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SOME COMPLETELY MONOTONIC FUNCTIONS INVOLVING THE GAMMA AND POLYGAMMA FUNCTIONS

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Abstract

The function $[\Gamma(x+1)]^{1/x}(1+1/x)^x/x$ is strictly logarithmically completely monotonic in $(0,\infty)$. The function $\psi''(x+2)+(1+x^2)/x^2(1+x)^2$ is strictly completely monotonic in $(0,\infty)$.

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1. Introduction

It is well known that the classical Euler gamma function $\Gamma(z)$ is defined for $\operatorname{Re} z > 0$ as

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,dt. \tag{1}$$

The psi or digamma function $\psi(x)=\Gamma'(x)/\Gamma(x)$, the logarithmic derivative of the gamma function, and the polygamma functions can be expressed for $x>0$ and $k\in\mathbb{N}$ as

$$\psi(x)=-\gamma+\sum_{n=0}^{\infty}\left(\frac{1}{1+n}-\frac{1}{x+n}\right), \tag{2}$$

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(3) $$\psi^{(k)}(x)=(-1)^{k+1}k!\sum_{i=0}^{\infty}\frac{1}{(x+i)^{k+1}},$$

(4) $$\psi(x)=-\gamma+\int_0^{\infty}\frac{e^{-t}-e^{-xt}}{1-e^{-t}}\,\mathrm{d}t,$$

(5) $$\psi^{(k)}(x)=(-1)^{k+1}\int_0^{\infty}\frac{t^k e^{-xt}}{1-e^{-t}}\,\mathrm{d}t,$$

where $\gamma=0.57721566490153286\dots$ is the Euler-Mascheroni constant.

DEFINITION 1. A function f is said to be *completely monotonic* on an interval I if f has derivatives of all orders on I which alternate successively in sign, that is,

(6) $$(-1)^n f^{(n)}(x)\geq 0$$

for $x\in I$ and $n\geq 0$. If inequality (6) is strict for all $x\in I$ and for all $n\geq 0$, then f is said to be strictly completely monotonic.

DEFINITION 2. A function f is said to be *logarithmically completely monotonic* on an interval I if its logarithm $\ln f$ satisfies

(7) $$(-1)^k[\ln f(x)]^{(k)}\geq 0$$

for $k\in\mathbb{N}$ on I. If inequality (7) is strict for all $x\in I$ and for all $k\in\mathbb{N}$, then f is said to be strictly logarithmically completely monotonic.

The concepts of (logarithmically) completely monotonic function are defined on an arbitrary interval I here, but the main case is when $I=(0,\infty)$, where the completely monotonic functions are characterized by Bernstein's Theorem [8, page 161] as the Laplace transforms of positive measure μ in $(0,\infty)$. Bernstein's Theorem states that a function f is completely monotonic in $(0,\infty)$ if and only if

(8) $$f(x)=\int_0^{\infty}e^{-xs}\,\mathrm{d}\mu(s),$$

where $\mu(s)$ is a nonnegative measure, or say that $\mu(s)$ is nondecreasing, on $(0,\infty)$ such that the integral converges for all $x>0$. Hence we conclude that a completely monotonic function which is non-identically zero cannot vanish at any point in $(0,\infty)$. It is clear that a completely monotonic function f in $(0,\infty)$ is strictly completely monotonic if and only if $\mu(s)$ has mass in the open interval $(0,\infty)$. Therefore the sharpenings with 'strict' in Definition 1 and Definition 2 are not very interesting.

To the best of our knowledge, the terminology or the notion 'logarithmically completely monotonic function' was explicitly introduced in [5, 6, 7] and it was also proved in [5, 6] that a logarithmically completely monotonic function is completely monotonic. However, it cannot be said to be new, since in [2] this notion appears implicitly in Lemma 2.4 (ii) which can be rephrased as [5, Theorem 1] or [6, Theorem 4].

Completely monotonic functions have applications in many branches. For example, they play a role in potential theory, probability theory, physics, numerical and asymptotic analysis, and combinatorics. Some related references are listed in [1].

It is easy to prove that the function $(1+1/x)^{-x}$ is completely monotonic in $(0,\infty)$ through proving that it is logarithmically completely monotonic in $(0,\infty)$. A stronger result that the function $(1+1/x)^{-x}$ is a Stieltjes transform in $(0,\infty)$ follows from [1, Remark 3, page 457]. A function f is called a Stieltjes transform if it is of the form

$$f(x)=a+\int_0^\infty \frac{\mathrm{d}\mu(s)}{s+x}, \tag{9}$$

where $a\ge 0$ and μ is a nonnegative measure on $[0,\infty)$ satisfying

$$\int_0^\infty \frac{1}{1+s}\,\mathrm{d}\mu(s)<\infty.$$

From (9) we can see directly that a Stieltjes transform is a completely monotonic function.

Among other things, the following results were obtained in [6]: For $\alpha\le 0$, the function $x^\alpha/[\Gamma(x+1)]^{1/x}$ is strictly logarithmically completely monotonic in $(0,\infty)$. For $\alpha\ge 1$, the function $[\Gamma(x+1)]^{1/x}/x^\alpha$ is strictly logarithmically completely monotonic in $(0,\infty)$. It should be noted that a similar but stronger result is contained in [2, Theorem 3.2]. The statement of [2] is that the function

$$\varphi(x)=\frac{1}{x[\Gamma(1+1/x)]^x}$$

is a Stieltjes transform and hence completely monotonic. However, it is well known (see, for example, [3, page 127]) that if $\varphi(x)$ is a Stieltjes transform, then so is $1/\varphi(1/x)$ and this is exactly the function $[\Gamma(x+1)]^{1/x}/x$, which is then completely monotonic, since it is a Stieltjes transform.

In [4] the following two inequalities are presented: For $x\in(0,1)$, we have

$$\frac{x}{[\Gamma(x+1)]^{1/x}}<\left(1+\frac{1}{x}\right)^x<\frac{x+1}{[\Gamma(x+1)]^{1/x}}.$$

For $x\ge 1$,

$$\left(1+\frac{1}{x}\right)^x\ge\frac{x+1}{[\Gamma(x+1)]^{1/x}}. \tag{10}$$

Equality in (10) occurs for $x = 1$.

It is easy to obtain, using the standard argument, that

$$\lim_{x\to\infty} \frac{[\Gamma(x+1)]^{1/x}}{x}\left(1+\frac{1}{x}\right)^x = 1.$$

Out of curiosity, the (logarithmically) completely monotonic property of the quotient between two (logarithmically) completely monotonic functions (Stieltjes transforms) $[\Gamma(x+1)]^{1/x}/x$ and $(1+1/x)^{-x}$ will be considered in this article. The main result of this consideration is

THEOREM 1.1. *The function* $x^{-1}(\Gamma(x+1))^{1/x}(1+1/x)^x$ *is strictly logarithmically completely monotonic in* $(0, \infty)$.

As a direct consequence of the proof of Theorem 1.1, we have

COROLLARY 1.2. *The function*

$$\psi''(x) + \frac{x^4+5x^3+7x^2+7x+2}{x^3(x+1)^3} = \psi''(x+2) + \frac{1+x^2}{x^2(1+x)^2}$$

is strictly completely monotonic in $(0, \infty)$.

2. Proof of Theorem 1.1

Define

(11) $$F(x) = \frac{[\Gamma(x+1)]^{1/x}}{x^c}\left(1+\frac{a}{x}\right)^{x+b}$$

for $x > 0$ and some fixed real numbers a, b and c.

Taking the logarithm of $F(x)$ and differentiating yields

$$\ln F(x) = (x+b)\ln\left(1+\frac{a}{x}\right) + \frac{\ln\Gamma(x+1)}{x} - c\ln x,$$

$$[\ln F(x)]' = \ln\left(1+\frac{a}{x}\right) - \frac{a(x+b)}{x(x+a)} + \frac{x\psi(x+1)-\ln\Gamma(x+1)}{x^2} - \frac{c}{x}, \quad \text{and}$$

$$\begin{aligned}[\ln F(x)]^{(n)} &= (-1)^{n-1}(n-1)!(x+b)\left[\frac{1}{(x+a)^n} - \frac{1}{x^n}\right] \\ &\quad + (-1)^n(n-2)!n\left[\frac{1}{(x+a)^{n-1}} - \frac{1}{x^{n-1}}\right] + \frac{h_n(x)}{x^{n+1}} + (-1)^n(n-1)!\frac{c}{x^n} \\ &= (-1)^n(n-2)!\left[\frac{(n-1)(b+c)-x}{x^n} + \frac{x+na-(n-1)b}{(x+a)^n}\right] + \frac{h_n(x)}{x^{n+1}},\end{aligned}$$

where $n \ge 2$, $\psi^{(-1)}(x+1) = \ln\Gamma(x+1)$, $\psi^{(0)}(x+1) = \psi(x+1)$, and

$$h_n(x) = \sum_{k=0}^{n} \frac{(-1)^{n-k} n! x^k \psi^{(k-1)}(x+1)}{k!},$$

$$h_n'(x) = x^n \psi^{(n)}(x+1) \begin{cases} > 0, & \text{if } n \text{ is odd;} \\ < 0, & \text{if } n \text{ is even.} \end{cases}$$

Therefore, we have

$$\begin{aligned}(-1)^n x^{n+1}[\ln F(x)]^{(n)} &+ (-1)^{n+1} h_n(x) \\ &= (n-2)! \left\{ (n-1)(b+c) - x + \frac{x^n[x+na-(n-1)b]}{(x+a)^n} \right\} x\end{aligned}$$

and

$$\begin{aligned}\frac{\mathrm{d}\left\{(-1)^n x^{n+1}[\ln F(x)]^{(n)}\right\}}{\mathrm{d}x} &\\ = (-1)^n x^n \psi^{(n)}(x+1) &+ (n-2)! \left\{ (n-1)(b+c) - 2x \right.\\ &\left. + \frac{x^n[a(b+an+an^2-bn^2)+(2a+b+2an-bn)x+2x^2]}{(x+a)^{n+1}} \right\} \\ = x^n \left\{ (-1)^n \psi^{(n)}(x+1) \right. &+ (n-2)! \left[\frac{(n-1)(b+c)-2x}{x^n} \right.\\ &\left.\left. + \frac{a(b+an+an^2-bn^2)+(2a+b+2an-bn)x+2x^2}{(x+a)^{n+1}} \right] \right\} \\ = x^n \left\{ (-1)^n \psi^{(n)}(x) + \frac{n!}{x^{n+1}} \right. &+ (n-2)! \left[\frac{(n-1)(b+c)-2x}{x^n} \right.\\ &\left.\left. + \frac{a(b+an+an^2-bn^2)+(2a+b+2an-bn)x+2x^2}{(x+a)^{n+1}} \right] \right\}.\end{aligned}$$

By letting $a = c = 1$ and $b = 0$, we have

$$\begin{aligned}\frac{\mathrm{d}\left\{(-1)^n x^{n+1}[\ln F(x)]^{(n)}\right\}}{\mathrm{d}x} &\\ = x^n \left\{ (-1)^n \psi^{(n)}(x) + \frac{n!}{x^{n+1}} \right. &\\ + (n-2)! \left[\frac{n-1-2x}{x^n} \right. &\left.\left. + \frac{n(n+1)+2(n+1)x+2x^2}{(x+1)^{n+1}} \right] \right\} \\ = x^n \left\{ (-1)^n \psi^{(n)}(x) + (n-2)! \right. &\left[\frac{n(n-1)+(n-1)x-2x^2}{x^{n+1}} \right.\\ &\left.\left. + \frac{n(n+1)+2(n+1)x+2x^2}{(x+1)^{n+1}} \right] \right\}\end{aligned}$$

$$\triangleq x^n\big\{(-1)^n\psi^{(n)}(x)+(n-2)!g_n(x)+(n-2)!h_n(x)\big\}.$$

By induction, it follows that $g_n'(x)=-(n-1)g_{n+1}(x)$ and $h_n'(x)=-(n-1)h_{n+1}(x)$. This implies $g_2^{(n-2)}(x)=(-1)^n(n-2)!g_n(x)$ and $h_2^{(n-2)}(x)=(-1)^n(n-2)!h_n(x)$. Therefore,

$$\frac{\mathrm{d}\big\{(-1)^n x^{n+1}[\ln F(x)]^{(n)}\big\}}{\mathrm{d}x}=(-1)^n x^n\big[\psi''(x)+g_2(x)+h_2(x)\big]^{(n-2)}.$$

It is a well-known fact that, for $x>0$ and $r>0$,

$$\frac{1}{x^r}=\frac{1}{\Gamma(r)}\int_0^\infty t^{r-1}e^{-xt}\,\mathrm{d}t. \tag{12}$$

From formulae (3), (5) and (12), for $x\in(0,\infty)$ and any nonnegative integer i, we have

$$\begin{aligned}
\phi(x)&\triangleq\psi''(x)+g_2(x)+h_2(x)=\psi''(x)+\frac{2+x-2x^2}{x^3}+\frac{2(3+3x+x^2)}{(x+1)^3}\\
&=\psi''(x)+\frac{x^4+5x^3+7x^2+7x+2}{x^3(x+1)^3}\\
&=\psi''(x)+\frac{2}{x^3}+\frac{1}{x^2}-\frac{2}{x}+\frac{2}{(1+x)^3}+\frac{2}{(1+x)^2}+\frac{2}{1+x}\\
&=\frac{1}{x^2}-\frac{2}{x}+\frac{2}{(1+x)^2}+\frac{2}{1+x}-2\sum_{i=2}^\infty\frac{1}{(x+i)^3}\\
&=\psi''(x+2)+\frac{1}{x^2}-\frac{2}{x}+\frac{2}{(1+x)^2}+\frac{2}{1+x}=\psi''(x+2)+\frac{1+x^2}{x^2(1+x)^2}\\
&=\int_0^\infty te^{-xt}\,\mathrm{d}t-2\int_0^\infty e^{-xt}\,\mathrm{d}t+2\int_0^\infty te^{-(x+1)t}\,\mathrm{d}t\\
&\quad+2\int_0^\infty e^{-(x+1)t}\,\mathrm{d}t-\int_0^\infty\frac{t^2e^{-(x+2)t}}{1-e^{-t}}\,\mathrm{d}t\\
&=\int_0^\infty\big[t-2+(t+4)e^{-t}-(t^2+2t+2)e^{-2t}\big]\frac{e^{-xt}}{1-e^{-t}}\,\mathrm{d}t\triangleq\int_0^\infty\frac{q(t)e^{-xt}}{1-e^{-t}}\,\mathrm{d}t,\\
\phi^{(i)}(x)&=(-1)^i\int_0^\infty q(t)\frac{t^ie^{-xt}}{1-e^{-t}}\,\mathrm{d}t,
\end{aligned}$$

and

$$\begin{aligned}
q'(t)&=\big(2+2t+2t^2-3e^t+e^{2t}-te^t\big)e^{-2t}\triangleq p(t)e^{-2t},\\
p'(t)&=2+4t-4e^t+2e^{2t}-te^t,\qquad p''(t)=4-5e^t+4e^{2t}-te^t,\\
p'''(t)&=\big(8e^t-t-6\big)e^t>0.
\end{aligned}$$

Hence, $p''(t)$ increases in $(0, \infty)$. Since $p''(0) = 3 > 0$, we have $p''(t) > 0$ and $p'(t)$ is increasing. Because $p'(0) = 0$, it follows that $p'(t) > 0$ in $(0, \infty)$, and then $p(t)$ is increasing. From $p(0) = 0$, it is deduced that $p(t) > 0$ and $q'(t) > 0$ in $(0, \infty)$, then $q(t)$ increases. As a result of $q(0) = 0$, we obtain $q(t) > 0$ in $(0, \infty)$. Therefore, we have $\phi(x) > 0$ in $(0, \infty)$, and then for all nonnegative integer i, we have $(-1)^i \phi^{(i)}(x) > 0$ in $(0, \infty)$. This means that the function $\psi''(x) + g_2(x) + h_2(x)$ is strictly completely monotonic in $(0, \infty)$.

Thus the function $(-1)^n x^{n+1} [\ln F(x)]^{(n)}$ is increasing in $x \in (0, \infty)$. Since

$$\lim_{x \to 0} \left\{ (-1)^n x^{n+1} [\ln F(x)]^{(n)} \right\} = 0,$$

we have $(-1)^n x^{n+1} [\ln F(x)]^{(n)} > 0$, then $(-1)^n [\ln F(x)]^{(n)} > 0$ for $n \geq 2$ in $(0, \infty)$. Since $[\ln F(x)]'' > 0$, the function $[\ln F(x)]'$ is increasing. It is not difficult to obtain $\lim_{x \to \infty} [\ln F(x)]' = 0$, so $[\ln F(x)]' < 0$ and $\ln F(x)$ is decreasing in $(0, \infty)$. In conclusion, the function $\ln F(x)$ is strictly completely monotonic in $(0, \infty)$. The proof is complete. □

3. An open problem

We would like to pose the following open problem:

OPEN PROBLEM. *Under what conditions on a, b and c is the function $F(x)$ defined by* (11) *completely monotonic, or logarithmically completely monotonic, or a Stieltjes transform on* $(0, \infty)$?

In some subsequent papers, we will discuss the above open problem and publish its solutions.

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