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THE FACTORIAL MOMENTS OF ADDITIVE FUNCTIONS WITH RATIONAL ARGUMENT

J. ŠIAULYS and G. STEPANAUSKAS

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Abstract

We consider the weak convergence of the set of strongly additive functions $f(q)$ with rational argument q. It is assumed that $f(p)$ and $f(1/p) \in \{0, 1\}$ for all primes. We obtain necessary and sufficient conditions of the convergence to the limit distribution. The proof is based on the method of factorial moments. Sieve results, and Halász's and Ruzsa's inequalities are used. We present a few examples of application of the given results to some sets of fractions.

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1. Introduction

Let $\{f_x : \mathbb{N} \to \mathbb{R},\ x \geq 2\}$ be a set of additive functions, A be some subset of natural numbers, and

$$\nu_x(A) := \frac{1}{[x]} \#\{n \leq x,\ n \in A\}$$

be a frequency of natural numbers $n \in A$. The set A is allowed to depend on x and other parameters.

The central problem of probabilistic number theory is to find conditions under which the frequencies $\nu_x(f_x(n) - \alpha(x) < u)$, with a suitably chosen centering function $\alpha(x)$, converge to the limit law as $x \to \infty$. Starting with Turán's proof of the Hardy-Ramanujan theorem on the normal order of prime factors in 1934, many works have been devoted to this problem. This problem is the main object of the monographs of

Kubilius [5] and Elliott [1, 2]. There are three different cases in the investigation of a set of additive functions (see, for example, [1, 2, 5]). The first case is when $f_x(n)$ does not depend on x. The Erdös-Wintner theorem (see, for example, [2, page 187]) is the most celebrated result of this case. The second case arises when $f_x(n) = f(n)/b(x)$, where $b(x)$ is some normalizing unbounded function satisfying some additional conditions. The well-known Levin-Timofeev theorem (see, for example, [2, pages 122–123]) is one such result about the weak convergence of distributions $\nu_x(f(n)/b(x) - \alpha(x) < u)$ to the limit law.

The third, most general case, we obtain when the additive function f_x depends on x in an arbitrary way. The first result in this direction was obtained by Rusza. He found necessary and sufficient conditions for the weak convergence of $\nu_x(f_x(n) - \alpha(x) < u)$ to the improper law (see [6]). Šiaulys continued the investigation of the set of such functions. He derived necessary and sufficient conditions for the convergence of the distributions $\nu_x(f_x(n) < u)$ to the Poisson law in the case when f_x are strongly additive and $f_x(p) \in \{0, 1\}$ (see [8]).

In the present paper we consider the additive function defined on the set of positive rational numbers $\mathbb{Q}_+$. We suppose throughout that the natural numbers in the representation of the rational number $q = m/n$, are coprime, that is, $(m, n) = 1$. Any rational number q has a unique representation as a product $q = p_1^{\alpha_1} \cdots p_s^{\alpha_s}$, where $p_1, \ldots, p_s$ are distinct prime numbers, and $\alpha_1, \ldots, \alpha_s$ are integers. The power of prime p in such a product for the rational q is denoted by $\alpha_p(q)$. We say that the rational number $q_1 = m_1/n_1$ *divides* $q_2 = m_2/n_2$ $(q_1|q_2)$ if $m_1|m_2$ and $n_1|n_2$, and that they are *coprime* if $(m_1, m_2) = (m_1, n_2) = (m_2, n_1) = (n_1, n_2) = 1$.

For any additive function $f : \mathbb{Q}_+ \to \mathbb{C}$ the equality $f(q) = \sum_p f(p^{\alpha_p(q)})$ holds. If, in addition, $f(p^\alpha) = f(p^{\operatorname{sgn}\alpha})$ for all integers α and primes p, the function f is called strongly additive. Thus for any strongly additive function f with rational argument $f(q) = \sum_{p^\delta|q} f(p^\delta)$, where $\delta \in \{-1, 1\}$.

For $x \geq 2$, for an interval $I := (\xi, \eta]$, and for some condition A, where ξ, η and A are allowed to depend on x, we write:

$$\mathbb{Q}_x^I := \left\{q = \frac{m}{n} : n \leq x,\ \xi < \frac{m}{n} \leq \eta\right\}, \quad \mathbb{E}_x^I(p^\alpha) := \left\{q \in \mathbb{Q}_x^I : \alpha_p(q) = \alpha\right\},$$
$$\nu_x^I(A) := \left(\#\mathbb{Q}_x^I\right)^{-1} \#\left\{q \in \mathbb{Q}_x^I : q \in A\right\}, \qquad \mathbb{P}_x^I := \left\{p^{\operatorname{sgn}\alpha} : \mathbb{E}_x^I(p^\alpha) \neq \emptyset\right\}.$$

In the expression for $\nu_x^I(A)$ we suppose $\#\mathbb{Q}_x^I > 0$. We call the elements from $\mathbb{P}_x^I$ the prime ones. The quantity $\nu_x^I(A)$ denotes the frequency of the rational numbers which satisfy the condition A. In the particular case $I = (0, 1]$ we omit the symbol I and instead of $\mathbb{Q}_x^I$, $\mathbb{E}_x^I(p^\alpha)$, $\mathbb{P}_x^I$, $\nu_x^I(A)$ we simply write $\mathbb{Q}_x$, $\mathbb{E}_x(p^\alpha)$, $\mathbb{P}_x$, $\nu_x(A)$. We observe that $\mathbb{Q}_x = \{m/n : n \leq x,\ m/n \leq 1\}$ is the classical set of Farey fractions.

The probabilistic model for solving problems on the value distribution of additive functions with rational arguments can be developed in analogy with the Kubilius model (see [4, 11, 13, 12]). In this work, we consider the distribution of the set of strongly additive functions $\nu_x(f_x(q) < u)$ using the factorial moments method.

Throughout the paper we use the following notations. The function $\varepsilon(x)$ is always vanishing as x tends to infinity. The absolute constants are denoted by $c_1, c_2, \ldots$. The expression $a \ll b$ is equivalent to $|a| \le cb$, with some positive constant c. If the vanishing function or bounding quantities depend on d, we write $\varepsilon_d(x)$, O_d, $\ll_d$.

Let $\widetilde{\mathbb{P}}_x := \{p^\delta \in \mathbb{P}_x : f_x(p^\delta) = 1\}$. For the sake of brevity, we use a star and a t above the summation sign $\sum^{*t}$ to denote a summation expanded over all collections of $p_1^{\delta_1}, p_2^{\delta_2}, \ldots, p_t^{\delta_t} \in \widetilde{\mathbb{P}}_x$ (with some fixed t) such that $p_i \ne p_j$, $1 \le i < j \le t$. The most frequent case is when $t = l$. In this case we will omit l and write $\sum^*$.

The main result of this paper is the following statement.

THEOREM 1.1. *Let $\{f_x,\ x \ge 2\}$ be a set of strongly additive functions with rational argument. Let $f_x(p^\delta) \in \{0, 1\}$ for every prime number p and exponent $\delta \in \{-1, 1\}$. Then the frequency $\nu_x(f_x(q) < u)$ converges weakly to some distribution function if and only if the limit*

$$\lim_{x\to\infty} \sum\nolimits^* \frac{\Delta_x\left(p_1^{\delta_1} p_2^{\delta_2} \cdots p_l^{\delta_l}\right)}{(p_1+1)(p_2+1)\cdots(p_l+1)} = g_l \tag{1.1}$$

exists for every fixed natural number l. Here

$$\Delta_x(q) := \begin{cases} 1 & \text{if it exists } q_1 \in \mathbb{Q}_x : q | q_1, \\ 0 & \text{if that } q_1 \text{ does not exist.} \end{cases}$$

Moreover, if the limit distribution exists, then its characteristic function is equal to $1 + \sum_{l=1}^{\infty} (g_l/l!)(\mathrm{e}^{it} - 1)^l$.

2. Examples

Using Theorem 1.1 we can calculate the asymptotic densities of some arithmetically interesting sets of fractions. Let us give a few examples.

EXAMPLE 1. Define the strongly additive function f by

$$f(p) = \begin{cases} 1 & \text{if } p = 2, 3, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad f(1/p) = \begin{cases} 1 & \text{if } p = 3, 5, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from (1.1) that $g_1 = 1$, $g_2 = 11/18$, $g_3 = 1/6$, $g_l = 0$, $l \ge 4$. Hence the limit law of $\nu_x(f(q) < u)$ has the characteristic function

$$\frac{5}{18} + \frac{17}{36}\mathrm{e}^{it} + \frac{2}{9}\mathrm{e}^{2it} + \frac{1}{36}\mathrm{e}^{3it}. \tag{2.1}$$

From the well-known asymptotics $\#\{q \in \mathbb{Q}_x\} \sim (3/\pi^2)x^2$, $x \to \infty$, and the structure of characteristic function (2.1), we have that

$$\#\{q \in \mathbb{Q}_x : \text{prime elements } 2, 3, 1/3, 1/5 \text{ do not divide } q\} \sim \frac{5}{6\pi^2}\, x^2,$$

$$\#\{q \in \mathbb{Q}_x : \text{exactly two elements from } \{2, 3, 1/3, 1/5\} \text{ divide } q\} \sim \frac{2}{3\pi^2}\, x^2,$$

$$\#\{q \in \mathbb{Q}_x : \text{exactly three elements from } \{2, 3, 1/3, 1/5\} \text{ divide } q\} \sim \frac{1}{12\pi^2}\, x^2,$$

as $x \to \infty$.

EXAMPLE 2. Define the strongly additive function f_x by

$$f_x(p) = f_x(1/p) = \begin{cases} 1 & \text{if } \log x < p \le \log^2 x, \\ 0 & \text{otherwise.} \end{cases}$$

In this case $g_1 = 2\log 2$, $g_2 = (2\log 2)^2$, $g_l = (2\log 2)^l$, $l \ge 3$. Thus the limit law of $\nu_x(f_x(q) < u)$ is the Poisson law with parameter $\lambda = 2\log 2$, and we have

$$\#\big\{q \in \mathbb{Q}_x : p \nmid q,\ (1/p) \nmid q \text{ for } p \in (\log x, \log^2 x]\big\} \sim \frac{3}{4\pi^2}\, x^2,$$

$$\#\big\{q \in \mathbb{Q}_x : q \text{ has exactly one prime divisor } p^\delta \text{ for } p \in (\log x, \log^2 x]\big\} \sim \frac{3\log 2}{2\pi^2}\, x^2,$$

$$\#\big\{q \in \mathbb{Q}_x : q \text{ has exactly two prime divisors } p^\delta \text{ for } p \in (\log x, \log^2 x]\big\} \sim \frac{3\log^2 2}{2\pi^2}\, x^2.$$

as $x \to \infty$.

EXAMPLE 3. Let the strongly additive function f_x be defined by

$$f_x(p) = f_x(1/p) = \begin{cases} 1 & \text{if } \sqrt[3]{x} < p \le x, \\ 0 & \text{otherwise.} \end{cases}$$

Put $\theta = \log 3 \log 2 + \mathrm{Li}_2(1/3) - \mathrm{Li}_2(2/3)$, where $\mathrm{Li}_2(u)$ is the polylogarithm of second order, that is,

$$\mathrm{Li}_2(u) = \sum_{k=1}^{\infty} \frac{u^k}{k^2}, \qquad |u| \le 1.$$

We have that $g_1 = 2\log 3$, $g_2 = 2\big((\log 3)^2 + \theta\big)$, $g_3 = 6\theta\log 3$, $g_4 = 6\theta^2$, $g_l = 0$, $l \ge 5$.

Since the limit law of $\nu_x(f_x(q) < u)$ has the characteristic function

$$(1 - \log 3)(1 - \log 3 + \theta) + \theta^2/4 + \big(2\log 3(1 - \log 3) + (3\log 3 - 2 - \theta)\theta\big)\, \mathrm{e}^{it} + \big(\log^2 3 + (1 - 3\log 3 + 3\theta/2)\theta\big)\, \mathrm{e}^{2it} + (\log 3 - \theta)\theta\, \mathrm{e}^{3it} + \theta^2\, \mathrm{e}^{4it}/4,$$

the following asymptotic expressions (for $x \to \infty$) are true:

$$\begin{aligned}
&\#\bigl\{q \in \mathbb{Q}_x : \text{prime elements from } \{p, 1/p,\ p \in (\sqrt[3]{x}, x]\} \text{ do not divide } q\bigr\}\\
&\qquad \sim \frac{3}{\pi^2}\bigl((1-\log 3)(1-\log 3+\theta)+\theta^2/4\bigr)\,x^2 \approx 0.00072x^2,\\
&\#\bigl\{q \in \mathbb{Q}_x :\ \text{exactly 3 elements from}\{p, 1/p,\ p \in (\sqrt[3]{x}, x]\} \text{ divide } q\bigr\}\\
&\qquad \sim \frac{3}{\pi^2}(\log 3-\theta)\theta x^2 \approx 0.07197x^2,\\
&\#\bigl\{q \in \mathbb{Q}_x :\ \text{exactly 4 elements from}\{p, 1/p,\ p \in (\sqrt[3]{x}, x]\} \text{ divide } q\bigr\}\\
&\qquad \sim \frac{3}{4\pi^2}\theta^2 x^2 \approx 0.00659x^2.
\end{aligned}$$

3. Auxiliary lemmas

The proof of Theorem 1.1 is based on the factorial moments method, but some sieve results (Lemma 3.1), and the inequalities of Halász (see [3]) and Ruzsa (see [7]) are of key importance as well. In this section we present the analogues, sufficient for our needs, of these inequalities for functions of rational argument (Lemma 3.2 and Lemma 3.3).

LEMMA 3.1 (see [12]). *Let $I = (\xi, \eta]$, $0 \le \xi < \eta$, be an interval of real numbers. Let N_0, N_1, N_2 be natural numbers, which do not have any common prime divisor. All quantities ξ, η, N_0, N_1, N_2 may depend on $x \ge 2$. Then*

$$\begin{aligned}
\#\left\{\frac{m}{n} \in \mathbb{Q}_x^I : (m, N_0N_1) = (n, N_0N_2) = 1\right\}
&= \frac{3}{\pi^2}(\eta-\xi)x^2 \prod_{p\mid N_0}\left(1-\frac{2}{p+1}\right)\prod_{p\mid N_1N_2}\left(1-\frac{1}{p+1}\right)\\
&\quad \times \left(1 + 2^{\omega(N_0N_1N_2)}\, O\left(\frac{\log x}{x} + \frac{1}{x(\eta-\xi)}\right)\right),
\end{aligned}$$

where $\omega(m)$ is the number of distinct primes dividing m.

LEMMA 3.2 (see [10]). *Let $I = (\xi, \eta]$ be an interval of real numbers, where ξ and η may depend on $x \ge 2$ and satisfy the limit conditions:*

$$\limsup_{x\to\infty} \frac{\xi}{x(\eta-\xi)} \le c_1, \quad \lim_{x\to\infty} \frac{1}{x(\eta-\xi)} = 0. \tag{3.1}$$

Let $f : \mathbb{Q} \to \mathbb{N} \cup \{0\}$ *be an integer-valued additive function. Then, for every* $L \in \mathbb{N} \cup \{0\}$,

$$\nu_x^I(f(q) = L) \le c_2 \left(\max \left\{ \sum_{p \in \mathbb{P}_x^I,\ f(p) \neq 0} 1/p, \sum_{(1/p) \in \mathbb{P}_x^I,\ f(1/p) \neq 0} 1/p \right\} \right)^{-1/2}, \tag{3.2}$$

where c_2 *depends on* c_1 *and on the convergence rate in* (3.1).

LEMMA 3.3 (see [9]). *Let* $I = (\xi, \eta]$ *be an interval satisfying the conditions* $\xi \le c_3 x(\eta - \xi)$, $x(\eta - \xi) \ge c_4$. *Then, for an arbitrary strongly additive function* $f : \mathbb{Q} \to \mathbb{C}$ *and for every natural number* l,

$$\sum_{q \in \mathbb{Q}_x^I} \left| f(q) - \sum_{p^\delta \in \mathbb{P}_x^I} \frac{f(p^\delta)}{p} \right|^l \ll x^2(\eta - \xi) \left(\left(\sum_{p^\delta \in \mathbb{P}_x^I} \frac{|f(p^\delta)|^2}{p} \right)^{l/2} + \sum_{p^\delta \in \mathbb{P}_x^I} \frac{|f(p^\delta)|^l}{p} \right),$$

where the constant implied in the symbol $\ll$ *may depend on* c_3, c_4 *and* l.

4. Boundedness of factorial moments

PROPOSITION 4.1. *Let* $I = (\xi, \eta]$ *be an interval of real numbers, where* ξ *and* η *may depend on* $x \ge 2$ *and satisfy the limit conditions*:

$$\limsup_{x \to \infty} \frac{\xi}{x(\eta - \xi)} < \infty, \quad \lim_{x \to \infty} \frac{1}{x(\eta - \xi)} = 0.$$

Let f_x *be a set of strongly additive functions with rational argument. Let* $f_x(p^\delta) \in \{0, 1\}$ *for every prime number* p *and exponent* $\delta \in \{-1, 1\}$. *Let*

$$\varphi(l, x) = \frac{1}{\#\mathbb{Q}_x^I} \sum_{q \in \mathbb{Q}_x^I} f_x(q)(f_x(q) - 1) \cdots (f_x(q) - l + 1) \tag{4.1}$$

for every natural number l.

If the distributions $\nu_x^I(f_x(q) < u)$ *have a weak limit as* $x \to \infty$, *then*

$$\limsup_{x \to \infty} \varphi(l, x) \ll 1.$$

Here the constant in $\ll$ *depends on* l *and on the structure of the limit law.*

PROOF. Suppose X is a random variable for which

$$\nu_x^I(f_x(q) < u) \underset{x \to \infty}{\Longrightarrow} P(X < u).$$

The random variable X is integer valued, hence there exists $L \in \{0\} \cup \mathbb{N}$ for which $P(X = L) > 0$. From the limit

$$\lim_{x\to\infty} \nu_x^I(f_x(q) = L) = P(X = L), \tag{4.2}$$

we have that $\nu_x^I(f_x(q) = L) \ge \frac{1}{2}P(X = L)$ for $x \ge c_5$, where c_5 depends on $P(X = L)$ and on the rate of convergence in (4.2).

It follows from Lemma 3.2 that

$$\max\left\{\sum_{p\in\mathbb{P}_x^I,\ f_x(p)=1} 1/p, \sum_{1/p\in\mathbb{P}_x^I,\ f_x(1/p)=1} 1/p\right\} \ll \frac{1}{P^2(X = L)}.$$

Hence

$$\limsup_{x\to\infty} \sum_{p^\delta\in\mathbb{P}_x^I,\ f_x(p^\delta)=1} 1/p \le c_6. \tag{4.3}$$

The constant c_6 depends on the structure of the limit random variable. According to Lemma 3.1, $\#\mathbb{Q}_x^I \gg x^2(\eta - \xi)$ for x sufficiently large. Therefore from (4.3) and Lemma 3.3 we obtain

$$\limsup_{x\to\infty} \frac{1}{\#\mathbb{Q}_x^I} \sum_{q\in\mathbb{Q}_x^I} \left| f_x(q) - \sum_{p^\delta\in\mathbb{P}_x^I} \frac{f_x(p^\delta)}{p}\right|^l \ll_l \left(c_6^{l/2} + c_6\right).$$

Since

$$\sum_{q\in\mathbb{Q}_x^I} f_x^l(q) \le 2^l \left(\sum_{q\in\mathbb{Q}_x^I} \left| f_x(q) - \sum_{p^\delta\in\mathbb{P}_x^I} \frac{f_x(p^\delta)}{p}\right|^l + \#\mathbb{Q}_x^I \left(\sum_{p^\delta\in\mathbb{P}_x^I} \frac{f_x(p^\delta)}{p}\right)^l\right),$$

we conclude finally from (4.3) that

$$\limsup_{x\to\infty} \varphi(l, x) \le \limsup_{x\to\infty} \frac{1}{\#\mathbb{Q}_x^I} \sum_{q\in\mathbb{Q}_x^I} f_x^l(q) \ll_l \max\left(c_6^{l/2}, 1\right).$$

Proposition 4.1 is proved. □

5. The factorial moments method

Proposition 5.1. *Let $I = (\xi, \eta)$ be an interval of real numbers and f_x be a set of strongly additive functions satisfying the condition of Proposition 4.1. Let X be an integer-valued random variable and*

$$g_l = \sum_{k=l}^{\infty} k(k-1)\cdots(k-l+1)P(X = k)$$

for every natural l*. If* $g_{l+2} < \infty$ *for some* l *and*

$$\nu_x^I(f_x(q) < u) \underset{x\to\infty}{\Longrightarrow} P(X < u), \tag{5.1}$$

then

$$\lim_{x\to\infty} \varphi(l, x) = g_l. \tag{5.2}$$

Furthermore, if (5.2) *is satisfied for every fixed natural* l *and* $\sum_{l=1}^{\infty}(2^l g_l/l!) < \infty$*, then* (5.1) *holds for a random variable* X *that has the characteristic function*

$$1 + \sum_{l=1}^{\infty} \frac{g_l}{l!}(e^{it} - 1)^l.$$

PROOF (necessity). Let condition (5.1) be satisfied. Since the random variable X is integer-valued, it follows from (5.1) that

$$\nu_x^I(f_x(q) = k) = P(X = k) + \varepsilon_k(x) \tag{5.3}$$

for each fixed $k = 0, 1, 2, \ldots$.

Let us split the factorial moment $\varphi(l, x)$ (see (4.1)) into two parts:

$$\varphi(l, x) = \beta_1(l, x, y) + \beta_2(l, x, y), \qquad y \geq l + 3, \tag{5.4}$$

where $\beta_1(l, x, y)$ is that part of sum (4.1) for which $f_x(q) < [y]$ and $\beta_2(l, x, y)$ is the part for which $f_x(q) \geq [y]$.

From (5.3) we have

$$\begin{aligned}
\beta_1(l, x, y) &= \sum_{k=l}^{[y]-1} k(k-1)\cdots(k-l+1)\nu_x^I(f_x(q) = k) \\
&= \sum_{k=l}^{[y]-1} k(k-1)\cdots(k-l+1)\big(P(X = k) + \varepsilon_k(x)\big) \\
&= g_l + \varepsilon_y(x) - \sum_{k=[y]}^{\infty} k(k-1)\cdots(k-l+1)P(X = k).
\end{aligned}$$

Since

$$\begin{aligned}
&\sum_{k=[y]}^{\infty} k(k-1)\cdots(k-l+1)P(X = k) \\
&\qquad \leq \sum_{k=[y]}^{\infty} k(k-1)\cdots(k-l+1) \\
&\qquad\quad \times \sum_{j=k}^{\infty} \frac{j(j-1)\cdots(j-l+1)(j-l)(j-l-1)}{j(j-1)\cdots(j-l+1)(j-l)(j-l-1)} P(X = j)
\end{aligned}$$

$$\leq \sum_{k=[y]}^{\infty} \frac{k(k-1)\cdots(k-l+1)}{k(k-1)\cdots(k-l+1)(k-l)(k-l-1)}$$
$$\times \sum_{j=l+2}^{\infty} j(j-1)\cdots(j-l-1)P(X=j)$$
$$\leq g_{l+2}\sum_{k=[y]}^{\infty}\frac{1}{(k-l-1)^2} = g_{l+2}\sum_{r=[y]-l-1}^{\infty}\frac{1}{r^2} \leq \frac{g_{l+2}}{[y]-l-2},$$

we find that

$$\beta_1(l,x,y) = g_l + \varepsilon_y(x) + O\left(\frac{g_{l+2}}{[y]-l-2}\right). \tag{5.5}$$

Applying the estimate of Proposition 4.1 we obtain, for x sufficiently large,

$$\beta_2(l,x,y) = \frac{1}{\#\mathbb{Q}_x^I}\sum_{\substack{q\in\mathbb{Q}_x^I\\ f_x(q)\geq[y]}} f_x(q)(f_x(q)-1)\cdots(f_x(q)-l+1)\frac{f_x(q)-l}{f_x(q)-l}$$
$$\leq \frac{\varphi(l+1,x)}{[y]-l} \ll_l \frac{1}{[y]-l}.$$

The last estimate and equalities (5.4) and (5.5) imply that

$$\varphi(l,x) = g_l + \varepsilon_y(x) + O\left(\frac{g_{l+2}}{[y]-l-2}\right) + O_l\left(\frac{1}{[y]-l}\right).$$

Consequently equality (5.2) holds. □

PROOF (sufficiency). Let equation (5.2) be satisfied for every fixed natural l. Let

$$\psi_x(t) = \frac{1}{\#\mathbb{Q}_x^I}\sum_{q\in\mathbb{Q}_x^I} \mathrm{e}^{itf_x(q)}$$

be the characteristic function of the distribution $\nu_x^I(f_x(q) < u)$.

For every $r \in \{0\}\cup\mathbb{N}$ and $L \in \mathbb{N}$

$$\left|\mathrm{e}^{itr} - 1 - \sum_{l=1}^{L}\binom{r}{l}(\mathrm{e}^{it}-1)^l\right| \leq \binom{r}{L+1}|\,\mathrm{e}^{it}-1|^{L+1}.$$

Consequently

$$\psi_x(t) = 1 + \sum_{l=1}^{L}\frac{\varphi(l,x)}{l!}(\mathrm{e}^{it}-1)^l + O\left(\frac{\varphi(L+1,x)}{(L+1)!}|\,\mathrm{e}^{it}-1|^{L+1}\right)$$

for every natural number L.

According to equality (5.2) we have

$$\lim_{x\to\infty}\psi_x(t) = 1+\sum_{l=1}^{L}\frac{g_l}{l!}(\mathrm{e}^{it}-1)^l + O\left(\frac{g_{L+1}}{(L+1)!}|\,\mathrm{e}^{it}-1|^{L+1}\right)$$
$$= 1+\sum_{l=1}^{\infty}\frac{g_l(\mathrm{e}^{it}-1)^l}{l!} + O\left(\sum_{l=L+1}^{\infty}\frac{2^l g_l}{l!}+\frac{2^{L+1}g_{L+1}}{(L+1)!}\right),$$

where $t\in\mathbb{R}$.

Letting L go to infinity, we can assert that

$$\lim_{x\to\infty}\psi_x(t) = 1+\sum_{l=1}^{\infty}\frac{g_l}{l!}\left(\mathrm{e}^{it}-1\right)^l$$

for every $t\in\mathbb{R}$.

Since the limit function is continuous, (5.1) holds for some random variable X, which has the above characteristic function. Proposition 5.1 is proved. □

6. Main term of the factorial moment

PROPOSITION 6.1. *Let f_x be a set of strongly additive functions with rational argument. Assume $f_x\left(p^{\delta}\right)\in\{0,1\}$ for each prime number p and exponent $\delta\in\{-1,1\}$. Then*

$$\varphi(1,x) = \sum_{\substack{p^{\delta}\in\mathbb{P}_x\\ f_x(p^{\delta})=1}}\frac{1}{p+1}+\varepsilon(x).$$

If

$$\sum_{\substack{p^{\delta}\in\mathbb{P}_x\\ f_x(p^{\delta})=1}}\frac{1}{p}\ll 1, \tag{6.1}$$

then for every natural l

$$\varphi(l,x) = {\sum}^{*}\frac{\Delta_x\left(p_1^{\delta_1}p_2^{\delta_2}\cdots p_l^{\delta_l}\right)}{(p_1+1)(p_2+1)\cdots(p_l+1)}+\varepsilon_l(x). \tag{6.2}$$

PROOF. First we consider the case $l=1$. It is evident that

$$\varphi(1,x) = \frac{1}{\#\mathbb{Q}_x}\sum_{q\in\mathbb{Q}_x}f_x(q) = \sum_{p^{\delta}\in\widetilde{\mathbb{P}}_x}\frac{1}{\#\mathbb{Q}_x}\sum_{p^{\delta}|q}1.$$

Assume $p \in \widetilde{\mathbb{P}}_x$ and x is sufficiently large ($x \geq c_7$). Using Lemma 3.1, we obtain

$$\begin{aligned}\frac{1}{\#\mathbb{Q}_x}\sum_{p|q} 1 &= (\#\mathbb{Q}_x)^{-1}\#\left\{\frac{m}{n} : \frac{mp}{n} \leq 1,\ n \leq x,\ (mp, n) = 1\right\}\\ &= (\#\mathbb{Q}_x)^{-1}\#\left\{\frac{m}{n} \in \mathbb{Q}_x^{(0,1/p]},\ (n, p) = 1\right\}\\ &= \frac{1}{p+1}\left(1 + O\left(\frac{\log x}{x} + \frac{p}{x}\right)\right) = \frac{1}{p+1} + O\left(\frac{\log x}{x}\right).\end{aligned}$$

If $1/p \in \widetilde{\mathbb{P}}_x$ and $x \geq c_7$, we have similarly from Lemma 3.1

$$\frac{1}{\#\mathbb{Q}_x}\sum_{(1/p)|q} 1 = \frac{1}{p+1}\left(1 + O\left(\frac{p\log(x/p)}{x} + \frac{1}{x}\right)\right) = \frac{1}{p+1} + O\left(\frac{\log x}{x}\right).$$

If $p^\delta \in \mathbb{P}_x$ and $p > \hat{x} := x^{1-1/\sqrt{\log x}}$, then

$$\frac{1}{\#\mathbb{Q}_x}\sum_{p^\delta|q} 1 \leq \frac{1}{\#\mathbb{Q}_x}\sum_{n\leq x}\sum_{\substack{m\leq n\\ p^\delta|(m/n)}} 1 \ll \frac{1}{p}$$

for $x \geq c_7$. Hence

$$\begin{aligned}\varphi(1, x) &= \sum_{\substack{p^\delta\in\widetilde{\mathbb{P}}_x\\ p\leq\hat{x}}}\left(\frac{1}{p+1} + O\left(\frac{\log x}{x}\right)\right) + O\left(\sum_{\substack{p^\delta\in\mathbb{P}_x\\ p>\hat{x}}}\frac{1}{p}\right)\\ &= \sum_{p^\delta\in\widetilde{\mathbb{P}}_x}\frac{1}{p+1} + O\left(\frac{\log x}{x}\sum_{p\leq\hat{x}}1 + \sum_{\hat{x}<p\leq x}\frac{1}{p}\right)\\ &= \sum_{p^\delta\in\widetilde{\mathbb{P}}_x}\frac{1}{p+1} + O\left(\log\left(1 - \frac{1}{\sqrt{\log x}}\right)^{-1} + \mathrm{e}^{-\sqrt{\log x}}\right).\end{aligned}$$

Finally $\varphi(1, x) = \sum_{p^\delta\in\widetilde{\mathbb{P}}_x}(p+1)^{-1} + \varepsilon(x)$.

Now let $l \geq 2$. It is easily seen that

$$\varphi(l, x) = \sum{}^{*}\frac{1}{\#\mathbb{Q}_x}\sum_{\substack{q\in\mathbb{Q}_x\\ p_1^{\delta_1}\cdots p_l^{\delta_l}|q}} 1. \tag{6.3}$$

We split (6.3) into four parts and denote them by $\{\varphi(l, x)\}_i$, $i = 1, 2, 3, 4$. Into the first and second sums we include all summands for which $\delta_1, \delta_2, \ldots, \delta_l = 1$ or -1, respectively. The third sum

$$\{\varphi(l, x)\}_3 = \sum_{k=1}^{l-1}\binom{l}{k}\sum_{\substack{\delta_1,\ldots,\delta_k=1\\ \delta_{k+1},\ldots,\delta_l=-1\\ p_{k+1}\cdots p_l\leq\hat{x}}}^{*}\frac{1}{\#\mathbb{Q}_x}\sum_{\substack{q\in\mathbb{Q}_x\\ \frac{p_1\cdots p_k}{p_{k+1}\cdots p_l}|q}} 1.$$

The fourth sum $\{\varphi(l,x)\}_4$ is constructed in the same manner as $\{\varphi(l,x)\}_3$ with the condition $p_{k+1}\cdots p_l \le \hat{x}$ replaced by its opposite $p_{k+1}\cdots p_l > \hat{x}$.

If $\Delta_x(p_1^{\delta_1}\cdots p_l^{\delta_l}) = 1$ for $p_1^{\delta_1}\cdots p_l^{\delta_l}$, we define

$$P_1 = P_1\left(p_1^{\delta_1}\cdots p_l^{\delta_l}\right) = \prod_{\substack{i=1\\ \delta_i=1}}^{l} p_i, \quad P_2 = P_2\left(p_1^{\delta_1}\cdots p_l^{\delta_l}\right) = \prod_{\substack{i=1\\ \delta_i=-1}}^{l} p_i,$$

Let $p_1, p_2, \dots, p_l$ be distinct prime numbers. It follows from Lemma 3.1 that, for x sufficiently large ($x \ge c_7$),

$$\begin{aligned}
(6.4)\qquad \frac{1}{\#\mathbb{Q}_x}\sum_{\substack{q\in\mathbb{Q}_x\\ p_1^{\delta_1}\cdots p_l^{\delta_l}\mid q}} 1 &= (\#\mathbb{Q}_x)^{-1}\#\left\{\frac{m}{n} : \frac{mP_1}{nP_2}\le 1,\ P_2 n\le x,\ (P_1 m, P_2 n)=1\right\}\\
&= (\#\mathbb{Q}_x)^{-1}\#\left\{\frac{m}{n}\in\mathbb{Q}_x^{(0,P_2/P_1]},\ (m,P_2)=(n,P_1)=1\right\}\\
&= \prod_{p\mid P_1P_2}\frac{1}{p+1}\left(1+O_l\left(\frac{P_2\log x}{x}+\frac{P_1}{x}\right)\right)\\
&= \prod_{p\mid P_1P_2}\frac{1}{p+1} + O_l\left(\frac{\log x}{xP_1}+\frac{1}{xP_2}\right).
\end{aligned}$$

On the other hand, for each $p_1^{\delta_1}\cdots p_l^{\delta_l} = P_1/P_2$ we have

$$(6.5)\qquad \frac{1}{\#\mathbb{Q}_x}\sum_{\substack{q\in\mathbb{Q}_x\\ (P_1/P_2)\mid q}} 1 \ll \frac{1}{x^2}\sum_{\substack{n\le x\\ P_2\mid n}}\sum_{\substack{m\le x\\ P_1\mid m}} 1 \le \frac{1}{P_1P_2}$$

for $x \ge c_7$.

Using condition (6.1), expression (6.4) and Landau's inequality (see, for example, [14]),

$$(6.6)\qquad \#\{m\le x : \omega(n)=j\} \ll \frac{x}{(j-1)!}\,\frac{(\log\log x)^{j-1}}{\log x},$$

where j is a fixed natural number, we obtain

$$\begin{aligned}
(6.7)\qquad \{\varphi(l,x)\}_1 &= \sum_{\delta_1,\dots,\delta_l=1}^{*}\frac{\Delta_x(p_1\cdots p_l)}{(p_1+1)\cdots(p_l+1)}\\
&\quad + O_l\left(\frac{\log x}{x}\sum_{\substack{\delta_1,\dots,\delta_l=1\\ p_1\cdots p_l\le x}}^{*}\frac{1}{p_1\cdots p_l}+\frac{1}{x}\sum_{\substack{\delta_1,\dots,\delta_l=1\\ p_1\cdots p_l\le x}}^{*} 1\right)
\end{aligned}$$

$$= \sum_{\delta_1,\ldots,\delta_l=1}^{*} \frac{\Delta_x(p_1\cdots p_l)}{(p_1+1)\cdots(p_l+1)} + \varepsilon_l(x),$$

because the first remainder term is $O_l\big(x^{-1}\log x\big(\sum_{p^\delta\in\widetilde{\mathbb{P}}_x} 1/p\big)^l\big) = O_l\big(x^{-1}\log x\big)$, and the second is $O_l\big(x^{-1}\#\{m\le x : \omega(m)=l\}\big) = O_l\big((\log x)^{-1}(\log\log x)^{l-1}\big)$.

Similarly from (6.1) and (6.4)–(6.6), we have

(6.8) $\{\varphi(l,x)\}_2$

$$= \sum_{\substack{\delta_1,\ldots,\delta_l=-1\\ p_1\cdots p_l\le\hat{x}}}^{*} \frac{\Delta_x(1/(p_1\cdots p_l))}{(p_1+1)\cdots(p_l+1)}$$

$$+ O_l\left(\frac{1}{x}\sum_{\substack{\delta_1,\ldots,\delta_l=-1\\ p_1\cdots p_l\le\hat{x}}}^{*}\left(\frac{1}{p_1\cdots p_l}+\log x\right) + \sum_{\substack{\delta_1,\ldots,\delta_l=-1\\ p_1\cdots p_l>\hat{x}}}^{*}\frac{\Delta_x(1/(p_1\cdots p_l))}{p_1\cdots p_l}\right)$$

$$= \sum_{\delta_1,\ldots,\delta_l=-1}^{*} \frac{\Delta_x(1/(p_1\cdots p_l))}{(p_1+1)\cdots(p_l+1)} + \varepsilon_l(x) + O_l\big(W_l(x)\big),$$

where

$$W_l(x) = \sum_{\substack{\delta_1,\ldots,\delta_l=-1\\ \hat{x}<p_1\cdots p_l\le x}}^{*} \frac{1}{p_1\cdots p_l}.$$

Using (6.1) and (6.4)–(6.6) again, we obtain

(6.9) $\{\varphi(l,x)\}_3$

$$= \sum_{k=1}^{l-1}\binom{l}{k}\sum_{\substack{\delta_1,\ldots,\delta_k=1\\ \delta_{k+1},\ldots,\delta_l=-1\\ p_{k+1}\cdots p_l\le\hat{x}}}^{*} \frac{\Delta_x\left((p_1\cdots p_k)/(p_{k+1}\cdots p_l)\right)}{(p_1+1)\cdots(p_l+1)}$$

$$+ O_l\left(\frac{1}{x}\sum_{k=1}^{l-1}\binom{l}{k}\sum_{\substack{\delta_1,\ldots,\delta_k=1\\ \delta_{k+1},\ldots,\delta_l=-1\\ p_{k+1}\cdots p_l\le\hat{x}}}^{*}\Delta_x\left(\frac{p_1\cdots p_k}{p_{k+1}\cdots p_l}\right)\left(\frac{\log x}{p_1\cdots p_k}+\frac{1}{p_{k+1}\cdots p_l}\right)\right)$$

$$= \sum_{k=1}^{l-1}\binom{l}{k}\sum_{\substack{\delta_1,\ldots,\delta_k=1\\ \delta_{k+1},\ldots,\delta_l=-1}}^{*} \frac{\Delta_x\left(\frac{p_1\cdots p_k}{p_{k+1}\cdots p_l}\right)}{(p_1+1)\cdots(p_l+1)} + \varepsilon_l(x) + O_l\left(\sum_{k=1}^{l-1}W_{l-k}(x)\right).$$

Finally, inequality (6.1) and estimate (6.5) imply that

$$\{\varphi(l,x)\}_4 = O_l\left(\sum_{k=1}^{l-1}W_{l-k}(x)\right). \tag{6.10}$$

Substituting (6.7)–(6.10) into (6.3), we can assert that

$$\varphi(l,x)=\sum{}^{*}\frac{\Delta_x\left(p_1^{\delta_1}p_2^{\delta_2}\cdots p_l^{\delta_l}\right)}{(p_1+1)(p_2+1)\cdots(p_l+1)}+\varepsilon_l(x)+O_l\left(\sum_{k=1}^{l-1}W_k(x)\right). \tag{6.11}$$

We have that

$$W_1=\sum_{\substack{(1/p_1)\in\widetilde{\mathbb{P}}_x\\ \hat{x}<p\le x}}\frac{1}{p}\le\sum_{\hat{x}<p\le x}\frac{1}{p}=\log\left(1-\frac{1}{\sqrt{\log x}}\right)^{-1}+O\left(\mathrm{e}^{-\sqrt{\log x}}\right).$$

Define $\hat{x}_l=x^{1-(\log x)^{-1/(l+1)}}$ for natural number l. Applying (6.1) for each fixed $k\ge 2$, we obtain

$$\begin{aligned}
&W_k(x)\\
&\le\sum_{\substack{\delta_1,\ldots,\delta_{k-1}=-1\\ p_1\cdots p_{k-1}\le\hat{x}_2}}^{*(k-1)}\frac{1}{p_1\cdots p_{k-1}}\left(\log\left(1+\frac{\log(x/\hat{x})}{\log(\hat{x}/(p_1\cdots p_{k-1}))}\right)\right.\\
&\quad\left.+O\left(\frac{1}{\log(\hat{x}/(p_1\cdots p_{k-1})}\right)\right)+O_k\left(\sum_{\substack{\delta_1,\ldots,\delta_{k-1}=-1\\ \hat{x}_2<p_1\cdots p_{k-1}\le x}}^{*(k-1)}\frac{1}{p_1\cdots p_{k-1}}\right)\\
&=O_k\left(\frac{1}{(\log x)^{1/6}}+\sum_{\substack{\delta_1,\ldots,\delta_{k-1}=-1\\ \hat{x}_2<p_1\cdots p_{k-1}\le x}}^{*(k-1)}\frac{1}{p_1\cdots p_{k-1}}\right)\\
&=O_k\left(\frac{1}{(\log x)^{1/6}}+\sum_{\substack{\delta_1,\ldots,\delta_{k-2}=-1\\ p_1\cdots p_{k-2}\le\hat{x}_3}}^{*(k-2)}\frac{1}{p_1\cdots p_{k-2}}\left(\log\left(1+\frac{\log(x/\hat{x}_2)}{\log(\hat{x}_2/(p_1\cdots p_{k-2}))}\right)\right.\right.\\
&\quad\left.\left.+O\left(\frac{1}{\log(\hat{x}_2/(p_1\cdots p_{k-2})}\right)\right)+\sum_{\substack{\delta_1,\ldots,\delta_{k-2}=-1\\ \hat{x}_3<p_1\cdots p_{k-2}\le x}}^{*(k-2)}\frac{1}{p_1\cdots p_{k-2}}\right)\\
&=O_k\left(\frac{1}{(\log x)^{1/12}}+\sum_{\substack{\delta_1,\ldots,\delta_{k-2}=-1\\ \hat{x}_3<p_1\cdots p_{k-2}\le x}}^{*(k-2)}\frac{1}{p_1\cdots p_{k-2}}\right)\\
&=\cdots=O_k\left(\frac{1}{(\log x)^{1/(k(k+1))}}+\sum_{\substack{(1/p_1)\in\widetilde{\mathbb{P}}_x\\ \hat{x}_k<p_1\le x}}\frac{1}{p_1}\right)=O_k\left(\frac{1}{(\log x)^{1/(k(k+1))}}\right).
\end{aligned}$$

Equality (6.2) follows from (6.11). This completes the proof of Proposition 6.1. □

7. Proof of Theorem 1.1

PROOF (Necessity). Let $\nu_x(f_x(q) < u) \Rightarrow P(X < u)$ for some integer-valued random variable X. It follows from Proposition 4.1 that

$$\varphi(1, x) = \frac{1}{\#\mathbb{Q}_x} \sum_{q \in \mathbb{Q}_x} f_x(q) \ll 1.$$

Hence, according to Proposition 6.1, $\sum_{p^\delta \in \widetilde{\mathbb{P}}_x} 1/p \le c_8$, where c_8 depends on the structure of the limit law of X.

Using (6.5), we have for x sufficiently large

$$\nu_x(f_x(q) = k) = \frac{1}{k!} \sum\nolimits^{*k} \frac{1}{\#\mathbb{Q}_x} \sum_{p_1^{\delta_1} \cdots p_k^{\delta_k} | q} 1 \ll \frac{1}{k!} \sum\nolimits^{*k} \frac{1}{p_1 \cdots p_k} \le \frac{1}{k!} \left(\sum_{p^\delta \in \widetilde{\mathbb{P}}_x} \frac{1}{p} \right)^k .$$

Since $\lim_{x\to\infty} \nu_x(f_x(q) = k) = P(X = k)$ for every $k = 0, 1, 2, \dots,$ we obtain that $P(X = k) \ll c_8^k/k!$, $k \in \mathbb{N}$. Hence

$$g_l = \sum_{k=l}^{\infty} k(k-1)\cdots(k-l+1) P(X = k) \ll \sum_{k=l}^{\infty} \frac{c_8}{(k-l)!} = c_8^l e^{c_8}$$

for each fixed natural l.

The necessity of the condition in Theorem 1.1 now follows from Propositions 5.1 and 6.1. □

PROOF (Sufficiency). Let all the limits in the statement of Theorem 1.1 exist. Since

$$\lim_{x\to\infty} \sum_{p^\delta \in \widetilde{\mathbb{P}}_x} \frac{1}{p+1} = g_1,$$

we have that

$$g_l = \lim_{x\to\infty} \sum\nolimits^{*} \frac{\Delta_x\left(p_1^{\delta_1} \cdots p_l^{\delta_l}\right)}{(p_1+1)\cdots(p_l+1)} \le \left(\lim_{x\to\infty} \sum_{p^\delta \in \widetilde{\mathbb{P}}_x} \frac{1}{p+1} \right)^l = g_1^l.$$

Therefore the series $\sum_{l=1}^{\infty} (2^l g_l / l!)$ converges.

On the other hand, Proposition 6.1 implies that $\lim_{x\to\infty} \varphi(l, x) = g_l$ for each natural l.

The statement of Theorem 1.1 now follows from Proposition 5.1. □

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Department of Probability
Theory and Number Theory
Vilnius University
Naugarduko 24
03225 Vilnius
Lithuania
e-mail: jonas.siaulys@mif.vu.lt

Department of Mathematical Informatics
Vilnius University
Naugarduko 24
03225 Vilnius
Lithuania
e-mail: gediminas.stepanauskas@maf.vu.lt