J. Aust. Math. Soc. **80** (2006), 173–178

GROUP ALGEBRAS WITH AN ENGEL GROUP OF UNITS

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(Received 2 February 2004; revised 11 November 2004)

Communicated by E. A. O'Brien

Abstract

Let $\mathbb{F}$ be a field of characteristic p and G a group containing at least one element of order p. It is proved that the group of units of the group algebra $\mathbb{F}G$ is a bounded Engel group if and only if FG is a bounded Engel algebra, and that this is the case if and only if G is nilpotent and has a normal subgroup H such that both the factor group G/H and the commutator subgroup H' are finite p-groups.

2000 *Mathematics subject classification*: primary 16S34, 16U60; secondary: 20C05.
Keywords and phrases: Group algebra, group of units, n-Engel groups, Engel groups.

1. Introduction

For elements a, b, c of a group G, let (a, b) denote the group commutator $a^{-1}b^{-1}ab$ and write $(a, b, c) = ((a, b), c)$. Further, let $(a, b, 0) = a$, and for positive integers n define (a, b, n) inductively by $(a, b, n) = ((a, b, n-1), b)$.

Recall that a group G is said to be an *Engel group* if for each pair of elements a and b of G there exists an n such that $(a, b, n) = 1$. If n can be chosen to be independent of the pair (a, b), then G is said to be a *bounded Engel group*; for any such choice of n, G is then also called an *n-Engel group*. Similar terminology applies to associative rings or algebras in terms of the ring commutator $[a, b] = ab - ba$.

We shall study group algebras over fields F of prime characteristic, to see whether the group of units is an Engel or bounded Engel group. The difficulties that arise have to do with the fact that we know very little about non-solvable Engel groups. For this reason, until now it has only been possible to obtain significant results under some additional hypothesis on the group G. Non-modular group algebras $\mathbb{F}G$ whose group

The research supported by OTKA No. T037202.

of units $U(\mathbb{F}G)$ is an Engel group were characterized by Bovdi and Khripta [1, 2] as follows.

THEOREM 1.1. *Suppose that the group G contains no elements of order equal to the characteristic of the field $\mathbb{F}$. If $U(\mathbb{F}G)$ is an Engel group, then G is an Engel group with one of the following conditions*:

(1) *the set $t(G)$ of torsion elements of G is a central subgroup of G.*

(2) *$\mathbb{F}$ is a prime field of characteristic $p = 2^t - 1$, $t(G)$ is an abelian group of exponent dividing $p^2 - 1$, and $gag^{-1} = a^p$ for all $a \in t(G)$ and $g \in G$ outside the centralizer of $t(G)$.*

Conversely, if G is an Engel group satisfying one of these conditions and $G/t(G)$ is a u.p.-group, then $U(\mathbb{F}G)$ is an Engel group.

Recall that a group G is called a *u.p.-group* (unique product group), if for any two nonempty finite subsets A and B of G there exists at least one element $v \in G$ which can be written uniquely as $v = ab$ with $a \in A$, $b \in B$. In [2], the second part of the Theorem 1.1 was proved under the assumption that $G/t(G)$ admits a right order, but the proof is easily adapted to the more general case of u.p.-groups. Indeed, Strojnowski [11] proved that any u.p.-group has the property that for any two nonempty finite subsets A and B of G with $|A| + |B| > 2$ there exist two distinct elements $v_1, v_2 \in G$ which can be written uniquely as $v_i = a_i b_i$ with $a_i \in A$ and $b_i \in B$. Using this, one can prove [2, Lemma 1.2] for the case when $G/t(G)$ is a u.p.-group, and repeat the rest of the proof of Theorem 1.1.

In [1, 2], it was also shown that in this theorem one may replace 'Engel group' by 'bounded Engel group' throughout. The referee has kindly pointed out that there is a solvable bounded Engel version of Theorem 1 which goes back to Fisher, Parmenter, Sehgal [3].

In the modular case, the groups G for which the units of $\mathbb{F}G$ form a bounded Engel group were characterized in [1, 2] only under the assumption that G is solvable. We now extend those results to arbitrary groups, using recent results of Liu, Passman [6] and Giambruno, Sehgal, Valenti [4] on group algebras whose units satisfy group identities.

2. Engel or bounded Engel groups

Recall that all finite Engel groups are nilpotent (see [8, 12.3.4]), and all solvable Engel groups are locally nilpotent ([8, 12.3.3]). For bounded Engel groups, we shall also need Lemma 2.4 from [2].

Lemma 2.1. *Let G be a bounded Engel group satisfying one of the following conditions*:

(1) *G is an extension of a nilpotent group by a finitely generated nilpotent group.*
(2) *G is an extension of a finite group by a nilpotent group.*

Then G is a nilpotent group.

3. Group of units which are Engel groups or bounded Engel groups

For an element $v = \sum_{g\in G} a_g g$ (with all $a_g \in \mathbb{F}$), the *support* of v is defined as

$$\mathrm{Supp}(v) = \{g \in G \mid a_g \neq 0\}.$$

Lemma 3.1. *Suppose that $\mathbb{F}$ is a field of characteristic p, that $v \in \mathbb{F}G$ is a square-zero element centralized by the elements g, h of G, and that $1 \in \mathrm{Supp}(v)$. If $(1 + vg, h, q) = 1$ for some power q of p, then $(g, h^q) \in \mathrm{Supp}(v)$.*

Proof. An easy calculation shows that

$$(1 + vg, h, q) = 1 + v\sum_{i=0}^{q}(-1)^i\binom{q}{i}g^{h^{q-i}} = 1 + v\big(g^{h^q} - g\big),$$

because $\binom{q}{i} \equiv 0 \pmod p$ whenever $0 < i < q$. Thus if $(1 + vg, h, q) = 1$ then $v(g^{h^q} - g) = 0$, and then $1 \in \mathrm{Supp}(v)$ yields that $(g, h^q) \in \mathrm{Supp}(v)$. □

We still cannot offer a general criterion for deciding, in terms of G, whether $U(\mathbb{F}G)$ is an Engel group. For solvable G, such a criterion was given in [1, 2]. Here we give a new proof for that, and show that it also applies to certain groups which are not assumed solvable. The choice of these groups may seem arbitrary here, but will serve us well when we turn our attention to bounded Engel groups.

Theorem 3.2. *Let $\mathbb{F}$ be a field of characteristic p and G a group with a nontrivial p-Sylow subgroup P. Suppose either that G itself is solvable or that P is solvable, normal in G, and contains a nontrivial finite subgroup which is normal in G. Then $U(\mathbb{F}G)$ is an Engel group if and only if G is locally nilpotent and its commutator subgroup G' is a p-group; in fact, in that case $U(\mathbb{F}G)$ is not only Engel but even locally nilpotent.*

Proof. Given any finite subset of $U(\mathbb{F}G)$, the union of the supports of its elements is a finite subset of G: thus each finitely generated subgroup of $U(\mathbb{F}G)$ lies in $U(\mathbb{F}H)$ for some finitely generated subgroup H of G. Similarly, each element of G' lies in H'

for some such H. In both cases, H can be chosen so as to meet P nontrivially and to contain also any given finite normal subgroup of G. Thus we may restrict attention to finitely generated G.

Suppose first that $U(\mathbb{F}G)$ is an Engel group. Finitely generated solvable Engel groups are nilpotent and therefore have finite and normal Sylow subgroups: hence if G is solvable then the alternate hypothesis also applies. Under that alternate hypothesis, assume that N is a nontrivial finite normal subgroup of G contained in P, and put $v = \sum_{x\in N} x$. Evidently, this v is a central square-zero element in $\mathbb{F}G$. Consider two arbitrary elements g, h of G. Since $U(\mathbb{F}G)$ is an Engel group, $(1 + vg, h, q) = 1$ for some power q of p, and then $(g, h^q) \in \mathrm{Supp}(v) = N$ by Lemma 3.1. Using that $(g', h, h^q) = (g', h^q, h)$ for every element g' of G, a simple induction on k shows that $(g', h^q) \in N$ whenever $g' = (g, h, k)$ for some nonnegative integer k.

Let m be the smallest positive integer such that $(g, h, m) \in P$. For an argument by contradiction, assume that $m - 2 \geq 0$. Set $g' = (g, h, m - 2)$ and note that $(g', h^q) \in N \leq P$. In view of $(g', h, h) \in P$, we have $(g', h^q) \equiv (g', h)^q \pmod{P}$, and as G/P has no nontrivial p-element, it follows that $(g, h, m - 1) = (g', h) \in P$. This contradicts the minimal choice of m. Therefore, $m \leq 1$, G/P is abelian, and so G is solvable group whose commutator subgroup is a p-group. Being a finitely generated solvable Engel group, G is also nilpotent. For the converse, we may assume that G is a finitely generated nilpotent group with G' a p-group; then G' must be finite, and the result of Khripta [5] gives that $U(\mathbb{F}G)$ is nilpotent. □

We are now ready to give a completely general result about modular group algebras FG whose group of units is a bounded Engel group. We shall use recent results on group algebras with units satisfying a group identity, and the following result of Shalev [10]: *Let A be an associative algebra over a field of characteristic p satisfying the n-Engel condition. Then the group of units $U(A)$ is m-Engel for some m depending on n.*

THEOREM 3.3. *Let $\mathbb{F}$ be a field of characteristic p, and let G be a group having a nontrivial p-Sylow subgroup P. Then $U(\mathbb{F}G)$ is a bounded Engel group if and only if G is a nilpotent group with a normal subgroup H of p-power index such that the commutator subgroup of H is a finite p-group, and in this case $\mathbb{F}G$ is a bounded Engel algebra.*

PROOF. Consider first the case when P is finite and $U(FG)$ is an Engel group. Lemma 1.1 on page 379 of Plotkin's book [7] (page 307 in the English translation) says that if K is a nilpotent subgroup generated by a finite set E of right Engel elements of an arbitrary group G and if the normalizer $N_G(K)$ of K in G does not coincide with G, then some element of E must have a conjugate in $N_G(K)\setminus K$. This guarantees that a finite Sylow subgroup in an Engel group is always normal. Thus in our case P

is normal and so Theorem 3.2 yields that G/P is abelian. Lemma 2.1 then confirms that G is nilpotent, and of course the commutator subgroup of G is a finite p-group, because P is finite. Therefore $H = G$ satisfies the assertion of the theorem.

Now let G be an n-Engel group having an infinite p-Sylow subgroup P. Use that $U(\mathbb{F}G)$ satisfies a nontrivial group identity, namely $(x, y, n) = 1$. If G is not torsion, then we can apply [4, Theorem 5.5]: G has a normal subgroup L of finite index whose commutator subgroup L' is a finite p-group. If G is torsion, this follows by [6, Theorems 1.1–1.2]. In either case, G/L is a finite Engel group, so it is nilpotent. Part (ii) of Lemma 2.1 yields that L is nilpotent, and therefore by part (i) of the same result G is also nilpotent. In particular, the p-Sylow subgroup P is normal and some nontrivial element z of order p is central in G.

Since $U(\mathbb{F}G)$ is a bounded Engel group, it is q-Engel for some power q of p. Let g, h be arbitrary elements of G, and apply Lemma 3.1 with $v = 1 + z + z^2 + \cdots + z^{p-1}$: it tells us that $(g, h^q) \in \mathrm{Supp}(v)$, so (g, h^q) is central and $(g, h^q)^p = 1$. It follows that $(g, h^{pq}) = 1$ for all g in G, that is, h^{pq} lies in the centre, $\zeta(G)$, for all $h \in G$. The normal subgroup $H = L \cdot \zeta(G)$ then satisfies the conclusions of the theorem, for $|G/H|$ is a finite group of p-power exponent and the commutator subgroup of H is the finite p-group L'.

Conversely, if G is nilpotent and has a normal subgroup H such that G/H and H' are finite p-groups, then $\mathbb{F}G$ is a bounded Engel algebra by [9, Theorem V.6.1], and so $U(\mathbb{F}G)$ is a bounded Engel group by the theorem of Shalev [10]. □

Acknowledgement

The author would like to thank Prof. L. G. Kovács for valuable comments and suggestions for clarifying the exposition.

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