sim$ *onto* $\mathrm{HB_c}(A)$.

Using the proof of Lemma 2.1, we obtain the following corollary.

COROLLARY 2.4. *For any σ-unital closed two-sided ideal I of $A\otimes\mathbb{K}$, there is a σ-unital closed two-sided ideal I_0 of A such that $I\cong I_0\otimes\mathbb{K}$.*

3. Conjugacy classes

For a C*-algebra A, let $\mathrm{HB}(A)^{-1}$ be the set of all invertible elements in $\mathrm{HB}(A)$. Then $\mathrm{Pic}(A)=\mathrm{HB}(A)^{-1}$. We consider the set of conjugacy classes of elements in $\mathrm{HB}(A)$ by an invertible element in $\mathrm{HB}(A)$. That is, for any $[X_1], [X_2]\in\mathrm{HB}(A)$, $[X_1]$

is equivalent to $[X_2]$, written $[X_1] \sim [X_2]$, if there is an element $[Y] \in \mathrm{Pic}(A)$ such that $[X_2] = [Y][X_1][Y]^{-1}$. We denote by $\mathrm{HB}(A)/\sim$ the quotient set by the above equivalence relation and denote by $[[X]]$ the equivalence class of $[X] \in \mathrm{HB}(A)$. By Brown, Green and Rieffel [4, Proposition 3.1], we can easily consider the following conjecture:

CONJECTURE. *There is a map from* $\mathrm{PAut}(A \otimes \mathbb{K})/\approx$ *to* $\mathrm{HB}(A)/\sim$ *induced by the homomorphism of* $\mathrm{PAut}(A \otimes \mathbb{K})/\sim$ *to* $\mathrm{HB}(A)$ *defined in Section 2 if we give an appropriate equivalence relation* $\sim$ *in* $\mathrm{PAut}(A \otimes \mathbb{K})/\sim$, *where* $\mathrm{PAut}(A \otimes \mathbb{K})/\approx$ *is the quotient set of* $\mathrm{PAut}(A \otimes \mathbb{K})/\sim$ *by the appropriate equivalence relation. Furthermore, the map from* $\mathrm{PAut}(A \otimes \mathbb{K})/\approx$ *to* $\mathrm{HB}(A)/\sim$ *is injective if A is σ-unital.*

In this section, we shall show this conjecture and obtain the main theorem. First, we shall give an equivalence relation $\sim$ in $\mathrm{PAut}(A \otimes \mathbb{K})/\sim$ as follows; for $\Theta_j = (I_j, J_j, \theta_j) \in \mathrm{PAut}(A \otimes \mathbb{K})$ $(j = 1, 2)$, $[\Theta_1] \sim [\Theta_2]$ if there is a $\beta \in \mathrm{Aut}(A \otimes \mathbb{K})$ such that $I_2 = \beta(I_1)$, $J_2 = \beta(J_1)$ and $[\Theta_1] = [(I_1, J_1, \beta^{-1} \circ \theta_2 \circ \beta)]$. We denote by $[[\Theta]]$ the equivalence class of $[\Theta] \in \mathrm{PAut}(A \otimes \mathbb{K})/\sim$.

LEMMA 3.1. *Let* $\Theta_j = (I_j, J_j, \theta_j) \in \mathrm{PAut}(A \otimes \mathbb{K})$ *for* $j = 1, 2$. *If* $[\Theta_1] \sim [\Theta_2]$, $[Y_{\Theta_1}] \sim [Y_{\Theta_2}]$.

PROOF. Since $[\Theta_1] \sim [\Theta_2]$, there is a $\beta \in \mathrm{Aut}(A \otimes \mathbb{K})$ such that $I_2 = \beta(I_1)$, $J_2 = \beta(J_1)$ and $[\Theta_1] = [(I_1, J_1, \beta^{-1} \circ \theta_2 \circ \beta)]$. By the definition of the product in $\mathrm{PAut}(A \otimes \mathbb{K})/\sim$,

$$\begin{aligned}[\Theta_1] &= [(I_1, J_1, \beta^{-1} \circ \theta_2 \circ \beta)] \\ &= [(A \otimes \mathbb{K}, A \otimes \mathbb{K}, \beta)][\Theta_2][(A \otimes \mathbb{K}, A \otimes \mathbb{K}, \beta^{-1})] = [\beta][\Theta_2][\beta]^{-1}.\end{aligned}$$

Thus using the homomorphism of $\mathrm{PAut}(A \otimes \mathbb{K})/\sim$ to $\mathrm{HB}(A)$ defined in Section 2, we see that $[Y_{\Theta_1}] = [Y_\beta][Y_{\Theta_2}][Y_\beta]^{-1}$, where $Y_\beta = \tilde{\mathbb{H}}_A \otimes_{A \otimes \mathbb{K}} X_\beta \otimes_{A \otimes \mathbb{K}} \mathbb{H}_A$ and X_β is an $A \otimes \mathbb{K}$-$A \otimes \mathbb{K}$-equivalence bimodule induced by β, which is defined in Brown, Green and Rieffel [4, Section 3]. Therefore, we obtain the conclusion. □

By Lemma 3.1, we can define a map

$$[[\Theta]] \in \mathrm{PAut}(A \otimes \mathbb{K})/\approx \ \to [[Y_\Theta]] \in \mathrm{HB}(A)/\sim. \tag{1}$$

LEMMA 3.2. *If A is a σ-unital C^*-algebra, the map (1) is injective.*

PROOF. Let $\Theta_j = (I_j, J_j, \theta_j) \in \mathrm{PAut}(A \otimes \mathbb{K})$ $(j = 1, 2)$ with $[Y_{\Theta_1}] \sim [Y_{\Theta_2}]$. Since A is σ-unital, by Brown, Green and Rieffel [4, Corollary 3.5], there is a $\beta \in \mathrm{Aut}(A \otimes \mathbb{K})$ such that $[Y_{\Theta_1}] = [Y_\beta][Y_{\Theta_2}][Y_\beta]^{-1}$ where Y_β is an A-A-equivalence bimodule defined in the proof of Lemma 3.1. Since the homomorphism $[\Theta] \in \mathrm{PAut}(A \otimes \mathbb{K})/\sim \ \to [Y_\Theta] \in \mathrm{HB}(A)$ is injective, we obtain that $[\Theta_1] = [\beta][\Theta_2][\beta]^{-1}$. □

THEOREM 3.3. *Let A be a σ-unital C*-algebra. Then the map*

$$[[\Theta]] \in \mathrm{PAut}_{\sigma}(A \otimes \mathbb{K})/\approx \ \to [[Y_{\Theta}]] \in \mathrm{HB}_{\mathrm{c}}(A)/\sim$$

is a bijection.

PROOF. The result follows immediately from Theorem 2.3 and Lemma 3.2. □

4. Crossed products

In this section, we shall consider a crossed product of a C*-algebra by a Hilbert C*-bimodule defined in Abadie, Eilers and Exel [1].

Let A be a C*-algebra and Y an A-A-Hilbert bimodule countably generated as equivalence bimodules. Then by Abadie, Eilers and Exel [1, Remark 3.4],

$$(A \times_Y \mathbb{Z}) \otimes \mathbb{K} \cong (A \otimes \mathbb{K}) \times_{\Theta_Y} \mathbb{Z},$$

where $(A \otimes \mathbb{K}) \times_{\Theta_Y} \mathbb{Z}$ is the crossed product of $A \otimes \mathbb{K}$ by Θ_Y which is defined in Exel [6] and Θ_Y is a partial automorphism of $A \otimes \mathbb{K}$ induced by Y which is defined in Section 2. Hence we obtain the following proposition.

PROPOSITION 4.1. *Let A be a C*-algebra and Y an A-A-Hilbert bimodule countably generated as equivalence bimodules. Then*

$$A \times_Y \mathbb{Z} \cong (1 \otimes e)((A \otimes \mathbb{K}) \times_{\Theta_Y} \mathbb{Z})(1 \otimes e),$$

where Θ_Y is the partial automorphism of $A \otimes \mathbb{K}$ induced by Y and 1 is the unit element in $M(A)$, e is a rank one projection in $\mathbb{K}$. In particular, if A is a simple σ-unital C-algebra, there is a $\beta \in \mathrm{Aut}(A \otimes \mathbb{K})$ which is uniquely determined up to multiplication by a generalized inner automorphism of $A \otimes \mathbb{K}$ such that*

$$A \times_Y \mathbb{Z} \cong (1 \otimes e)((A \otimes \mathbb{K}) \times_{\beta} \mathbb{Z})(1 \otimes e).$$

COROLLARY 4.2. *Let A be a simple σ-unital C*-algebra and Y an A-A-equivalence bimodule. Then $A \times_Y \mathbb{Z}$ is simple if and only if for any $n \in \mathbb{Z} \setminus \{0\}$, $[Y]^n \neq [A]$ in* $\mathrm{Pic}(A)$.

PROOF. Let β be an automorphism of $A \otimes \mathbb{K}$ such that

$$A \times_Y \mathbb{Z} \cong (1 \otimes e)((A \otimes \mathbb{K}) \times_{\beta} \mathbb{Z})(1 \otimes e).$$

Hence $A \times_{\beta} \mathbb{Z}$ is simple if and only if $\beta^n \notin \mathrm{Int}(A \otimes \mathbb{K})$ for any $n \in \mathbb{Z} \setminus \{0\}$ by Pedersen [8, Corollary 8.9.10]. By the proof of Lemma 2.1, $Y \cong \widetilde{\mathbb{H}}_A \otimes_{A\otimes\mathbb{K}} X_{\beta} \otimes_{A\otimes\mathbb{K}} \mathbb{H}_A$ where X_{β} is the $A \otimes \mathbb{K}$-$A \otimes \mathbb{K}$-equivalence bimodule induced by β. Therefore, by Brown, Green and Rieffel [4, Corollary 3.5], we obtain the conclusion. □

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