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PRODUCTS OF IDEMPOTENTS IN ALGEBRAIC MONOIDS

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Abstract

Let M be a reductive algebraic monoid with zero and unit group G. We obtain a description of the submonoid generated by the idempotents of M. In particular, we find necessary and sufficient conditions for $M\backslash G$ to be idempotent generated.

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Introduction

Let S be a semigroup. It has long been recognized that an important tool in understanding the structure of S is to consider the semigroup $\langle E(S)\rangle$ generated by the idempotent set $E(S)$ of S, see, for example, [3, 4, 5, 6]. In particular for a regular semigroup S, Hall [5] constructs from the semigroup $\langle E(S)\rangle$ a universal fundamental semigroup T_E containing the fundamental image S/μ of S.

Our interest is in linear algebraic monoids M with unit group G. In earlier papers [8, 10], we have found sufficient conditions for $M\backslash G$ to be idempotent generated. In this paper we find complete answers. We begin by studying $\langle E(M)\rangle$ for any irreducible algebraic monoid M. For each regular $\mathscr{J}$-class J of M we associate a normal subgroup G_J of G so that for any idempotent e in J, $J \cap \langle E(M)\rangle = G_J e G_J$. When M is a regular irreducible monoid with zero (equivalently G is reductive), we find necessary and sufficient conditions for J to be idempotent generated. The conditions are of a discrete nature, associated with the Weyl group of G.

1. Preliminaries

Let M be a strongly π-regular monoid. This means that some power of each element lies in a subgroup. If $X \subseteq M$, let $E(X)$ denote the set of idempotents in X. Let $\mathscr{J} = \mathscr{D}, \mathscr{R}, \mathscr{L}, \mathscr{H}$ denote the usual Green's relations on M. A $\mathscr{J}$-class J is *regular* if $E(J) \neq \emptyset$. M is regular if all $\mathscr{J}$-classes are regular. Let $\mathscr{U}(M)$ denote the partially ordered set of regular $\mathscr{J}$-classes of M. If $J \in \mathscr{U}(M)$, then $J^0 = J \cup \{0\}$ with

$$a \circ b = \begin{cases} ab & \text{if } ab \in J; \\ 0 & \text{otherwise} \end{cases}$$

is a completely 0-simple semigroup. We are interested in the products of idempotents. It has been noted by Hall [5, Lemma 1] that the property of being a product of idempotents is local.

PROPOSITION 1.1. *If $J \in \mathscr{U}(M)$, then $J \cap \langle E(M) \rangle \subseteq \langle E(J) \rangle$.*

COROLLARY 1.2. *$\langle E(M) \rangle$ is a strongly π-regular monoid.*

PROOF. Let $a \in \langle E(M) \rangle$. Then $a^m \mathscr{H} a^{2m}$ for some positive integer m. If J is the $\mathscr{J}$-class of a^m, then $a^m \in J \cap \langle E(M) \rangle \subseteq \langle E(J) \rangle$. Since J^0 is completely 0-simple, $a^m \mathscr{H} a^{2m}$ in $\langle E(J^0) \rangle$ and hence in $\langle E(M) \rangle$. □

Let $J \in \mathscr{U}(M)$. We will say that J is *idempotent generated* if $J \subseteq \langle E(M) \rangle$. In such a case J is a regular $\mathscr{J}$-class of $\langle E(M) \rangle$. If $e \in E(J)$ and if H is the $\mathscr{H}$-class of e (unit group of eMe), then J is idempotent generated if and only if $H \subseteq \langle E(M) \rangle$ and any two idempotents in J are $\mathscr{J}$-related in $\langle E(M) \rangle$. The unit group of M, if non-trivial, is never idempotent generated. Both the full transformation semigroup of a finite set and the multiplicative monoid of $n \times n$ matrices over a field have the property that the non-units are products of idempotents, see, for example, [3, 6].

2. Algebraic monoids

Let M be an algebraic monoid over an algebraically closed field k. This means that M is an affine variety with the product map being a morphism. By [9, Theorem 3.18], M is a strongly π-regular monoid. Let M^c denote the irreducible component of 1. We will assume that M is an irreducible monoid, that is, $M = M^c$. By [9, Theorem 5.10], $\mathscr{U}(M)$ is a finite lattice. Let G denote the unit group of M. For $e \in E(M)$,

$$G_e^r = \{x \in G \mid xe = e\}, \qquad G_e^l = \{x \in G \mid ex = e\},$$
$$G_e = \{x \in G \mid ex = e = xe\}, \qquad C_G(e) = \{x \in G \mid ex = xe\}$$

are closed subgroups of G and $C_G(e)$ is also connected. For $J \in \mathscr{U}(M)$, $e \in E(J)$, let

$$G_J = \{x \in G \mid ex \in \langle E(M)\rangle\}. \tag{2.1}$$

THEOREM 2.1. (i) *G_J is a closed normal subgroup of G and is independent of the choice of the idempotent e.*

(ii) *If $e \in E(J)$, then $G_J = \langle G_e^r, G_e^l\rangle$ and is also equal to the normal subgroup of G generated by G_e.*

(iii) *$J \cap \langle E(M)\rangle = J \cap \overline{G}_J = G_J e G_J$ is a closed irreducible subset of J for all $e \in E(J)$.*

(iv) *J is idempotent generated if and only if $G = G_J$.*

(v) *If $J_1, J_2 \in \mathscr{U}(M)$ with $J_1 \leq J_2$, then $G_{J_2} \subseteq G_{J_1}$.*

PROOF. Let $e \in E(J)$, $x \in G_J$. If $e\mathscr{L}e_1 \in E(J)$, then

$$e_1 x = e_1 ex \in e_1\langle E(J)\rangle \subseteq \langle E(J)\rangle. \tag{2.2}$$

If $e\mathscr{R}e_1 \in E(J)$, then

$$e_1 x = ee_1 x = (ex)(x^{-1}e_1 x) \in e_1\langle E(J)\rangle(x^{-1}e_1 x) \subseteq \langle E(J)\rangle. \tag{2.3}$$

If $f \in E(J)$, then by [9, Theorem 5.9],

$$e\mathscr{L}e_1\mathscr{R}e_2\mathscr{L}f \quad \text{for some } e_1, e_2 \in E(J). \tag{2.4}$$

By (2.2)–(2.4), we see that

$$E(J)G_J \subseteq \langle E(J)\rangle. \tag{2.5}$$

It follows that G_J is independent of the choice of the idempotent e. If $g \in G$, then by (2.5),

$$eg^{-1}xg = g^{-1}(geg^{-1} \cdot x)g \subseteq g^{-1}\langle E(J)\rangle g = \langle E(J)\rangle.$$

Hence $g^{-1}xg \in G_J$. Thus

$$g^{-1}G_J g \subseteq G_J \quad \text{for all } g \in G. \tag{2.6}$$

Let $a, b \in G_J$. Then $ea, eb \in \langle E(J)\rangle$. So

$$eab = (eb)b^{-1}(ea)b \in \langle E(J)\rangle b^{-1}\langle E(J)\rangle b = \langle E(J)\rangle^2 = \langle E(J)\rangle.$$

Hence $ab \in G_J$. Thus

$$G_J G_J \subseteq G_J \tag{2.7}$$

Now $E(J)$ is a closed irreducible subset of M by [9, Proposition 5.8]. Hence we have an ascending chain of closed irreducible sets $E(J) \subseteq \overline{E(J)^2} \subseteq \overline{E(J)^3} \subseteq \cdots$. Hence for some positive integer i,

$$S = \overline{\langle E(J) \rangle} = \overline{E(J)^i} = \overline{E(J)^{i+1}} = \cdots \tag{2.8}$$

is an irreducible algebraic semigroup. By (2.4), $J \cap S$ is the $\mathscr{J}$-class of e in S. By [9, Lemma 3.27], $X = \{a \in M \mid e \notin MaM\}$ is closed. Hence $S \cap J = SeS \backslash X$ is irreducible. Let H denote the $\mathscr{H}$-class of e in S. Since H is open in eSe, we see that there exists a non-empty open subset U of H such that $U \subseteq eE(J)^i e$. Since H is a connected group, $U^2 = H$. Hence $H \subseteq \langle E(J) \rangle$. By (2.4), $J \cap S \subseteq \langle E(J) \rangle$. Thus

$$J \cap S = J \cap \langle E(J) \rangle \tag{2.9}$$

is closed in J. It follows that G_J is closed in G. Hence by (2.6) and (2.7), G_J is a closed normal subgroup of G, proving (i).

If $e \in E(J)$, then $G_e \subseteq G_J$ and hence by [9, Theorem 6.11], $e \in \overline{G_e} \subseteq \overline{G_J}$. Thus $E(J) \subseteq \overline{G_J}$. So by (2.4), $J \cap \overline{G_J}$ is the $\mathscr{J}$-class of $\overline{G_J}$. Hence by [7, Theorem 1],

$$J \cap \overline{G_J} = G_J e G_J. \tag{2.10}$$

If $a, b \in G_J$, then by (2.5) $aeb \in aea^{-1} \cdot ab \in \langle E(J) \rangle$. So,

$$G_J e G_J \subseteq \langle E(J) \rangle \subseteq \overline{G_J}. \tag{2.11}$$

By (2.9)–(2.11) we see that (iii) and (iv) are valid.

Clearly $G_e^r, G_e^l \subseteq G_J$. So $\langle G_e^r, G_e^l \rangle \subseteq G_J$. Conversely let $x \in G_J$. Then $ex = e_1 \cdots e_m$ for some $e_1, \cdots, e_m \in E(J)$. Then $ex = ee_1 \cdots e_m$. By [9, Corollary 6.8], $e_1 = yey^{-1}$ for some $y \in G$. Since $ee_1 \in J$, $eye \mathscr{H} e$. By [9, Theorem 6.33], $y \in G_e^l C_G(e) G_e^r = G_e^l G_e^r C_G(e)$. Thus we may assume without loss of generality that $y \in G_e^l G_e^r$. So $eye = e$. Hence $ee_1 = ey^{-1}$. Then

$$ee_1 e_2 = ey^{-1} e_2 = ey^{-1} e_2 y y^{-1}.$$

As above, $e \cdot y^{-1} e_2 y = ez^{-1}$ for some $z \in G_e^l G_e^r$. So $ee_1 e_2 = ez^{-1} y^{-1}$. Continuing we see that there exists $u \in \langle G_e^r, G_e^l \rangle$ such that $ex = ee_1 \cdots e_m = eu$. So $exu^{-1} = e$ and $xu^{-1} \in G_e^l$. It follows that $x \in \langle G_e^l, G_e^r \rangle$. Thus $G_J = \langle G_e^l, G_e^r \rangle$.

Let N denote the normal subgroup of G generated by G_e. Then $N \subseteq G_J$. Now $e \in \overline{G_e} \subseteq \overline{N}$. Since all idempotents in J are conjugate and $N \lhd G$, we see that

$E(J) \subseteq \overline{N}$. By [7], $E(J) \subseteq \overline{N}^c$. Let $a \in G_e^r$. Then $ae = e$. Let $f = ea \in E(J)$. Then $e\mathscr{R}f$. So by [9, Corollary 6.8], $f = eb$ for some $b \in N^c$ with $be = e$. So $ab^{-1} \in G_e \subseteq N$. So $a \in N$. Hence $G_e^r \subseteq N$. Similarly $G_e^l \subseteq N$. Hence $\langle G_e^r, G_e^l \rangle \subseteq N$. Thus $N = G_J$, proving (ii).

Let $J_1, J_2 \in \mathscr{U}(M)$, $J_1 \leq J_2$. Then there exists $e_1 \in E(J_1)$, $e_2 \in E(J_2)$ with $e_1 \leq e_2$. Let $a \in G_{J_2}$. Then $e_2 a \in \langle E(M) \rangle$. So

$$e_1 a = e_1 e_2 a \in e_1 \langle E(M) \rangle \subseteq \langle E(M) \rangle.$$

Hence $a \in G_{J_1}$. Thus $G_{J_2} \subseteq G_{J_1}$. This proves (v), completing the proof. □

COROLLARY 2.2. *If M is a regular irreducible algebraic monoid, then $\langle E(M) \rangle$ is closed.*

PROOF. Let $J, J' \in \mathscr{U}(M)$, $J \geq J'$. Then by Theorem 2.1,

$$J' \cap \overline{G}_J \subseteq J' \cap \overline{G}_{J'} \subseteq \langle E(M) \rangle. \tag{2.12}$$

Choose $e_J \in E(J)$, $J \in \mathscr{U}(M)$. Then by (2.12), $\overline{G_J e G_J} \subseteq \langle E(M) \rangle$. So by Theorem 2.1, $\langle E(M) \rangle = \bigcup_{J \in \mathscr{U}(M)} \overline{G_J e_J G_J}$ is closed. □

If M is not irreducible then $\langle E(M) \rangle$ need not be closed.

EXAMPLE 1. Let J consist of all matrices of the form

$$\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} a & a \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} a & 0 \\ a & 0 \end{pmatrix}, \quad \begin{pmatrix} a & a \\ a & a \end{pmatrix},$$

where $a \in \mathbb{C}$, $a \neq 0$. Let

$$M = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cup J \cup \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

Then M is a non-irreducible, regular algebraic monoid with $J \in \mathscr{U}(M)$ and

$$E(J) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, 1/2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}.$$

So

$$\langle E(M) \rangle = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \cup \bigcup_{n \in \mathbb{Z}} 2^n E(J)$$

is not closed (in the Zariski topology).

The following is extracted from the proof of [9, Theorem 6.33].

LEMMA 2.3. *Let $x \in M$ and $e \in E(M)$. If $exe = e$, then $x \in G_e^l G_e^r$. If $exe\mathscr{H}e$, then $x \in G_e^l G_e^r C_G(e) = G_e^l C_G(e) G_e^r$.*

PROOF. Suppose $exe = e$. Then $e\mathscr{R}ex \in E(M)$, so $ex = ey = y^{-1}ey$ for some $y \in G$, by [9, Corollary 6.8]. Hence $exy^{-1} = e$, so $xy^{-1} \in G_e^l$. Also $ye = yexe = yy^{-1}eye = eye = exe = e$, so $y \in G_e^r$, giving $x = (xy^{-1})y \in G_e^l G_e^r$. Now suppose $exe\mathscr{H}e$. By [9, Theorem 6.16 (iii)], $e = exec = exce$ for some $c \in C_G(e)$. By the previous part, $xc \in G_e^l G_e^r$, so $x \in G_e^l G_e^r C_G(e)$, and the lemma is proved. □

If $E(J)$ is a semigroup, then it is a rectangular band and hence [2] J is a direct product of $E(J)$ and a group. J is then called a *rectangular group*. The following generalizes a result of Renner [13, Theorem 2] concerning completely regular algebraic monoids with solvable unit groups.

COROLLARY 2.4. *Let $e \in E(J)$. Then J is a rectangular group if and only if $G_e^r G_e^l = G_e^l G_e^r$.*

PROOF. Suppose J is a rectangular group. Let $a \in G_e^r$, $b \in G_e^l$. Let $e_1 = ea$, $e_2 = be \in E(J)$. So $eabe = a_1 e_2 = e$. By Lemma 2.3, $ab \in G_e^l G_e^r$. So $G_e^r G_e^l \subseteq G_e^l G_e^r$. Taking inverses we see that $G_e^r G_e^l = G_e^l G_e^r$.

Conversely suppose that $G_e^r G_e^l = G_e^l G_e^r$. Since all idempotents in J are conjugate, $G_f^l G_f^r = G_f^r G_f^l$ for all $f \in E(J)$. By [9, Theorem 5.9] there exist $e_1, e_2 \in E(J)$ such that $e\mathscr{R}e_1\mathscr{L}e_2\mathscr{R}f$. By [9, Corollary 6.8] $e = e_1 x$, $e_2 = ye_1$ for some $x \in G_{e_1}^r$, $y \in G_{e_1}^l$. So $xy \in G_{e_1}^r G_{e_1}^l = G_{e_1}^l G_{e_1}^r$. So $e_1 xye_1 = e_1$. Hence $e_2 e = ye_1 x \in E(J)$. The same argument shows that $ee_2 \in E(J)$. So $ee_2 = e_1$. Similarly, $e_1 f \in E(J)$. So $ef = ee_2 f = e_1 f \in E(J)$. Hence J is a rectangular group. □

REMARK. For the monoid of all triangular matrices, Bauer [1] has shown that a regular $\mathscr{J}$-class is a rectangular group if and only if the diagonal idempotent in it has the property that all the 1's are together.

COROLLARY 2.5. *Let $J_1, J_2 \in \mathscr{U}(M)$. If J_1 and J_2 are rectangular groups, then so is $J_1 \wedge J_2$.*

PROOF. Let $J = J_1 \wedge J_2$. Let $e \in E(J)$. Then by [9, Theorem 6.7, Corollary 6.10], there exist $e_1 \in E(J_1)$, $e_2 \in E(J_2)$ such that $e = e_1 e_2 = e_2 e_1$. Let $x \in G$. Then $e_1 x e_1 \in J_1$. By Lemma 2.3, $x \in G_{e_1}^l C_G(e_1) G_{e_1}^r$. So $x = abc$ for some $a \in G_{e_1}^l$, $b \in C_G(e_1)$, $c \in G_{e_1}^r$. So

$$\begin{aligned} exex^{-1}e &= eabcec^{-1}b^{-1}a^{-1}e \\ &= e_2 e_1 abce_1 e_2 c^{-1} b^{-1} a^{-1} e = e_2 e_1 b e_1 e_2 c^{-1} b^{-1} a^{-1} e. \end{aligned}$$

Now $c^{-1}b^{-1}a^{-1}b \in G^r_{e_1}b^{-1}G^l_{e_1}b = G^r_{e_1}G^l_{e_1} = G^l_{e_1}G^r_{e_1}$. So $c^{-1}b^{-1}a^{-1}b = a'c'$ for some $a' \in G^l_{e_1}$, $c' \in G^r_{e_1}$. So

$$\begin{aligned} exex^{-1}e &= e_2e_1be_1e_2c^{-1}b^{-1}a^{-1}e_1e_2 = e_2e_1be_1e_2a'c'b^{-1}e_1e_2 \\ &= e_2e_1be_2e_1a'c'e_1b^{-1}e_2 = e_1e_2be_2e_1b^{-1}e_2 \\ &= e_1e_2be_2b^{-1}e_1e_2 = e_1e_2be_2b^{-1}e_2e_1. \end{aligned}$$

Now $e_2be_2 \mathscr{J} e_2$ and hence by Lemma 2.3, $b \in G^l_{e_2}C_G(e_2)G^r_{e_2}$. So $b = vwu$ for some $v \in G^l_{e_2}$, $w \in C_G(e_2)$, $u \in G^r_{e_2}$. So

$$e_2be_2b^{-1}e_2 = e_2vwue_2u^{-1}w^{-1}v^{-1}e_2 = we_2u^{-1}w^{-1}v^{-1}e_2.$$

Now $u^{-1}w^{-1}v^{-1}w \in G^r_{e_2}w^{-1}G^l_{e_2}w = G^r_{e_2}G^l_{e_2} = G^l_{e_2}G^r_{e_2}$. So $u^{-1}w^{-1}v^{-1}w = v'u'$ for some $v' \in G^l_{e_2}$, $u' \in G^r_{e_2}$. So

$$e_2be_2b^{-1}e_2 = we_2v'u'w^{-1}e_2 = we_2v'u'e_2w^{-1} = we_2w^{-1} = e_2.$$

Hence $exex^{-1}e = e_1e_2be_2b^{-1}e_2e_1 = e_1e_2e_1 = e$. Since all idempotents in J are conjugate, we see that $E(J)$ is a semigroup. Hence J is a rectangular group. □

3. Reductive monoids

We will assume in this section that M is a regular, irreducible algebraic monoid with zero. Equivalently the unit group G of M is reductive. Then the commutator subgroup (G, G) is semisimple and $G = (G, G)Z$, where $Z = Z(G)$ is the center of G. If $\dim Z = 1$, we say that M is a *semisimple monoid*. Now by [9, Theorem 6.20], all maximal chains in $\mathscr{U}(M)$ have the same length. This gives rise to a rank function in $\mathscr{U}(M)$ and hence on M. By [9, Theorem 7.9], the fundamental image M/μ is obtained by factoring the maximal subgroups of M by their centers. By [9, Chapter 9], there is an idempotent cross-section $e_J (J \in \mathscr{U}(M))$ such that for $J_1, J_2 \in \mathscr{U}(M)$,

$$J_1 \le J_2 \quad \text{if and only if} \quad e_{J_1} \le e_{J_2}.$$

Then $\Lambda = \{e_J \mid J \in \mathscr{U}(M)\}$ is called a *cross-section lattice* of M and is unique up to conjugacy. By [9, Chapter 9] $B = \{g \in G \mid ge = ege \text{ for all } e \in \Lambda\}$ is a Borel subgroup of G containing the maximal torus

$$T = \{g \in G | ge = eg \text{ for all } e \in \Lambda\}.$$

Let $W = N_G(T)/T$ denote the Weyl group of G with generating set S of simple reflections. The subgroups containing B are called parabolic subgroups and are of the form $P_I = BW_IB$, $I \subseteq S$. Here W_I is the subgroup W generated by I. Let U, U_I denote respectively the unipotent radicals of B and P_I, $I \subseteq S$. If $s \in S$, $I = \{s\}$, then denote U_I by X_s. Then $X_s \cong k$ and is called a root subgroup. Let $J \in \mathscr{U}(M)$. As in [12], the *type* of J is defined as $\lambda(J) = \{s \in S \mid se_J = e_J s\}$. Let

$$\lambda^*(J) = \bigcap_{J' \geq J} \lambda(J') \quad \text{and} \quad \lambda_*(J) = \bigcap_{J' \leq J} \lambda(J').$$

Then $W_{\lambda(J)} = W_{\lambda^*(J)} \times W_{\lambda_*(J)}$. Now S has the structure of a Coxeter graph where for $s, t \in S$, s and t are adjacent if $st \neq ts$. Let S_J denote the union of components of S not contained in $\lambda^*(J)$.

THEOREM 3.1. *If $J \in \mathscr{U}(M)$, then $W(G_J^c) = W_{S_J}$.*

PROOF. Let $e = e_J$, $I = \lambda(J)$. Let S' be a component of S. First suppose that $S' \subseteq S_J$. Then $S' \not\subseteq \lambda^*(J)$. So there exists $s \in S'$ such that $s \notin \lambda^*(J)$. Suppose $s \notin I$. Then $X_s \subseteq U_I$ and hence $X_s e = \{e\}$. So $X_s \subseteq G_e^r \subseteq G_J$. Thus $X_s \subseteq G_J^c$. Since $G_J^c \lhd G$, it is a reductive group. So $s \in W(G_J^c)$. Since $G_J^c \lhd G$, $S' \subseteq W(G_J^c)$. Next suppose that $s \in \lambda(J)$. Since $s \notin \lambda^*(J)$, $s \in \lambda_*(J)$. So $se = e = es$. Since G_e^c is a reductive group, $X_s \subseteq G_e^c \subseteq G_J^c$. So again $s \in W(G_J^c)$ and $S' \subseteq W(G_J^c)$.

Assume conversely that $S' \subseteq W(G_J^c)$. We claim that $S' \subseteq S_J$. Otherwise, $S' \subseteq \lambda^*(J)$. There exists a closed connected normal subgroup G_1 of G contained in G_J^c such that $W(G_1) = W_{S'}$. Since G is a reductive group, there exists a closed connected normal subgroup G_2 of G such that $G = G_1G_2$ and G_2 centralizes G_1. Since $S' \subseteq \lambda(J)$ and $W(G_1) = W_{S'}$, we see that $G_1 \subseteq C_G(e)$. So if $f \in E(J)$, then $f = xex^{-1}$ for some $x \in G_2$. So G_1 centralizes f. Hence G_1 centralizes $\langle E(J) \rangle$. Since $G_1 \subseteq G_J$, $eG_1 \subseteq \langle E(J) \rangle$. So eG_1 is commutative and $W(eG_1) = 1$. So $S' \subseteq \lambda_*(J)$, a contradiction. Thus $S' \subseteq S_J$, completing the proof. □

COROLLARY 3.2. *Let $J \in \mathscr{U}(M)$. Then the image of J in M/μ is idempotent generated if and only if no component of S is contained in $\lambda^*(J)$.*

COROLLARY 3.3. *Let $J \in \mathscr{U}(M)$, $e = e_J$. Then J is idempotent generated if and only if*

(i) *no component of S is contained in $\lambda^*(J)$; and*
(ii) $G = (G, G)T_e$.

PROOF. Suppose first that J is idempotent generated. Then (i) is true by Theorem 3.1. Let $H = (G, G)T_e$. Then $H^c = (G, G)T_e^c$ is a reductive group and $e \in \overline{H^c}$. Now $Z \subseteq T$ and $G = (G, G)Z$. Let $f \in E(J)$. Then f is conjugate to e and hence there exists $x \in (G, G)$ such that $f = x^{-1}ex$. Hence $f \in \overline{H^c}$. Thus $E(J) \subseteq \overline{H^c}$. Let

$z \in Z$. Then $ez \in J \subseteq \langle E(J)\rangle \subseteq \overline{H^c}$. So there exists $t \in H^c \cap T$ such that $ez = et$. So $zt^{-1} \in T_e \subseteq H$ and hence $z \in H$. Thus $Z \subseteq H$. Since $G = (G, G)Z$, we see that $G = H$.

Assume conversely that (i), (ii) are valid. Then by Theorem 3.1, $W(G_J^c) = W$. Hence $(G, G) \subseteq G_J$. Since $T_e \subseteq G_J$, $G = G_J$. By Theorem 2.1, J is idempotent generated. This completes the proof. □

Let $J \in \mathscr{U}(M)$. Then by Theorem 2.1, the $\mathscr{J}$-class $J \cap \overline{G_J^c} = J \cap \langle E(M)\rangle$ of $\overline{G_J^c}$ is idempotent generated. By Theorem 3.1, (G_J^c, G_J^c) is the unique closed connected normal subgroup of (G, G) with Weyl group W_{S_J}. We have, by Corollary 3.3,

COROLLARY 3.4. *Let $J \in \mathscr{U}(M)$, $e = e_J$. Then $J \cap \langle E(M)\rangle = (G_J^c, G_J^c)e(G_J^c, G_J^c)$.*

COROLLARY 3.5. *Let $J \in \mathscr{U}(M)$. If J is idempotent generated then the dimension of the center of G is at most equal to the corank of J.*

PROOF. Let $e = e_J$. Then $rk J = \dim eT$ and $\dim T_e$ is the corank of J. By Corollary 3.3, $G = (G, G)T_e$. Since $G = (G, G)Z$, we see that $\dim Z \leq \dim T_e$. □

Following [11], we will say that a nilpotent element a is *standard* if $a^m \neq 0$, where m is the rank of a. We have shown in [11] that the number of conjugacy classes of regular nilpotent elements is finite. In the monoid of all $n \times n$ matrices, a standard nilpotent element is one with almost one non-zero Jordan block.

COROLLARY 3.6. *Let $J \in \mathscr{U}(M)$. If J has a standard nilpotent element, then it is idempotent generated.*

PROOF. Let $e = e_J$. By [11], there exists $x \in W$ such that ex is a standard nilpotent element. Now $T_e^c \subseteq G_J$ and by Theorem 2.1, $E(J) \subseteq \overline{G_J^c}$. We also have the following maximal chain of $E(\overline{T_e^c})$ contained in $\overline{G_J^c}$:

$$e > e \cdot xex^{-1} > exex^{-1}x^2ex^{-2} > \cdots .$$

So $\overline{G_J^c}$ contains a maximal chain of $E(\overline{T})$. Hence $T \subseteq G_J$. Since $G_J \lhd G$, $G = G_J$. Thus by Theorem 2.1, J is idempotent generated. □

We are now able to solve [8, Problem 2.10].

THEOREM 3.7. *$M \backslash G$ is idempotent generated if and only if*

(i) *For any maximal $\mathscr{J}$-class $J \neq G$, no component of S is contained in $\lambda(J)$; and*

(ii) *M is semisimple.*

PROOF. First suppose that $M\backslash G$ is idempotent generated. Then (i) follows by Corollary 3.3 and (ii) follows by Corollary 3.5. Assume conversely that (i) and (ii) are true. Let J be a maximal $\mathscr{J}$-class in $M\backslash G$, $e = e_J$. By Theorem 3.1, $(G, G) \subseteq G_J$. By (ii), $\dim G = 1 + \dim(G, G)$. Now $T_e \subseteq G_J$. Since (G, G) is closed in M and $e \in \overline{T_e^c}$, we see that $T_e^c \not\subseteq (G, G)$. So $G = (G, G)T_e$ and $G = G_J$. By Theorem 2.1 (iv), J is idempotent generated. So by Theorem 2.1 (v), $M\backslash G$ is idempotent generated. □

EXAMPLE 2. Let $G = \{\alpha A \oplus \beta A \mid A \in SL_2(k),\ \alpha, \beta \in k^*\}$ and let M denote the Zariski closure of G in $M_4(k)$. Then $S = \{(12)\}$. The non-trivial elements of the cross-section lattice Λ are given by

$$e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$e_1' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad e_2' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Let the corresponding $\mathscr{J}$-classes be J_1, J, J_2, J_1', J_2'. Then $S \subseteq \lambda^*(J_1)$, $S \subseteq \lambda^*(J_2)$. So by Corollary 3.2, the images of J_1, J_2 are not idempotent generated in M/μ. By Corollary 3.6, J_1', J_2' are idempotent generated in M. Now $S \not\subseteq \lambda^*(J)$ and so by Corollary 3.2, the image of J is idempotent generated in M/μ. However, J is not idempotent generated in M by Corollary 3.5. In fact,

$$J \cap \langle E(M) \rangle = \{A \oplus A \in M \mid rkA = 1\}$$

while $J = \{A \oplus B \in M \mid rkA = 1,\ B = \alpha A \text{ for some } \alpha \in k^*\}$.

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