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GROWTH PROPERTIES AND SEQUENCES OF ZEROS OF ANALYTIC FUNCTIONS IN SPACES OF DIRICHLET TYPE

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Abstract

For $0 < p < \infty$, we let $\mathcal{D}^p_{p-1}$ denote the space of those functions f that are analytic in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy $\int_\Delta (1 - |z|)^{p-1} |f'(z)|^p \, dx \, dy < \infty$. The spaces $\mathcal{D}^p_{p-1}$ are closely related to Hardy spaces. We have, $\mathcal{D}^p_{p-1} \subset H^p$, if $0 < p \le 2$, and $H^p \subset \mathcal{D}^p_{p-1}$, if $2 \le p < \infty$. In this paper we obtain a number of results about the Taylor coefficients of $\mathcal{D}^p_{p-1}$-functions and sharp estimates on the growth of the integral means and the radial growth of these functions as well as information on their zero sets.

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1. Introduction and main results

We denote by Δ the unit disc $\{z \in \mathbb{C} : |z| < 1\}$. If f is a function which is analytic in Δ and $0 < r < 1$, we set

$$M_p(r, f) = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{it})|^p \, dt \right)^{1/p}, \quad 0 < p < \infty,$$
$$I_p(r, f) = M_p^p(r, f), \quad 0 < p < \infty,$$
$$M_\infty(r, f) = \sup_{|z|=r} |f(z)|.$$

For $0 < p \le \infty$, the *Hardy space* H^p consists of all analytic functions f in the disc for which $\|f\|_{H^p} \stackrel{\text{def}}{=} \sup_{0<r<1} M_p(r, f) < \infty$. We refer the reader to [10] and [13] for the theory of Hardy spaces.

If $0 < p < \infty$ and $\alpha > -1$, we let A^p_α denote the (standard) *weighted Bergman space*, that is, the set of analytic functions f in Δ such that

$$\int_\Delta (1-|z|)^\alpha |f(z)|^p \, dA(z) < \infty.$$

Here, $dA(z) = (1/\pi)\, dx\, dy$ denotes the normalized Lebesgue area measure in Δ. The standard unweighted Bergman space A^p_0 is simply denoted by A^p. We mention [11] and [17] as general references for the theory of Bergman spaces.

The space $\mathscr{D}^p_\alpha$ ($p > 0$, $\alpha > -1$) consists of all functions f which are analytic in Δ such that $f' \in A^p_\alpha$. The space $\mathscr{D}^2_0$ is the classical Dirichlet space $\mathscr{D}$. For other values of p and α the spaces $\mathscr{D}^p_\alpha$ have been extensively studied in a number papers such as [27, 28, 30, 33] for $p = 2$ and [4, 8, 34, 36] for other values of p. If $p < \alpha + 1$, it is well known that $\mathscr{D}^p_\alpha = A^p_{\alpha - p}$ with equivalence of norms (see [12, Theorem 6]). For $\alpha = p - 2$, the space $\mathscr{D}^p_\alpha$ is the Besov space B^p (compare to [3]).

The space $\mathscr{D}^p_\alpha$ is said to be a Dirichlet space if $p \ge \alpha + 1$. In this paper we shall be primarily interested in the 'limit case' $p = \alpha + 1$, that is, in the spaces $\mathscr{D}^p_{p-1}$, $0 < p < \infty$, which are closely related to Hardy spaces. Indeed, a classical result of Littlewood and Paley [19] (see also [20]) asserts that

$$H^p \subset \mathscr{D}^p_{p-1}, \quad 2 \le p < \infty. \tag{1}$$

On the other hand, we have

$$\mathscr{D}^p_{p-1} \subset H^p, \quad 0 < p \le 2, \tag{2}$$

(see [34, Lemma 1.4]). Notice that, in particular, we have $\mathscr{D}^2_1 = H^2$. However, we remark that if $p \neq 2$ then

$$H^p \neq \mathscr{D}^p_{p-1}. \tag{3}$$

This can be seen using the characterization of power series with Hadamard gaps which belong to the spaces $\mathscr{D}^p_{p-1}$.

Proposition A. *If f is an analytic function in Δ which is given by a power series with Hadamard gaps, $f(z) = \sum_{k=1}^\infty a_k z^{n_k}$ ($z \in \Delta$) with $n_{k+1} \ge \lambda n_k$ for all k ($\lambda > 1$), then, for every $p \in (0, \infty)$, $f \in \mathscr{D}^p_{p-1}$ if and only if $\sum_{k=1}^\infty |a_k|^p < \infty$.*

Since for Hadamard gap series as above we have, for $0 < p < \infty$, $f \in H^p$ if and only of $\sum_{k=1}^\infty |a_k|^2 < \infty$, we immediately deduce that $\mathscr{D}^p_{p-1} \neq H^p$ if $p \neq 2$. We remark that Proposition A follows from [7, Proposition 2.1]. In Section 2 we shall see that Proposition A can also be deduced from the following theorem which gives a condition on the Taylor coefficients of a function f, analytic in Δ, which implies that $f \in \mathscr{D}^p_{p-1}$.

THEOREM 1.1. *Let f be an analytic function in Δ, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(z \in \Delta)$.*

(i) *If $0 < p < \infty$ and*

$$\sum_{n=0}^{\infty} \left(\sum_{k \in I(n)} |a_k| \right)^p < \infty, \tag{4}$$

then $f \in \mathscr{D}^p_{p-1}$.

(ii) *If $0 < p \leq 2$ and*

$$\sum_{n=1}^{\infty} \left(\sum_{k \in I(n)} |a_k|^2 \right)^{p/2} < \infty, \tag{5}$$

then $f \in \mathscr{D}^p_{p-1}$.

Here and throughout the paper, for $n = 0, 1, \ldots, I(n)$ is the set of the integers k such that $2^n \leq k < 2^{n+1}$.

If $0 < p \leq 2$, then (4) implies (5). Hence, for $p \in (0, 2]$, (ii) is stronger than (i). We remark also that if $0 < p \leq 2$, then the condition $\sum_{n=0}^{\infty} |a_n|^p < \infty$ implies (5). Consequently, (ii) improves [34, Lemma 1.5].

In Theorem 1.2 we give a condition on the Taylor coefficients of an analytic function f which is necessary for its membership in $\mathscr{D}^p_{p-1}$ if $2 \leq p < \infty$.

THEOREM 1.2. *Let f be an analytic function in Δ, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(z \in \Delta)$. If $2 \leq p < \infty$ and $f \in \mathscr{D}^p_{p-1}$, then*

$$\sum_{n=1}^{\infty} \left(\sum_{k \in I(n)} |a_k|^2 \right)^{p/2} < \infty. \tag{6}$$

If $0 < p < 2$ then (3) can be seen in some other ways. Rudin proved in [29] that there exists a Blaschke product B which does not belong to $\mathscr{D}^1_0$ (see also [24]). Vinogradov [34] extended this result showing that for every $p \in (0, 2)$ there exist Blaschke products B which do not belong to $\mathscr{D}^p_{p-1}$. This clearly gives that $\mathscr{D}^p_{p-1} \neq H^p$ if $0 < p < 2$, a fact which can be also deduced from the results of [9] and [14]. In contrast with what happens for $0 < p < 2$, it is not easy to give examples of functions $f \in \mathscr{D}^p_{p-1} \setminus H^p$ for a certain $p \in (2, \infty)$ that are not given by power series by Hadamard gaps. Since $H^p \subset \mathscr{D}^p_{p-1}$ if $p \geq 2$, any Blaschke product belongs to $\bigcap_{2 \leq p < \infty} \mathscr{D}^p_{p-1}$. Also, for a number of classes $\mathscr{F}$ of analytic functions in Δ we have $\mathscr{F} \cap \mathscr{D}^p_{p-1} = \mathscr{F} \cap H^p$ $(0 < p < \infty)$. For example, it is very easy to prove the following lemma.

LEMMA 1.3. (i) *If $\alpha > 0$, $0 < p < \infty$, and $f(z) = 1/(1-z)^{\alpha}$, $(z \in \Delta)$, then $f \in H^p$ if and only if $f \in \mathscr{D}^p_{p-1}$ if and only if $\alpha p < 1$.*

(ii) *If* $\alpha, \beta > 0$, $p \in (0, \infty)$, *and*

$$f(z) = \frac{1}{(1-z)^{\alpha}(\log(2/(1-z)))^{\beta}}, \quad (z \in \Delta),$$

then $f \in H^p$ *if and only if* $f \in \mathscr{D}^p_{p-1}$ *if and only if* $\alpha p < 1$ *and* $\beta > 0$ *or* $\alpha p = 1$ *and* $\beta p > 1$.

A much deeper result is stated in [6, Theorem 1] which asserts that, if $\mathscr{U}$ denotes the class of all univalent (holomorphic and one-to-one) functions in Δ, then $\mathscr{U} \cap H^p = \mathscr{U} \cap \mathscr{D}^p_{p-1}$ for all $p > 0$ (see also [25] for the case $p = 1$).

In spite of these facts we shall prove that, for every $p \in (2, \infty)$, there are a lot of differences between the space H^p and the space $\mathscr{D}^p_{p-1}$. In Section 3, we shall be mainly concerned in obtaining sharp estimates on the growth of the integral means of $\mathscr{D}^p_{p-1}$-functions. If $0 < p \leq 2$ and $f \in \mathscr{D}^p_{p-1}$, then $f \in H^p$ and hence, the integral means $M_p(r, f)$ are bounded. This is no longer true for $p > 2$. Our main results in Section 3 are stated in the following two theorems.

THEOREM 1.4. *If* $2 < p < \infty$ *and* $f \in \mathscr{D}^p_{p-1}$, *then*

(i)

$$M_p(r, f) = \mathrm{O}\left(\left(\log \frac{1}{1-r}\right)\right), \quad \textit{as } r \to 1. \tag{7}$$

(ii)

$$M_2(r, f) = \mathrm{O}\left(\left(\log \frac{1}{1-r}\right)^{1/2-1/p}\right), \quad \textit{as } r \to 1. \tag{8}$$

THEOREM 1.5. *If* $2 < p < \infty$ *and* $0 < \beta < 1/2 - 1/p$, *then there exists a function* $f \in \mathscr{D}^p_{p-1}$ *such that*

$$\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{it})|\,dt\right) \neq \mathrm{o}\left(\left(\log\frac{1}{1-r}\right)^{\beta}\right), \quad \textit{as } r \to 1^{-}. \tag{9}$$

Since

$$\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{it})|\,dt\right) \leq M_2(r, f),$$

Theorem 1.5 shows that part (ii) of Theorem 1.4 is sharp in a very strong sense.

REMARK. Using Theorem 1.4 we can obtain an upper bound on the integral means $M_q(r, f)$, $2 < q < p$, of a function $f \in \mathscr{D}^p_{p-1}$. Indeed, if $q \in (2, p)$, then $q = p\lambda + 2(1-\lambda)$, where $\lambda = (q-2)/(p-2) \in (0, 1)$. Consequently, using Theorem 1.4 and Hölder's inequality with exponents $1/\lambda$ and $1/(1-\lambda)$ we see that, if $f \in \mathscr{D}^p_{p-1}$ and $2 < q < p$, then

$$M_q(r, f) = \left(\left(\log \frac{1}{1-r}\right)^{\eta}\right), \quad \text{as } r \to 1,$$

where $\eta = \eta(p, q) = p\lambda/q + (p-2)(1-\lambda)/pq$ and $\lambda = (q-2)/(p-2)$.

In Section 4 we study properties of the sequences of zeros of non trivial $\mathscr{D}^p_{p-1}$-functions. If $0 < p \le 2$ then $\mathscr{D}^p_{p-1} \subset H^p$ and hence, the sequence of zeros of a non-identically zero $\mathscr{D}^p_{p-1}$-function satisfies the Blaschke condition. This does not remain true for $p > 2$. Our main results about the sequences of zeros of functions f in the space $\mathscr{D}^p_{p-1}$, $2 < p < \infty$, are stated in Theorem 1.6 and Theorem 1.7

THEOREM 1.6. *Suppose that $2 < p < \infty$ and let f be a function which belongs to the space $\mathscr{D}^p_{p-1}$ with $f(0) \neq 0$. Let $\{z_k\}_{k=1}^{\infty}$ be the sequence zeros of f ordered so that $|z_k| \le |z_{k+1}|$ for all k. Then*

$$\prod_{k=1}^{N} \frac{1}{|z_k|} = \mathrm{o}\left((\log N)^{1/2-1/p}\right), \quad \text{as } N \to \infty. \tag{10}$$

From now on, if f is a non-identically zero analytic function of zeros and $\{z_k\}_{k=1}^{\infty}$ is the sequence zeros of f ordered so that $|z_k| \le |z_{k+1}|$ for all k, we shall say that $\{z_k\}_{k=1}^{\infty}$ is the sequence of ordered zeros of f. Theorem 1.7 asserts that Theorem 1.6 is best possible.

THEOREM 1.7. *If $2 < p < \infty$ and $0 < \beta < 1/2 - 1/p$, then there exists a function $f \in \mathscr{D}^p_{p-1}$ with $f(0) \neq 0$ such that if $\{z_k\}_{k=1}^{\infty}$ is the sequence of ordered zeros of f, then*

$$\prod_{k=1}^{N} \frac{1}{|z_k|} \neq \mathrm{o}\left((\log N)^{\beta}\right), \quad \text{as } N \to \infty. \tag{11}$$

As a consequence of Theorem 1.6 and Theorem 1.7, we obtain the following result.

COROLLARY 1.8. *If $2 \le p < q < \infty$ then there exists a sequence $\{z_k\} \subset \Delta$ that is the sequence of zeros of a $\mathscr{D}^q_{q-1}$-function but is not the sequence of zeros of any $\mathscr{D}^p_{p-1}$-function.*

Hence the situation in this setting is similar to that in the setting of Bergman spaces (see [18, Theorem 1]).

Next we shall get into the proofs of these and some other results. We shall be using the convention that $C_{p,\alpha,\dots}$ denotes a positive constant which depends only upon the displayed parameters $p, \alpha, \dots$ but is not necessarily the same at different occurrences.

2. Taylor coefficients of $\mathscr{D}^p_{p-1}$ functions.

We start by recalling the following useful result due to Mateljevic and Pavlovic [21] (see also [5, Lemma 3] and [22]) which will be basic in the proofs of Theorem 1.1 and Theorem 1.2.

LEMMA B. *Let $\alpha > 0$ and $p > 0$. There exists a constant K that depends only on p and α such that, if $\{a_n\}_{n=1}^{\infty}$ is a sequence of non-negative numbers, $t_n = \sum_{k\in I(n)} a_n$ $(n \ge 0)$, and $f(x) = \sum_{n=1}^{\infty} a_n x^{n-1}$ $(x \in (0,1))$, then*

$$K^{-1}\sum_{n=0}^{\infty} 2^{-n\alpha} t_n^p \le \int_0^1 (1-x)^{\alpha-1} f(x)^p \, dx \le K \sum_{n=0}^{\infty} 2^{-n\alpha} t_n^p .$$

PROOF OF THEOREM 1.1. Take $p \in (0,\infty)$ and let f be analytic in Δ,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \Delta. \tag{12}$$

Suppose that (4) holds. Using Lemma B and (4) we see that

$$\begin{aligned}
\int_\Delta |f'(z)|^p (1-|z|^2)^{p-1} \, dA(z) &\le C_p \int_0^1 (1-r)^{p-1} \left(\sum_{n=1}^{\infty} n |a_n| r^{n-1} \right)^p dr \\
&\le C_p \sum_{n=0}^{\infty} 2^{-np} \left(\sum_{k\in I(n)} k|a_k| \right)^p \\
&\le C_p \sum_{n=0}^{\infty} 2^{-np} 2^{(n+1)p} \left(\sum_{k\in I(n)} |a_k| \right)^p \\
&\le C_p \sum_{n=0}^{\infty} \left(\sum_{k\in I(n)} |a_k| \right)^p < \infty.
\end{aligned}$$

Hence, $f \in \mathscr{D}^p_{p-1}$ and the proof of (i) is finished.

Suppose now that $0 < p \leq 2$, f is as in (12) and satisfies (5). Using the fact that $M_p(r, f') \leq M_2(r, f')$ for all $r \in (0, 1)$, making the change of variable $r^2 = s$ and using Lemma B, we obtain

$$\begin{aligned}
\int_\Delta |f'(z)|^p(1-|z|^2)^{p-1}\,dA(z) &= 2\int_0^1 r(1-r^2)^{p-1}M_p(r, f')^p\,dr \\
&\leq 2\int_0^1 r(1-r^2)^{p-1}M_2(r, f')^p\,dr \\
&= 2\int_0^1 r(1-r^2)^{p-1}\left(\sum_{n=1}^\infty n^2|a_n|^2 r^{2n-2}\right)^{p/2} dr \\
&\leq C\int_0^1 (1-s)^{p-1}\left(\sum_{n=1}^\infty n^2|a_n|^2 s^{n-1}\right)^{p/2} ds \\
&\leq C_p\sum_{n=0}^\infty 2^{-np}\left(\sum_{k\in I(n)} k^2|a_k|^2\right)^{p/2} \\
&\leq C_p\sum_{n=0}^\infty \left(\sum_{k\in I(n)} |a_k|^2\right)^{p/2} < \infty.
\end{aligned}$$

Hence, $f \in \mathscr{D}^p_{p-1}$. This finishes the proof of (ii). □

Next we see that Proposition A can be deduced from Theorem 1.1 as announced.

Proof of Proposition A. Let f be an analytic function in Δ given by a power series with Hadamard gaps

$$f(z) = \sum_{j=1}^\infty a_j z^{n_j} \quad \text{with} \quad \frac{n_{j+1}}{n_j} \geq \lambda > 1 \quad \text{for all } j, \tag{13}$$

and suppose that $\sum_{j=1}^\infty |a_j|^p < \infty$. Using the gap condition, we see that there are at most $C_\lambda = \log_\lambda 2 + 1$ of the $n'_j s$ in the set $I(n)$. Then there exists a constant $C_{\lambda,p} > 0$ such that

$$\sum_{n=0}^\infty \left(\sum_{j\in I(n)} |a_j|\right)^p \leq C_{\lambda,p}\sum_{j=1}^\infty |a_j|^p < \infty,$$

and consequently, using Theorem 1.1, we deduce that $f \in \mathscr{D}^p_{p-1}$.

To prove the other implication suppose that f is as in (13) and $f \in \mathscr{D}^p_{p-1}$ for a certain $p > 0$. It is well known (see [38, Chapter V, Vol. I]) that there exist constants $A(\lambda, p)$ and $B(\lambda, p)$ such that

$$A(\lambda, p)M_2^p(r, f') \leq M_p^p(r, f') \leq B(\lambda, p)M_2^p(r, f'), \quad 0 < r < 1.$$

This and Lemma B give

$$\begin{aligned}
\infty > \int_{\Delta} |f'(z)|^p (1-|z|^2)^{p-1}\, dA(z) &= \int_0^1 r(1-r^2)^{p-1} M_p^p(r, f')\, dr \\
&\geq A(\lambda, p) \int_0^1 r(1-r^2)^{p-1} M_2^p(r, f')\, dr \\
&\geq A(\lambda, p) \int_0^1 r(1-r^2)^{p-1} \left(\sum_{j=1}^{\infty} {n_j}^2 |a_j|^2 r^{2n_j-2} \right)^{p/2} dr \\
&\geq A(\lambda, p) \int_0^1 t(1-t)^{p-1} \left(\sum_{j=1}^{\infty} {n_j}^2 |a_j|^2 t^{j-1} \right)^{p/2} dt \\
&\geq C_p A(\lambda, p) \sum_{n=0}^{\infty} 2^{-np} \left(\sum_{n_j \in I(n)} {n_j}^2 |a_j|^2 \right)^{p/2} \\
&\geq C_p A(\lambda, p) \sum_{n=0}^{\infty} 2^{-np} 2^{np} \left(\sum_{n_j \in I(n)} |a_j| \right)^{p} \geq C_{\lambda,p} A(\lambda, p) \sum_{j=0}^{\infty} |a_j|^p.
\end{aligned}$$

The last inequality is obvious if $p \geq 1$ and, in the case $0 < p < 1$, follows again using the fact that there are at most $C_\lambda = \log_\lambda 2 + 1$ of the $n'_j s$ in the set $I(n)$. Thus, we have $\sum_{j=0}^{\infty} |a_j|^p < \infty$. This finishes the proof. □

PROOF OF THEOREM 1.2. Suppose that $2 \leq p < \infty$ and $f \in \mathscr{D}^p_{p-1}$,

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad z \in \Delta.$$

Using Lemma B, bearing in mind that $k \asymp 2^n$ if $k \in I(n)$, making a change of variable, and using that since $p \geq 2$, $M_2(r, f') \leq M_p(r, f')$, we obtain

$$\begin{aligned}
\sum_{n=1}^{\infty} \left(\sum_{k \in I(n)} |a_k|^2 \right)^{p/2} &\leq \sum_{n=1}^{\infty} 2^{-np} \left(\sum_{k \in I(n)} k^2 |a_k|^2 \right)^{p/2} \\
&\leq C_p \int_0^1 (1-t)^{p-1} \left(\sum_{n=1}^{\infty} n^2 |a_n|^2 t^{n-1} \right)^{p/2} dt \\
&\leq C_p \int_0^1 (1-r^2)^{p-1} \left(\sum_{n=1}^{\infty} n^2 |a_n|^2 r^{2n-2} \right)^{p/2} dt \\
&\leq C_p \int_0^1 (1-r)^{p-1} M_p(r, f')^p < \infty.
\end{aligned}$$

□

3. Growth properties of $\mathscr{D}^p_{p-1}$-functions

In this section we are mainly interested in obtaining sharp estimates on the growth of functions f in the spaces $\mathscr{D}^p_{p-1}$ $(2 < p < \infty)$.

3.1. Integral means estimates

Let us start with estimates on the growth of the maximum modulus $M_\infty(r, f)$. We can prove the following result.

THEOREM 3.1. *Let f be an analytic function in Δ. If $f \in \mathscr{D}^p_{p-1}$, $0 < p < \infty$, then*

$$M_\infty(r, f) = \mathrm{o}\left(\frac{1}{(1-r)^{1/p}}\right), \quad \text{as } r \to 1^-. \tag{14}$$

PROOF. Let $f \in \mathscr{D}^p_{p-1}$ and $z \in \Delta$. Let $D(z)$ denote the open disc

$$\left\{ w \in \mathbb{C} : |z - w| < \frac{1-|z|}{2} \right\}.$$

Clearly, $D(z) \subset \Delta$. Since the function $z \to |f'(z)|^p$ is subharmonic in Δ, we have

$$|f'(z)|^p \le \frac{C}{|D(z)|} \int_{D(z)} |f'(\omega)|^p \, dA(\omega) \le \frac{C}{(1-|z|^2)^2} \int_{D(z)} |f'(\omega)|^p \, dA(\omega). \tag{15}$$

It is clear that $(1-|z|^2) \asymp (1-|\omega|^2)$, $\omega \in D(z)$, $z \in \Delta$. Using this and (15) we obtain

$$\begin{aligned} |f'(z)|^p &\le \frac{C_p}{(1-|z|^2)^2} \int_{D(z)} \left[\frac{1-|\omega|}{1-|z|}\right]^{p-1} |f'(\omega)|^p \, dA(\omega) \\ &= \frac{C_p}{(1-|z|^2)^{p+1}} \int_{D(z)} (1-|\omega|)^{p-1} |f'(\omega)|^p \, dA(\omega). \end{aligned} \tag{16}$$

On the other hand, since $f \in \mathscr{D}^p_{p-1}$, it follows that

$$\int_{D(z)} (1-|\omega|)^{p-1} |f'(\omega)|^p \, dA(\omega) = \mathrm{o}(1), \quad \text{as } |z| \to 1^-,$$

which, with (16), implies

$$M_\infty(r, f') = \mathrm{o}\left(\frac{1}{(1-r)^{1+1/p}}\right), \quad \text{as } r \to 1^-, \tag{17}$$

and (14) follows by integration. $\square$

REMARK. We observe that for any $p \in (0, \infty)$, the exponent $1/p$ in (14) is the best possible. Moreover, if we take

$$f_{p,\beta}(z) = (1-z)^{-1/p}\left(\log\frac{2}{1-z}\right)^{-\beta}, \quad z \in \Delta,$$

with $\beta > \frac{1}{p}$ then, as we noticed in Lemma 1.3, $f_{p,\beta} \in \mathscr{D}^p_{p-1}$ and it is easy to see that

$$M_\infty(r, f) \approx (1-r)^{-1/p}\left(\log\frac{1}{1-r}\right)^{-\beta}, \quad 0 < r < 1.$$

So condition (14) in Theorem 3.1 cannot be substituted by the condition

$$M_\infty(r, f) = \mathrm{o}\left(\frac{1}{(1-r)^{1/p}(\log(1/(1-r))^{1/p+\varepsilon}}\right), \qquad \text{as } r \to 1^-,$$

for any $\varepsilon > 0$.

Now we turn to the proofs of Theorem 1.4 and Theorem 1.5.

PROOF OF THEOREM 1.4. Suppose that $2 < p < \infty$ and $f \in \mathscr{D}^p_{p-1}$. Then

(18) $$\lim_{r\to 1^-}\int_r^1 (1-s)^{p-1}M_p^p(s, f')\,ds = 0.$$

Since $M_p(s, f')$ is an increasing function of s

$$\int_r^1 (1-s)^{p-1}M_p^p(s, f')\,ds \geq M_p^p(r, f')\int_r^1 (1-s)^{p-1}\,ds \geq C_p M_p^p(r, f')(1-r)^p,$$

which, together with (18), yields

(19) $$M_p(r, f') = \mathrm{o}\left((1-r)^{-1}\right), \quad \text{as } r \to 1^-,$$

which, using Minkowski's integral inequality, implies (7).

Using (19) and the fact that for any fixed r with $0 < r < 1$ the integral means $M_p(r, f')$ increase with p, we deduce that

$$I_2(r, f') = \mathrm{o}\left((1-r)^{-2}\right), \quad \text{as } r \to 1^-.$$

and then using the well-known inequality (see [26, pages 125–126])

$$\frac{d^2}{dr^2}\left(I_2(r, f)\right) \leq 4I_2(r, f'), \quad 0 < r < 1,$$

we obtain

$$\frac{d^2}{dr^2}\big(I_2(r,\,f)\big) = \mathrm{o}\left((1-r)^{-2}\right) \quad \text{as } r \to 1^-,$$

which, integrating twice, gives

$$M_2(r,\,f) = \mathrm{o}\left(\left(\log(1/(1-r)\right)^{1/2}\right), \quad \text{as } r \to 1.$$

This is worse than (8). To obtain this we use Theorem 1.2.

Say that $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $(z \in \Delta)$. Suppose, without loss of generality that $a_0 = 0$. Using Hölder's inequality with the exponents $p/2$ and $p/(p-2)$ and Theorem 1.2, we obtain

$$\begin{aligned}
M_2(r,\,f)^2 &= \sum_{n=1}^{\infty} |a_n|^2 r^{2n} = \sum_{n=0}^{\infty} \sum_{k\in I(n)} |a_k|^2 r^{2k} \le \sum_{n=0}^{\infty} r^{2^{n+1}} \left(\sum_{k\in I(n)} |a_k|^2\right) \\
&\le \left[\sum_{n=0}^{\infty}\left(\sum_{k\in I(n)} |a_k|^2\right)^{p/2}\right]^{2/p} \left[\sum_{n=0}^{\infty} r^{2^{n+1}p/(p-2)}\right]^{1-2/p} \\
&\le C_{f,p}\left(\log\frac{1}{1-r}\right)^{1-2/p}.
\end{aligned}$$

□

Since

$$\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\theta})|\,d\theta\right) \le M_2(r,\,f), \quad 0 < r < 1,$$

we trivially have the following result.

COROLLARY 3.2. *If $2 < p < \infty$ and $f \in \mathscr{D}_{p-1}^p$, then*

$$\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\theta})|\,d\theta\right) = \mathrm{O}\left(\left(\log\frac{1}{1-r}\right)^{1/2-1/p}\right), \quad \text{as } r \to 1.$$

Theorem 1.5 shows that Corollary 3.2 and the estimate (8) are sharp in a very strong sense. The following lemma, whose proof is simple and is omitted, will be used in the proof of Theorem 1.5.

LEMMA 3.3. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an analytic function in Δ. If $0 < \beta \le 1$ and $\sum_{k=0}^{N} |a_k|^2 \approx (\log N)^{\beta}$, as $N \to \infty$, then $I_2(r,\,f) \approx (\log(1-r)^{-1})^{\beta}$ as $r \to 1^-$.*

We make use of the technique introduced by Ullrich in [32]. Let us start introducing some notation.

Let $\omega = [0,1]^{\mathbb{N}}$ and $\omega_1, \omega_2, \ldots$ be 'the coordinate functions' $\omega_j : \Omega \to [0,1]$. Let $d\omega$ denote the product measure Ω derived from the Lebesgue measure on $[0,1]$. Now

$\omega_1, \omega_2, \dots$ are the Steinhaus variables (independent, identically distributed random variables uniformly distributed on $[0,1]$). Note that $\{e^{2\pi i\omega_j}\}_{j=1}^{\infty}$ is an orthonormal set in $L^2(\Omega)$, hence, if $\sum_{j=1}^{\infty}|a_j|^2<\infty$, then $\sum_{j=1}^{\infty}a_je^{2\pi i\omega_j}$ is a well defined element of $L^2(\Omega)$ with L^2-norm $\big(\sum_{j=1}^{\infty}|a_j|^2\big)^{1/2}$. The following theorem is [32, Theorem 1].

THEOREM C. *There exists $C>0$ such that for any sequence of complex numbers $\{a_j\}_{j=1}^{\infty}$ with $\sum_{j=1}^{\infty}|a_j|^2<\infty$, we have*

$$\exp\left[\int_{\Omega}\log\left|\sum_{j=1}^{\infty}a_je^{2\pi i\omega_j}\right|d\omega\right]\geq C\left(\sum_{j=1}^{\infty}|a_j|^2\right)^{1/2}.$$

PROOF OF THEOREM 1.5. Suppose that $2<p<\infty$ and $0<\beta<1/2-1/p$. Set $\varepsilon=1/2-1/p-\beta$, hence, $\varepsilon>0$. We define the sequence $\{b_j\}_{j=1}^{\infty}$ as $b_j=j^{-1/p-\varepsilon}$, $j=1,2,\dots$. Now, for every $\omega\in\Omega$ we define

$$f_\omega(z)=\sum_{j=1}^{\infty}b_je^{2\pi i\omega_j}z^{2^j}=\sum_{k=1}^{\infty}a_{k,\omega}z^k,\quad z\in\Delta. \tag{20}$$

Since $\sum_{j=1}^{\infty}|b_j|^p<\infty$, using Proposition A we deduce that $f_\omega\in\mathcal{D}^p_{p-1}$ for every $\omega\in\Omega$.

We will see that for a.e. $\omega\in\Omega$

$$\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f_\omega(re^{it})|\,dt\right)\neq \mathrm{o}\left(\big(\log(1/(1-r))\big)^{\beta}\right),\quad \text{as } r\to 1^-. \tag{21}$$

This will finish the proof.

Suppose that (21) is false. Then there exists a measurable set $E\subset\Omega$ with positive measure and such that for all $\omega\in E$

$$\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f_\omega(re^{it})|\,dt\right)= \mathrm{o}\left(\big(\log(1/(1-r))\big)^{\beta}\right),\quad \text{as } r\to 1^-. \tag{22}$$

This is equivalent to saying that

$$\lim_{r\to1^-}\frac{1}{2\pi}\int_{-\pi}^{\pi}\log\left[\frac{|f_\omega(re^{it})|}{\big(\log(1/(1-r))\big)^{\beta}}\right]dt=-\infty,\quad \omega\in E. \tag{23}$$

On the other hand,

$$\begin{aligned}\left(\sum_{j=1}^{N}|b_j|^2\right)^{1/2}&=\left(\sum_{j=1}^{N}\frac{1}{j^{2/p+2\varepsilon}}\right)^{1/2}\\&\sim\left(\int_1^N\frac{1}{x^{2/p+2\varepsilon}}\,dx\right)^{1/2}\sim N^{1/2-1/p-\varepsilon},\quad \text{as } N\to\infty.\end{aligned}$$

Thus, there exist $C > 0$ and $N_0 > 0$ such that

$$(24)\qquad \left(\sum_{k=1}^{N} |a_{k,\omega}|^2\right)^{1/2} \leq C\bigl(\log N\bigr)^{1/2-1/p-\varepsilon}, \quad N \geq N_0.$$

Using (24) and Lemma 3.3, we deduce that

$$M_2(r,\, f_\omega) = I_2(r,\, f_\omega)^{1/2} \leq C\left[\log\frac{1}{1-r}\right]^{1/2-1/p-\varepsilon}, \quad 0 < r < 1, \quad \omega \in \Omega,$$

which implies that for $0 < r < 1$ and $\omega \in \Omega$,

$$(25)\qquad \exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f_\omega(re^{it})|\,dt\right) \leq C\left[\log\frac{1}{1-r}\right]^{1/2-1/p-\varepsilon}.$$

From this we deduce as in (23), that there exists $C > 0$ such that

$$(26)\qquad \int_{-\pi}^{\pi} \log\left[\frac{|f_\omega(re^{it})|}{\bigl(\log(1/(1-r))\bigr)^{\beta}}\right] dt \leq C, \quad 0 < r < 1, \quad \omega \in \Omega.$$

Bearing in mind that E has positive measure, (26) and (23) imply

$$(27)\qquad \lim_{r\to 1^-} \int_\Omega \left[\int_{-\pi}^{\pi} \log\frac{|f_\omega(re^{it})|}{\bigl(\log(1/(1-r))\bigr)^{\beta}}\,dt\right] d\omega = -\infty.$$

For $N = 1, 2, \ldots$, let $\Omega_N = [0, 1]^N$ and m_N be the Lebesgue measure on Ω_N. Observe now that, for any N, we have

$$\begin{aligned}
&\int_{\Omega_N} \log|f_\omega(re^{it})|\,dm_N(\omega) \\
&\quad = \int_0^1 \cdots \int_0^1 \log\left|\sum_{j=1}^{N} b_j r^{2^j} e^{i[2\pi\omega_j + 2^j t]} + \sum_{j=N+1}^{\infty} b_j r^{2^j} e^{i[2\pi\omega_j + 2^j t]}\right| d\omega_1 d\omega_2 \cdots d\omega_N \\
&\quad = \int_0^1 \cdots \int_0^1 \log\left|\sum_{j=1}^{N} b_j r^{2^j} e^{2\pi i\omega_j} + \sum_{j=N+1}^{\infty} b_j r^{2^j} e^{i[2\pi\omega_j + 2^j t]}\right| d\omega_1 d\omega_2 \cdots d\omega_N, \text{ a.s.}
\end{aligned}$$

Letting N tend to ∞, we deduce that $\int_\Omega \log|f_\omega(re^{it})|\,d\omega$ is independent of t. Then using (27) and Fubini's Theorem we obtain

$$(28)\qquad \lim_{r\to 1^-} \int_\Omega \log\frac{|f_\omega(r)|}{\bigl(\log(1/(1-r))\bigr)^{\beta}}\,d\omega = -\infty.$$

However, if we set $r_N = 1 - 1/2^N$, $N = 1, 2, \ldots$, by Theorem C and the inequality

$$e^{-1} \leq r_N^{2^N} \leq r_N^{2^j}, \quad 1 \leq j \leq N,$$

we deduce that

$$\begin{aligned}
&\exp\left[\int_\Omega \log|f_\omega(r_N)|\,d\omega\right] \\
&\quad = \exp\left[\int_\Omega \log\left|\sum_{j=1}^{\infty} b_j e^{2\pi i\omega_j} r_N^{2^j}\right|\right] \\
&\quad \geq C\left(\sum_{j=1}^{\infty}|b_j|^2\left(r_N^{2^j}\right)^2\right)^{1/2} \geq C\left(\sum_{j=1}^{N}|b_j|^2\right)^{1/2} = C\left(\sum_{j=1}^{N}\frac{1}{j^{2/p+2\varepsilon}}\right)^{1/2} \\
&\quad \geq C\frac{1}{N^{1/p+\varepsilon-1/2}} \geq C\left(\log\frac{1}{1-r_N}\right)^{1/2-1/p-\varepsilon} = C\left(\log\frac{1}{1-r_N}\right)^{\beta},
\end{aligned}$$

which implies

$$\int_\Omega \log\frac{|f_\omega(r_N)|}{\left(\log(1-r_N)^{-1}\right)^{\beta}}\,d\omega \geq \log C, \quad \text{for all } N,$$

which contradicts (28). Consequently, (21) is true and the proof is finished. □

3.2. Radial growth of $\mathscr{D}^p_{p-1}$-functions In this section we obtain some estimates on the radial growth of $\mathscr{D}^p_{p-1}$-functions. If $0 < p \leq 2$ and $f \in \mathscr{D}^p_{p-1}$, then $f \in H^p$ and so f has nontangential limit a.e. $\mathbb{T}$. Therefore, we have: If $0 < p \leq 2$ and $f \in \mathscr{D}^p_{p-1}$, then $|f(re^{i\theta})| = \mathrm{O}(1)$, as $r \to 1^-$ for a.e. $e^{it} \in \partial\Delta$.

Zygmund proved in [37] that if f is an analytic function in Δ, then

$$\int_0^r |f'(\rho e^{it})|\,d\rho = \mathrm{o}\left[\left(\log\frac{1}{1-r}\right)^{1/2}\right], \quad \text{as } r \to 1^-. \tag{29}$$

for almost every point e^{it} in the Fatou set of f, F_f, which consists of those $e^{it} \in \mathbb{T}$ such that f has finite nontangential limit at e^{it}. Obviously, (29) implies

$$|f(re^{it})| = \mathrm{o}\left[\left(\log\frac{1}{1-r}\right)^{1/2}\right], \quad \text{as } r \to 1^-, \tag{30}$$

If $2 < p < \infty$, there are functions $f \in \mathscr{D}^p_{p-1}$ such that F_f has Lebesgue measure equal to zero. Indeed, an analytic function f given by a power series with Hadamard gaps whose sequence of Taylor coefficients $\{a_k\}$ belongs to $l^p \setminus l^2$, is a $\mathscr{D}^p_{p-1}$-function by Proposition A and F_f has null Lebesgue measure (see [38, Chapter V]). In spite of this, we can prove the following result for $\mathscr{D}^p_{p-1}$-functions.

THEOREM 3.4. *If* $2 < p < \infty$ *and* $f \in \mathscr{D}^p_{p-1}$, *then*

$$|f(re^{it})| = \mathrm{o}\left[\left(\log\frac{1}{1-r}\right)^{1-1/p}\right], \quad \text{as } r \to 1^- \text{ for a. e. } e^{it} \in \partial\Delta. \tag{31}$$

This is better that the a.e. estimate which can be deduced from (17).

PROOF OF THEOREM 3.4. Let p and f be as in the statement of the theorem. Then

$$\int_{-\pi}^{\pi}\left(\int_0^1 (1-r)^{p-1}|f'(re^{it})|^p\,dt\right)dr < \infty,$$

and it follows that the set A of points $e^{it} \in \partial\Delta$ for which

$$\int_0^1 (1-r)^{p-1}|f'(re^{it})|^p\,dt < \infty,$$

has Lebesgue measure equal to 2π.

Take and fix $e^{it} \in A$. Take also $\varepsilon > 0$. Then there exists $r_\varepsilon \in (0,1)$ such that

$$\int_{r_\varepsilon}^1 (1-s)^{p-1}|f'(se^{it})|^p\,ds < \varepsilon. \tag{32}$$

Using (32) and Hölder's inequality with exponents p and $p/(p-1)$, we obtain for $r_\varepsilon < r < 1$,

$$\begin{aligned}\int_0^r |f'(se^{it})|\,ds &= \int_0^{r_\varepsilon} |f'(se^{it})|\,ds + \int_{r_\varepsilon}^r |f'(se^{it})|\,ds \\ &\le C_{f,\varepsilon} + \int_{r_\varepsilon}^r \frac{(1-s)^{1-1/p}}{(1-s)^{1-1/p}}|f'(se^{it})|\,ds \\ &\le C_{f,\varepsilon} + \left[\int_{r_\varepsilon}^r (1-s)^{p-1}|f'(se^{it})|^p\,ds\right]^{1/p}\left[\int_{r_\varepsilon}^r \frac{ds}{(1-s)}\right]^{1-1/p} \\ &\le C_{f,\varepsilon} + \varepsilon\left(\log\frac{1}{1-r}\right)^{1-1/p}.\end{aligned} \tag{33}$$

Consequently, we have proved that

$$\limsup_{r\to 1}\left(\log\frac{1}{1-r}\right)^{1/p-1}\int_0^r |f'(se^{it})|\,ds \le \varepsilon.$$

Since $\varepsilon > 0$ and $e^{it} \in A$ are arbitrary, we have

$$\int_0^r |f'(se^{it})|\,ds = \mathrm{o}\left[\left(\log\frac{1}{1-r}\right)^{1-1/p}\right], \quad \text{as } r \to 1^-,$$

for all $e^{it} \in A$. This implies that (31) holds for all $e^{it} \in A$, which has Lebesgue measure equal to 2π. This finishes the proof. □

We do not know whether or not the exponent $1 - 1/p$ in Theorem 3.4 is sharp but we know that it cannot be substitutes by any exponent smaller than $1/2 - 1/p$. Indeed, we can prove the following result.

THEOREM 3.5. *If $2 < p < \infty$, then there exists a function $f \in \mathscr{D}^p_{p-1}$ such that*

$$\lim_{r\to 1^-} \frac{|f(re^{it})|}{\left(\log \frac{1}{1-r}\right)^{1/2-1/p}\left(\log\log \frac{1}{1-r}\right)^{-1}} = \infty, \quad \text{for a.e. } e^{it} \in \partial\Delta. \tag{34}$$

PROOF. Take $p > 2$. Define

$$a_k = \frac{1}{k^{1/p}\log 2k}, \quad k = 1, 2, \dots, \quad \text{and} \quad f(z) = \sum_{k=1}^{\infty} a_k z^{2^k}, \quad z \in \Delta.$$

Since $\sum_{k=1}^{\infty} |a_k|^p < \infty$, by Proposition A, we have that $f \in \mathscr{D}^p_{p-1}$.

On the other hand,

$$\left(\sum_{k=1}^{N} |a_k|^2\right)^{1/2} = \left(\sum_{k=1}^{N} \frac{1}{k^{2/p}\log^2 2k}\right)^{1/2} \sim \left(\int_1^N \frac{1}{x^{2/p}\log^2 2x}\,dx\right)^{1/2} \sim \frac{N^{1/2-1/p}}{\log N}, \quad \text{as } N \to \infty,$$

and then it is easy to see that

$$M_2(r, f) = I_2(r, f)^{1/2} \sim \frac{(\log \frac{1}{1-r})^{1/2-1/p}}{\log\log \frac{1}{1-r}}, \quad \text{as } r \to 1^-. \tag{35}$$

Now, by the law of the iterated logarithm for lacunary series (see [35]) we have that

$$\lim_{r\to 1^-} \frac{|f(re^{it})|}{\left[I_2(r, f)\log\log\log I_2(r, f)\right]^{1/2}} = 1, \quad \text{for a.e. } e^{it} \in \partial\Delta. \tag{36}$$

Now we observe that (36) and (35) imply (34). This finishes the proof. □

4. Zeros of $\mathscr{D}^p_{p-1}$ functions

4.1. Products of the zeros of $\mathscr{D}^p_{p-1}$ functions We start by recalling the the following result due to Horowitz, (see [18, page 65]).

LEMMA D. *Let f be an analytic function in Δ with $f(0) \neq 0$ and let $\{z_k\}$ be the sequence of ordered zeros of f. If $0 < p < \infty$, $0 \leq r < 1$, and N is a positive integer, then*

$$(37) \qquad |f(0)|^p \prod_{k=1}^{N} \frac{r^p}{|z_k|^p} \leq M_p(r, f)^p.$$

This lemma and the estimates for the integral means of $\mathscr{D}^p_{p-1}$-functions obtained in Section 3.1 are the basic ingredients in the proofs of Theorem 1.6 and Theorem 1.7. This method was used by Horowitz in [18] for the Bergman spaces and later by the first author of this paper, Nowak, and Waniurski in [15] for the Bloch space $\mathscr{B}$ and some other related spaces.

PROOF OF THEOREM 1.6. Let p, f, and $\{z_k\}_{k=1}^{\infty}$ be as in the statement of Theorem 1.6. Using Theorem 1.4, we see that f satisfies (8) and using Lemma D with $p = 2$, we deduce that

$$(38) \qquad \prod_{k=1}^{N} \frac{r}{|z_k|} \leq C M_2(r, f) \leq C \left(\log \frac{1}{1-r}\right)^{1/2-1/p}, \quad \text{if } r \text{ is close enough to } 1.$$

Now, taking $r = 1 - 1/N$ with N big enough in (38) and bearing in mind that $(1 - 1/N)^N > 1/2e$, we deduce that

$$(39) \qquad \prod_{k=1}^{N} \frac{1}{|z_k|} \leq C (\log N)^{1/2-1/p}.$$

This finishes the proof. ☐

Our next objective is to prove Theorem 1.7 which asserts that Theorem 1.6 is sharp. We start recalling some notation and facts from Nevanlinna theory (see [16, 23] or [31]) which will be needed in our proof.

Let f be a non-constant analytic function in Δ. For any $a \in \mathbb{C}$ and $0 < r < 1$, we denote by $n(r, a, f)$ the number of zeros $f - a$ in the disc $\{|z| \leq r\}$, where each zero is counted according to its multiplicity. We define also

$$(40) \qquad N(r, a, f) \stackrel{\text{def}}{=} \int_0^r \frac{n(t, a, f) - n(0, a, f)}{t}\, dt + n(0, a, f) \log r, \quad 0 < r < 1.$$

For simplicity, we shall write $n(r, f) = n(r, 0, f)$, $N(r, f) = N(r, 0, f)$. The *Nevanlinna characteristic function* $T(r, f)$ is defined by

$$T(r, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})|\, d\theta, \quad 0 < r < 1.$$

The *proximity function* $m(r, a, f)$ is given by

$$m(r, a, f) \stackrel{\text{def}}{=} \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^{+} \frac{1}{|f(re^{it}) - a|}\, dt, \quad 0 < r < 1.$$

Now we can state the *First Fundamental Theorem of Nevanlinna*.

THEOREM E. *Let f be a non-constant analytic function in Δ. Then*

$$m(r, a, f) + N(r, a, f) = T(r, f) + O(1), \quad \text{as } r \to 1^{-}.$$

for every $a \in \mathbb{C}$.

Now we can prove the following result.

PROPOSITION 4.1. *If $2 < p < \infty$ and f is a non-constant $\mathscr{D}^p_{p-1}$-function, then*

$$n(r, a, f) = \mathrm{O}\left(\frac{1}{1-r}\log\log\frac{1}{1-r}\right), \quad \text{as } r \to 1^{-}, \text{ for all } a \in \mathbb{C}. \tag{41}$$

PROOF. Using the arithmetic-geometric mean inequality we obtain

$$\begin{aligned} T(r, f) &\le \frac{1}{4\pi}\int_{-\pi}^{\pi} \log\left(|f(re^{it})|^2 + 1\right) dt \\ &\le \frac{1}{2}\log\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\left(|f(re^{it})|^2 + 1\right) dt\right) \le \frac{1}{2}\log\left(I_2(r, f) + 1\right), \end{aligned}$$

which, with part (ii) of Theorem 1.4, gives

$$T(r, f) = \mathrm{O}\left(\log\log\frac{1}{1-r}\right), \quad \text{as } r \to 1^{-}. \tag{42}$$

Using Theorem E, we deduce that

$$N(r, a, f) = \mathrm{O}\left(\log\log\frac{1}{1-r}\right), \quad \text{as } r \to 1^{-}, \text{ for all } a \in \mathbb{C}. \tag{43}$$

Now, it is well known (see [2, page 22]) that this implies (41). □

Now, we can proceed with the proof of Theorem 1.7.

PROOF OF THEOREM 1.7. Take p and β with $2 < p < \infty$ and $0 < \beta < 1/2 - 1/p$. Take $f \in \mathscr{D}^p_{p-1}$ with $f(0) \ne 0$ and

$$\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f(re^{it})|\, dt\right) \ne \mathrm{o}\left(\left(\log\frac{1}{1-r}\right)^{\beta}\right), \quad \text{as } r \to 1^{-}, \tag{44}$$

such a function exists by Theorem 1.5. Using (44) we see that there exist a sequence $\{r_j\}_{j=1}^{\infty} \subset (0,1)$ with $r_j \uparrow 1$ and a positive constant C (independent of j), such that

$$\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(r_j e^{it})|\,dt\right) \geq C\left(\log\frac{1}{1-r_j}\right)^{\beta}, \quad j=1,2\dots. \tag{45}$$

We shall write $n(r)$ instead of $n(r, f)$ for simplicity. Using Jensen's formula (see [1, page 206]) and (45) we deduce that

$$|f(0)|\prod_{k=1}^{n(r_j)}\frac{r_j}{|z_k|} \geq C\left(\log\frac{1}{1-r_j}\right)^{\beta}, \quad j=1,2\dots, \tag{46}$$

which implies that

$$n(r_j) \to \infty, \quad \text{as } j \to \infty. \tag{47}$$

On the other hand, Proposition 4.1 implies that there exists $C > 0$ such that

$$n(r) \leq C\frac{1}{1-r}\log\log\frac{1}{1-r}, \quad \text{if } r \text{ is sufficiently close to } 1.$$

This implies that

$$\log n(r) \leq C\log\frac{1}{1-r}, \quad \text{if } r \text{ is sufficiently close to } 1,$$

which, together with (46), shows that there exists $j_0 \in \mathbb{N}$ such that for every $j \geq j_0$

$$|f(0)|\prod_{k=1}^{n(r_j)}\frac{r_j}{|z_k|} \geq C\big[\log n(r_j)\big]^{\beta}.$$

This finishes the proof. □

4.2. A substitute of Blaschke condition If $2 < p < \infty$ the sequence $\{z_k\}$ of ordered zeros of a non trivial $\mathcal{D}^p_{p-1}$ function need not satisfy the Blaschke condition. Indeed, the Blaschke condition is equivalent to saying that $\prod_{n=1}^{N}(1/|z_n|) = O(1)$ and we have seen that this is not always true. Using Theorem 1.6 and arguing exactly as in the proof of [15, Theorem 5] we can prove the following result.

THEOREM 4.2. *Let $2 < p < \infty$ and $f \in \mathcal{D}^p_{p-1}$ with $f \not\equiv 0$. Let $\{z_k\}_{k=1}^{\infty}$ be the sequence of zeros of f. Then*

$$\sum_{|z_k|>1-1/e}(1-|z_k|)\left(\log\log\frac{1}{1-|z_k|}\right)^{-\alpha} < \infty \tag{48}$$

for all $\alpha > 1$.

Next, we shall prove that the condition $\alpha > 1$ is needed in Theorem 4.2.

THEOREM 4.3. *Let $2 < p < \infty$. Then there exists a function $f \in \mathcal{D}^p_{p-1}$ with $f \not\equiv 0$, whose sequence of zeros $\{z_k\}_{k=1}^{\infty}$ satisfies*

$$\sum_{|z_k|>1-1/e} (1-|z_k|)\left(\log\log\frac{1}{1-|z_k|}\right)^{-1} = \infty. \tag{49}$$

PROOF. Set $g(z) = \sum_{k=1}^{\infty} k^{-(p+2)/4p} z^{2^k}$, $z \in \Delta$. Since g is given by a power series with Hadamard gaps and $\sum_{k=1}^{\infty} k^{-(p+2)/4} < \infty$, it follows that $g \in \mathcal{D}^p_{p-1}$.

We shall follow the argument of the proof of [15, Theorem 6]. Set

$$r_n = 1 - 2^{-n}, \quad n = 1, 2, 3, \ldots. \tag{50}$$

It is easy to see that, for all sufficiently large n, $I_2(r_n, g) \ge Cn^{1/2-1/p}$, which, since $\log(1/(1-r_n)) = n\log 2$, implies that

$$I_2(r_n, g) \ge C\left(\log\frac{1}{1-r_n}\right)^{1/2-1/p} \quad \text{if } n \text{ is sufficiently large.} \tag{51}$$

Now, since $\log(1/(1-r_n)) \sim \log(1/(1-r_{n+1}))$, as $n \to \infty$, and since $I_2(r, g)$ and $(\log(1/(1-r)))^{1/2-1/p}$ are increasing functions of r, we deduce

$$I_2(r, g) \ge C\left(\log\frac{1}{1-r}\right)^{1/2-1/p}, \tag{52}$$

if r is sufficiently close to 1.

Using this and arguing as in [15, page 126] we deduce that there exist a complex number a with $g(0) \ne a$, a positive constant β, and a number $r_0 \in (0, 1)$ such that

$$N(r, a, g) \ge \beta\log\log\frac{1}{1-r} \quad r \in (r_0, 1). \tag{53}$$

Take such an $a \in \mathbb{C}$ and set $f(z) = g(z) - a$, $z \in \Delta$. Then $f \in \mathcal{D}^p_{p-1}$ and $f(0) \ne 0$. Also (53) can be written as

$$N(r, f) \ge \beta\log\log\frac{1}{1-r}, \quad r \in (r_0, 1). \tag{54}$$

Let $\{z_n\}$ be the sequence of zeros of f. Using Proposition 4.1 and arguing as in [15, page 127], we obtain (49). □

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