

## SOME NONOSCILLATION CRITERIA FOR INCLUSIONS

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### Abstract

New nonoscillatory criteria are presented for second order differential inclusions. The theory relies on Ky Fan's fixed point theorem for upper semicontinuous multifunctions.

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### 1. Introduction

In [1] we initiated the study of nonoscillatory solutions to the differential inclusion

$$(1.1) \quad (a(t)y'(t))' \in F(t, y(t)).$$

Recall a nontrivial solution of (1.1) is called oscillatory if it has arbitrarily large zeros, otherwise it is called nonoscillatory. In the single valued case many nonoscillatory results are available in the literature; see [6, 7, 8, 9, 10] and the references therein. In this paper, by looking at the asymptotic behaviour at infinity, we are able to establish new nonoscillatory criteria for (1.1). We first discuss (1.1) when  $\int_t^\infty ds/a(s) < \infty$  and we establish the existence of a nonoscillatory solution  $y$  (here  $y \geq 0$ ) to (1.1) with  $\lim_{t \rightarrow \infty} y(t) = 0$ . Two results will be presented, one where  $\lim_{t \rightarrow \infty} y(t)/\pi(t) > 0$  and the other where  $\lim_{t \rightarrow \infty} y(t)/\pi(t) = \infty$ ; here  $\pi(t) = \int_t^\infty ds/a(s)$ . The theory relies on Ky Fan's fixed point theorem in the Banach space setting. It is also possible (and we present this in Section 2) to discuss the case when  $\int_t^\infty ds/a(s)$  is infinity if we use Ky Fan's fixed point theorem in the Fréchet setting.

Solutions to (1.1) will be sought in  $B[T, \infty)$  and  $C[T, \infty)$ ;  $T \geq 0$  will be suitably chosen. Recall  $B[T, \infty)$  denotes the Banach space of all continuous, bounded real

valued functions on  $[T, \infty)$  endowed with the usual supremum norm, that is,  $|u|_\infty = \sup_{t \in [T, \infty)} |u(t)|$  for  $u \in B[T, \infty)$ .  $C[T, \infty)$  denotes the space of continuous real valued functions on  $[T, \infty)$ , the topology being that of uniform convergence on compact intervals on  $[T, \infty)$ .

We next state Ky Fan's fixed point theorem [2] and we also state a compactness criterion [5] in  $B[T, \infty)$ .

**THEOREM 1.1.** *Let  $Q$  be a nonempty, closed, convex subset of a Fréchet space  $E$  and  $N : Q \rightarrow CK(Q)$  an upper semicontinuous, compact map; here  $CK(Q)$  denotes the family of nonempty convex compact subsets of  $Q$ . Then there exists  $x \in Q$  with  $x \in N(x)$ .*

**THEOREM 1.2.** *Let  $E$  be an equicontinuous and uniformly bounded subset of the Banach space  $B[T, \infty)$ . If  $E$  is equiconvergent at  $\infty$ , it is also relatively compact.*

## 2. Differential inclusions

In this section a variety of nonoscillation results will be presented for the differential inclusion

$$(2.1) \quad (a(t)y'(t))' \in F(t, y(t)), \quad t \geq t_0 \geq 0,$$

where the function  $a$  is single valued and  $F$  is a multifunction. Throughout this section the following conditions will be satisfied:

$$(2.2) \quad a \in C([t_0, \infty), \mathbb{R}^+)$$

and

$$(2.3) \quad \left\{ \begin{array}{l} F : [t_0, \infty) \times \mathbb{R} \rightarrow CK(\mathbb{R}) \text{ is an } L^1\text{-Carathéodory multifunction: by} \\ \text{this we mean} \\ \text{(a) for each measurable } u : [t_0, \infty) \rightarrow \mathbb{R}, \text{ the map } t \mapsto F(t, u(t)) \text{ has} \\ \text{measurable single valued selections;} \\ \text{(b) for a.e. } t \in [t_0, \infty), \text{ the map } u \mapsto F(t, u) \text{ is upper semicontinuous;} \\ \text{(c) for each } r > 0, \text{ there exists } h_r \in L^1[t_0, \infty) \text{ with } |F(t, u)| \leq h_r(t) \\ \text{for a.e. } t \in [t_0, \infty) \text{ and all } u \in \mathbb{R} \text{ with } |u| \leq r; \text{ here } |F(x, u)| = \\ \text{sup}\{|v| : v \in F(x, u)\}. \end{array} \right.$$

In [1] we initiated the study of nonoscillatory solution to (2.1), and we continue this study here. In particular, different types (that is, different asymptotic behaviour at infinity than that in [1]) of nonoscillatory solutions are discussed. In our first two results we will assume  $\int_{t_0}^\infty ds/a(s) < \infty$ .

**THEOREM 2.1.** *Suppose (2.2) and (2.3) hold and in addition assume the following five conditions are satisfied*

$$(2.4) \quad F : [t_0, \infty) \times (0, \infty) \rightarrow CK((-\infty, 0])$$

$$(2.5) \quad \pi(t_0) < \infty \quad \text{where} \quad \pi(t) = \int_t^\infty \frac{ds}{a(s)}$$

$$(2.6) \quad \left\{ \begin{array}{l} \text{there exists a single valued } L^1\text{-Carathéodory function} \\ G : [t_0, \infty) \times \mathbb{R} \rightarrow [0, \infty) \text{ with } |F(u, z)| \leq G(u, z) \\ \text{for } (u, z) \in [t_0, \infty) \times (0, \infty) \end{array} \right.$$

$$(2.7) \quad \exists \lambda > 0 \quad \text{with} \quad \int_{t_0}^\infty G(s, \lambda\pi(s)) ds < \infty$$

and

$$(2.8) \quad \left\{ \begin{array}{l} \text{there exists a positive constant } M \text{ such that if} \\ 0 < x \leq y \leq \lambda\pi(t_0), \text{ then } G(t, x) \leq MG(t, y) \\ \text{for } t \geq t_0. \end{array} \right.$$

Then there is a nonoscillatory solution  $y$  of  $(a(t)y'(t))' \in F(t, y(t))$ , a.e.  $t \geq T$ , with  $\lim_{t \rightarrow \infty} y(t) = 0$  and  $y(t)/\pi(t) \in [\lambda/2, \lambda]$  for  $t \geq T$ ; here  $T$  is chosen as in (2.9). We also have

$$\lim_{t \rightarrow \infty} \frac{y(t)}{\pi(t)} = c_0 \in \left[ \frac{\lambda}{2}, \lambda \right].$$

**REMARK 2.1.** If  $G$  is nondecreasing in the second variable (that is, if  $0 < x \leq y$  implies  $G(t, x) \leq G(t, y)$  for  $t \geq t_0$ ) then clearly (2.8) holds with  $M = 1$ .

**PROOF.** From (2.7) there exists  $T \geq t_0$  with

$$(2.9) \quad \int_T^\infty G(s, \lambda\pi(s)) ds \leq \frac{\lambda}{2M}.$$

We wish to apply Theorem 1.1 with  $E = (B[T, \infty), |\cdot|_\infty)$  and

$$Q = \{y \in B[T, \infty) : \lambda\pi(t)/2 \leq y(t) \leq \lambda\pi(t) \text{ for } t \geq T\}.$$

Define a mapping  $N : Q \rightarrow \mathcal{P}(E)$  (the power set of  $E$ ) by (here  $y \in Q$ ),

$$N(y)(t) = \frac{\lambda}{2} \int_t^\infty \frac{ds}{a(s)} - \int_t^\infty \frac{1}{a(s)} \int_T^s F(u, y(u)) du ds \quad \text{for } t \geq T.$$

Note [4, Proposition 1.1, page 777] guarantees that  $N : Q \rightarrow C(E)$ ; here  $C(E)$  denotes the family of nonempty, convex subsets of  $E$ . We first show that

$$(2.10) \quad N : Q \rightarrow C(Q).$$

For notational purposes for any  $y \in Q$ , let

$$\mathcal{F}(y) = \left\{ u \in L^1[T, \infty) : u(t) \in F(t, y(t)) \text{ for a.e. } t \in [T, \infty) \right\}.$$

Let  $y \in Q$ , and take  $w \in N(y)$ . Then there exists  $\tau \in \mathcal{F}(y)$  with

$$w(t) = \frac{\lambda}{2} \int_t^\infty \frac{ds}{a(s)} - \int_t^\infty \frac{1}{a(s)} \int_T^s \tau(u) du ds \quad \text{for } t \geq T.$$

From (2.4) we have immediately that

$$w(t) \geq \frac{\lambda}{2} \int_t^\infty \frac{ds}{a(s)} = \frac{\lambda}{2} \pi(t) \quad \text{for } t \geq T.$$

Also (2.6), (2.8), (2.9) and  $y \in Q$  implies

$$\begin{aligned} w(t) &\leq \frac{\lambda}{2} \pi(t) + \int_t^\infty \frac{1}{a(s)} \int_T^s G(u, y(u)) du ds \\ &\leq \frac{\lambda}{2} \pi(t) + M \int_t^\infty \frac{1}{a(s)} \int_T^s G(u, \lambda\pi(u)) du ds \\ &\leq \frac{\lambda}{2} \pi(t) + \frac{\lambda}{2} \int_t^\infty \frac{ds}{a(s)} = \lambda\pi(t) \quad \text{for } t \geq T. \end{aligned}$$

As a result  $\lambda\pi(t)/2 \leq w(t) \leq \lambda\pi(t)$  for  $t \geq T$  for each  $w \in N(y)$ . Thus (2.10) holds. Next we show

$$(2.11) \quad N : Q \rightarrow C(Q) \text{ is a compact map.}$$

To see this we will use Theorem 1.2. Take any  $y \in Q$  and  $w \in N(y)$ . Then there exists  $\tau \in \mathcal{F}(y)$  with

$$w(t) = \frac{\lambda}{2} \int_t^\infty \frac{ds}{a(s)} - \int_t^\infty \frac{1}{a(s)} \int_T^s \tau(u) du ds \quad \text{for } t \geq T.$$

Thus

$$|w(t)| \leq \frac{\lambda}{2} \pi(t) + M \int_t^\infty \frac{1}{a(s)} \int_T^s G(u, \lambda\pi(u)) du ds \leq \lambda\pi(t)$$

for  $t \geq T$ , and so for each  $w \in N(y)$  we have  $|w|_\infty \leq \lambda\pi(t_0)$ . Thus the set

$$Y = \{Ny : y \in Q\} \text{ is a uniformly bounded subset of } B[T, \infty).$$

Since for each  $t \geq T$  we have  $|w(t)| \leq \lambda\pi(t)$  for  $w \in N(y)$ , then the set  $Y$  is equiconvergent at  $\infty$ . Next if  $t_1, t_2 \in [T, \infty)$  with  $t_1 \leq t_2$  we have

$$\begin{aligned} |w(t_2) - w(t_1)| &\leq \frac{\lambda}{2} \int_{t_1}^{t_2} \frac{ds}{a(s)} + M \int_{t_1}^{t_2} \frac{1}{a(s)} \int_T^s G(u, \lambda\pi(u)) du ds \\ &\leq \lambda \int_{t_1}^{t_2} \frac{ds}{a(s)} \end{aligned}$$

for  $w \in N(y)$ . Now Theorem 1.2 guarantees that  $Y$  is relatively compact in  $B[T, \infty)$ , and as a result (2.11) holds. Essentially the same reasoning as in [1] implies that

$$(2.12) \quad N : Q \rightarrow CK(Q) \text{ is an upper semicontinuous map.}$$

Theorem 1.1 guarantees that there exists  $y \in Q$  with  $y \in N(y)$ . That is for every  $t \geq T$ , we have

$$y(t) \in \frac{\lambda}{2} \pi(t) - \int_t^\infty \frac{1}{a(s)} \int_T^s F(u, y(u)) du ds.$$

Also since  $y \in Q$  we have  $y(t)/\pi(t) \in [\lambda/2, \lambda]$  for  $t \geq T$ , and  $\lim_{t \rightarrow \infty} y(t) = 0$  since (2.5) implies

$$\begin{aligned} |y(t)| &\leq \frac{\lambda}{2} \pi(t) + M \int_t^\infty \frac{1}{a(s)} \int_T^s G(u, \lambda\pi(u)) du ds \\ &\leq \lambda\pi(t) \rightarrow 0 \quad \text{as } t \rightarrow \infty. \end{aligned}$$

Now there exists  $\tau \in \mathcal{F}(y)$  with

$$y(t) = \frac{\lambda}{2} \pi(t) + \int_t^\infty \frac{1}{a(s)} \int_T^s [-\tau(u)] du ds \quad \text{for } t \geq T,$$

so

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{y(t)}{\pi(t)} &= \frac{\lambda}{2} + \lim_{t \rightarrow \infty} \frac{\int_t^\infty \left( \int_T^s [-\tau(u)] du \right) ds/a(s)}{\int_t^\infty ds/a(s)} \\ &= \frac{\lambda}{2} + \int_T^\infty [-\tau(x)] dx. \end{aligned}$$

□

**REMARK 2.2.** Minor adjustments in the analysis are needed if (2.4) is replaced by

$$(2.13) \quad F : [t_0, \infty) \times (0, \infty) \rightarrow CK([0, \infty)).$$

We leave the details to the reader.

**THEOREM 2.2.** *Suppose (2.2)–(2.6) hold and in addition assume the following conditions are satisfied*

$$(2.14) \quad \exists \lambda > 0 \quad \text{with} \quad \int_{t_0}^\infty \frac{1}{a(s)} \int_{t_0}^s G(u, \lambda) du ds < \infty$$

and

$$(2.15) \quad \begin{cases} \text{there exists a positive constant } M \text{ such that if } 0 < x \leq \lambda, \\ \text{then } G(t, x) \leq MG(t, \lambda) \text{ for } t \geq t_0. \end{cases}$$

Then there is a nonoscillatory solution  $y$  of  $(a(t)y'(t))' \in F(t, y(t))$ , a.e.  $t \geq T$  with  $\lim_{t \rightarrow \infty} y(t) = 0$  and  $y(t) \leq \lambda$  for  $t \geq T$ ; here  $T$  is chosen as in (2.20). If in addition

$$(2.16) \quad \begin{cases} \text{there exists a single valued } L^1\text{-Carathéodory function} \\ G_1 : [t_0, \infty) \times \mathbb{R} \rightarrow [0, \infty) \text{ with } |F(u, z)| \leq G_1(u, z) \\ \text{for } (u, z) \in [t_0, \infty) \times (0, \lambda), \end{cases}$$

$$(2.17) \quad \begin{cases} \text{there exists a positive constant } M_1 \text{ such that if } 0 < x \leq y \leq \lambda, \\ \text{then } G_1(t, x) \leq M_1 G_1(t, y) \text{ for } t \geq t_0 \end{cases}$$

and

$$(2.18) \quad \int_{t_0}^{\infty} G_1(s, \mu\pi(s)) ds = \infty \quad \text{for all } \mu > 0$$

hold, then  $\lim_{t \rightarrow \infty} y(t)/\pi(t) = \infty$ .

**PROOF.** Choose  $\mu > 0$  so that

$$(2.19) \quad \mu[1 + \pi(t_0)] \leq \lambda$$

and choose  $T \geq t_0$  so that

$$(2.20) \quad \int_T^{\infty} \frac{1}{a(s)} \int_T^s G(u, \lambda) du ds \leq \frac{\mu}{2M}.$$

Let  $E = B[T, \infty)$ ,  $Q = \{y \in B[T, \infty) : \mu\pi(t)/2 \leq y(t) \leq \lambda \text{ for } t \geq T\}$  and let  $N : Q \rightarrow \mathcal{P}(E)$  be (here  $y \in Q$ ),

$$N(y)(t) = \frac{\mu}{2} \int_t^{\infty} \frac{ds}{a(s)} - \int_t^{\infty} \frac{1}{a(s)} \int_T^s F(u, y(u)) du ds \quad \text{for } t \geq T.$$

It is easy to see from (2.4), (2.6), (2.15), (2.19) and (2.20) that

$$(2.21) \quad N : Q \rightarrow C(Q).$$

Similarly as in Theorem 2.1 one can also deduce that  $N : Q \rightarrow CK(Q)$  is an upper semicontinuous, compact map. Theorem 1.1 guarantees that there exists  $y \in Q$  with

$$y(t) \in \frac{\mu}{2} \pi(t) - \int_t^{\infty} \frac{1}{a(s)} \int_T^s F(u, y(u)) du ds \quad \text{for } t \geq T.$$

Also since  $y \in Q$  we have  $y(t) \in [\mu\pi(t)/2, \lambda]$  for  $t \geq T$ , and  $\lim_{t \rightarrow \infty} y(t) = 0$  since

$$|y(t)| \leq \frac{\mu}{2} \pi(t) + M \int_t^{\infty} \frac{1}{a(s)} \int_T^s G(u, \lambda) du ds \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Next suppose (2.16)–(2.18) hold. We now show that

$$\lim_{t \rightarrow \infty} \frac{y(t)}{\pi(t)} = \infty.$$

To see this notice there exists  $\tau \in \mathcal{F}(y)$  with

$$y(t) = \frac{\mu}{2}\pi(t) + \int_t^\infty \frac{1}{a(s)} \int_T^s [-\tau(u)] du ds \quad \text{for } t \geq T.$$

Thus

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{y(t)}{\pi(t)} &= \frac{\mu}{2} + \lim_{t \rightarrow \infty} \left[ - \int_T^t \tau(u) du \right] \\ &\geq \frac{\mu}{2} + \lim_{t \rightarrow \infty} \int_T^t G_1(u, y(u)) du \\ &\geq \frac{\mu}{2} + \frac{1}{M_1} \lim_{t \rightarrow \infty} \int_T^t G_1\left(u, \frac{\mu}{2}\pi(u)\right) du = \infty. \quad \square \end{aligned}$$

**REMARK 2.3.** There is also an analogue of Theorem 2.2 if (2.4) is replaced by (2.13). We leave the details to the reader.

If we don't assume the condition  $\pi(t_0) < \infty$ , then it is also possible to obtain an analogue of Theorem 2.1 and Theorem 2.2. The idea here is to work with  $C[T, \infty)$  instead of  $B[T, \infty)$ . For our next theorem, for notational purposes, let  $R[t, x] = \int_x^t ds/a(s)$  and  $R(t) = R[t, t_0]$ .

**THEOREM 2.3.** *Suppose (2.2)–(2.4) and (2.6) hold and in addition assume the following conditions are satisfied*

$$(2.22) \quad \exists t_1 \geq t_0 \quad \text{and} \quad \lambda > 0 \quad \text{with} \quad \int_{t_1}^\infty G(s, \lambda R[s, t_1]) ds < \infty$$

and

$$(2.23) \quad \begin{cases} \text{there exists a positive constant } M \text{ such that if } 0 < x \leq \lambda, \text{ then} \\ G(t, xR[t, t_1]) \leq MG(t, \lambda R[t, t_1]) \text{ for } t \geq t_1. \end{cases}$$

Then there is a nonoscillatory solution  $y$  of  $(a(t)y'(t))' \in F(t, y(t))$ , a.e.  $t \geq T$  with

$$\lim_{t \rightarrow \infty} \frac{y(t)}{R(t)} = c_0 \in (0, \lambda);$$

here  $T$  is chosen as in (2.24).

**REMARK 2.4.** (i) If  $G$  is nondecreasing in the second variable (as in Remark 2.1) then clearly (2.23) holds with  $M = 1$ .

(ii) There is also an analogue of Theorem 2.3 if (2.13) replaces (2.4).

**PROOF.** Choose  $T \geq t_1$  so that

$$(2.24) \quad \int_T^\infty G(s, \lambda R[s, t_1]) ds \leq \frac{\lambda}{2M}.$$

Let  $E = C[T, \infty)$ ,

$$Q = \left\{ y \in C[T, \infty) : \frac{\lambda}{2} R[t, T] \leq y(t) \leq \lambda R[t, t_1] \text{ for } t \geq T \right\}$$

and let  $N : Q \rightarrow \mathcal{P}(E)$  be (here  $y \in Q$ ),

$$N(y)(t) = \lambda \int_T^t \frac{ds}{a(s)} + \int_T^t \frac{1}{a(s)} \int_T^s F(u, y(u)) du ds \quad \text{for } t \geq T.$$

It is easy to see from (2.4), (2.6), (2.23) and (2.24) that

$$(2.25) \quad N : Q \rightarrow C(Q).$$

A slight modification of the argument in [3, Theorem 1.1, page 1293] guarantees that  $N : Q \rightarrow CK(Q)$  is an upper semicontinuous, compact map. Theorem 1.1 guarantees that there exists  $y \in Q$  with

$$y(t) \in \lambda \int_T^t \frac{ds}{a(s)} + \int_T^t \frac{1}{a(s)} \int_T^s F(u, y(u)) du ds \quad \text{for } t \geq T.$$

Now there exists  $\tau \in \mathcal{F}(y)$  with

$$y(t) = \int_T^t \frac{1}{a(s)} \left[ \lambda + \int_T^s \tau(u) du \right] ds \quad \text{for } t \geq T.$$

There are two cases to consider, namely  $\int_T^\infty ds/a(s) < \infty$  and  $\int_T^\infty ds/a(s) = \infty$ . If  $\int_T^\infty ds/a(s) < \infty$  then

$$0 \leq \int_T^\infty \frac{1}{a(s)} \left[ \lambda + \int_T^s \tau(u) du \right] ds \leq \lambda \int_T^\infty \frac{ds}{a(s)} < \infty,$$

so

$$\lim_{t \rightarrow \infty} \frac{y(t)}{R(t)} = \frac{\int_T^\infty [\lambda + \int_T^s \tau(u) du] ds/a(s)}{\int_{t_0}^\infty ds/a(s)} = c_0.$$

Notice that  $c_0 \in (0, \lambda]$  since

$$\begin{aligned} \int_T^\infty \frac{1}{a(s)} \left[ \lambda + \int_T^s \tau(u) du \right] ds &\geq \int_T^\infty \frac{1}{a(s)} \left[ \lambda - M \frac{\lambda}{2M} \right] ds \\ &= \frac{\lambda}{2} \int_T^\infty \frac{ds}{a(s)}, \end{aligned}$$



and so

$$\frac{(\lambda/2) \int_T^\infty ds/a(s)}{\int_{t_0}^\infty ds/a(s)} \leq c_0 \leq \frac{\lambda \int_T^\infty ds/a(s)}{\int_{t_0}^\infty ds/a(s)} \leq \lambda.$$

If  $\int^\infty ds/a(s) = \infty$  then

$$\int_T^\infty \frac{1}{a(s)} \left[ \lambda + \int_T^s \tau(u) du \right] ds \geq \frac{\lambda}{2} \int_T^\infty \frac{ds}{a(s)} = \infty,$$

so l'Hopital's rule implies

$$\lim_{t \rightarrow \infty} \frac{y(t)}{R(t)} = \lambda + \int_T^\infty \tau(u) du = c_0.$$

Notice that  $\lambda/2 \leq c_0 \leq \lambda$  since  $\lambda + \int_T^\infty \tau(u) du \geq \lambda - M(\lambda/2M) = \lambda/2$ .  $\square$

**THEOREM 2.4.** *Suppose (2.2)–(2.4) and (2.6) hold and in addition assume the following conditions are satisfied*

$$(2.26) \quad \exists \lambda > 0 \quad \text{with} \quad \int^\infty \frac{1}{a(s)} \int_{t_0}^s G(u, \lambda) du ds < \infty$$

and

$$(2.27) \quad \begin{cases} \text{there exists a positive constant } M \text{ such that if } \lambda/2 \leq x \leq \lambda, \text{ then} \\ G(t, x) \leq MG(t, \lambda) \text{ for } t \geq t_0. \end{cases}$$

Then there is a nonoscillatory solution  $y$  of  $(a(t)y'(t))' \in F(t, y(t))$ , a.e.  $t \geq T$  with  $\lim_{t \rightarrow \infty} y(t) = c_0 \in [\lambda/2, \lambda]$ . Here  $T$  is chosen as in (2.28).

**PROOF.** Choose  $T \geq t_0$  with

$$(2.28) \quad \int_T^\infty \frac{1}{a(s)} \int_T^s G(u, \lambda) du ds \leq \int_T^\infty \frac{1}{a(s)} \int_{t_0}^s G(u, \lambda) du ds \leq \frac{\lambda}{2M}.$$

Let  $E = C[T, \infty)$ ,  $Q = \{y \in C[T, \infty) : \lambda/2 \leq y(t) \leq \lambda \text{ for } t \geq T\}$  and let  $N : Q \rightarrow \mathcal{P}(E)$  be (here  $y \in Q$ ),

$$N(y)(t) = \lambda + \int_T^t \frac{1}{a(s)} \int_T^s F(u, y(u)) du ds \quad \text{for } t \geq T.$$

It is easy to see from (2.4), (2.6), (2.27) and (2.28) that

$$(2.29) \quad N : Q \rightarrow C(Q).$$

Also (as in Theorem 2.3) we have that  $N : Q \rightarrow CK(Q)$  is an upper semicontinuous, compact map. Theorem 1.1 guarantees that there exists  $y \in Q$  with

$$y(t) \in \lambda + \int_T^t \frac{1}{a(s)} \int_T^s F(u, y(u)) \, du \, ds \quad \text{for } t \geq T.$$

Notice that  $y(t) \in [\lambda/2, \lambda]$  for  $t \geq T$ . Now there exists  $\tau \in \mathcal{F}(y)$  with

$$y(t) = \lambda + \int_T^t \frac{1}{a(s)} \int_T^s \tau(u) \, du \, ds \quad \text{for } t \geq T.$$

Since  $y$  is monotonic,  $\lim_{t \rightarrow \infty} y(t)$  exists and

$$\lim_{t \rightarrow \infty} y(t) = \lambda + \int_T^\infty \frac{1}{a(s)} \int_T^s \tau(u) \, du \, ds. \quad \square$$

**REMARK 2.5.** (i) There is an analogue of Theorem 2.4 if (2.13) replaces (2.4).

(ii) In Theorem 2.4, it is possible to replace (2.6), (2.26) and (2.27) with

$$(2.30) \quad \exists \lambda > 0 \quad \text{with} \quad \int_{t_0}^\infty \frac{1}{a(s)} \int_{t_0}^s \sup_{w \in [\lambda/2, \lambda]} |F(u, w)| \, du \, ds < \infty,$$

and the result is again true. The proof only involves a slight modification of the above argument.

**THEOREM 2.5.** *Suppose (2.2)–(2.4) and (2.6) hold and in addition assume the following conditions are satisfied*

$$(2.31) \quad \exists \lambda > 0, \mu > 0 \quad \text{with} \quad \int_{t_0}^\infty G \left( s, \mu + \lambda \int_{t_0}^s \frac{dx}{a(x)} \right) ds < \infty$$

and

$$(2.32) \quad \text{for } 0 < x \leq y \text{ we have } G(t, x) \leq G(t, y) \text{ for } t \geq t_0.$$

Then there is a nonoscillatory solution  $y$  of  $(a(t)y'(t))' \in F(t, y(t))$ , a.e.  $t \geq T$ . Here  $T$  is chosen as in (2.34).

If, in addition,

$$(2.33) \quad \int_T^\infty \frac{1}{a(s)} \int_s^\infty G \left( u, \mu + \lambda \int_{t_0}^u \frac{dx}{a(x)} \right) du \, ds = \infty,$$

then  $\lim_{t \rightarrow \infty} y(t) = \infty$ .

**PROOF.** Choose  $T \geq t_0$  so that

$$(2.34) \quad \int_T^\infty G \left( s, \mu + \lambda \int_{t_0}^s \frac{dx}{a(x)} \right) ds \leq \lambda.$$

Let  $E = C[T, \infty)$ ,

$$Q = \left\{ y \in C[T, \infty) : \mu \leq y(t) \leq \mu + \lambda \int_{t_0}^t \frac{dx}{a(x)} \text{ for } t \geq T \right\}$$

and let  $N : Q \rightarrow \mathcal{P}(E)$  be (here  $y \in Q$ ),

$$N(y)(t) = \mu + \int_T^t \frac{1}{a(s)} \left[ \lambda + \int_T^s F(u, y(u)) du \right] ds \text{ for } t \geq T.$$

It is easy to see from (2.4), (2.6), (2.32) and (2.34) that

$$(2.35) \quad N : Q \rightarrow C(Q).$$

Also (as in Theorem 2.3) we have that  $N : Q \rightarrow CK(Q)$  is an upper semicontinuous, compact map. Theorem 1.1 guarantees that there exists  $y \in Q$  with

$$y(t) \in \mu + \int_T^t \frac{1}{a(s)} \left[ \lambda + \int_T^s F(u, y(u)) du \right] ds \text{ for } t \geq T.$$

Now there exists  $\tau \in \mathcal{F}(y)$  with

$$y(t) = \mu + \int_T^t \frac{1}{a(s)} \left[ \lambda + \int_T^s \tau(u) du \right] ds \text{ for } t \geq T.$$

Suppose (2.33) holds. Then

$$\begin{aligned} y(t) &\geq \mu + \int_T^t \frac{1}{a(s)} \left[ \int_T^\infty G \left( u, \mu + \lambda \int_{t_0}^u \frac{dx}{a(x)} \right) du + \int_T^s \tau(u) du \right] ds \\ &\geq \mu + \int_T^t \frac{1}{a(s)} \left[ \int_T^\infty G \left( u, \mu + \lambda \int_{t_0}^u \frac{dx}{a(x)} \right) du \right. \\ &\quad \left. - \int_T^s G(u, y(u)) du \right] ds \\ &\geq \mu + \int_T^t \frac{1}{a(s)} \left[ \int_T^\infty G \left( u, \mu + \lambda \int_{t_0}^u \frac{dx}{a(x)} \right) du \right. \\ &\quad \left. - \int_T^s G \left( u, \mu + \lambda \int_{t_0}^u \frac{dx}{a(x)} \right) du \right] ds \\ &= \mu + \int_T^t \frac{1}{a(s)} \left[ \int_s^\infty G \left( u, \mu + \lambda \int_{t_0}^u \frac{dx}{a(x)} \right) du \right] ds \xrightarrow{t \rightarrow \infty} \infty. \quad \square \end{aligned}$$

**REMARK 2.6.** (i) There is an analogue of Theorem 2.5 if (2.13) replaces (2.4).

(ii) Notice that if  $\int_T^\infty \left( \int_s^\infty G(u, \mu) du \right) ds / a(s) = \infty$ , then (2.33) holds since (2.32) guarantees that  $G(s, \mu + \lambda \int_{t_0}^s dx/a(x)) \geq G(s, \mu)$  for  $s \geq T$ .

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