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RADIAL DISTRIBUTIONS OF JULIA SETS OF MEROMORPHIC FUNCTIONS

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Abstract

We consider a meromorphic function of finite lower order that has ∞ as its deficient value or as its Borel exceptional value. We prove that the set of limiting directions of its Julia set must have a definite range of measure.

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1. Introduction

Let f be a meromorphic function defined in the complex plane $\mathbb{C}$ or on the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. The *Fatou set* $F(f)$ of f is the subset of $\overline{\mathbb{C}}$ where the iterates f^n $(n = 1, 2, \dots)$ of f are defined and $\{f^n\}$ forms a normal family. The complement of $F(f)$ is called the *Julia set*. It is obvious that $F(f)$ is an open set and $J(f)$ is closed. In general, the Julia set is very complicated.

Let $f(z)$ be a transcendental meromorphic function in the complex plane. Suppose that $\arg z = \theta$ is a ray from the origin. We say that θ is a *limiting direction of* $J(f)$ if, for any $\varepsilon > 0$ and any $R > 0$, the domain $\{z : \theta - \varepsilon < \arg z < \theta + \varepsilon, |z| > R\}$ has nonempty intersection with $J(f)$. We define the set $E \in [0, 2\pi)$ to be all the limiting directions of $J(f)$.

Baker first proved in [3] that, for a transcendental entire function f, the set E contains infinitely many points. Later Qiao [6] proved that if the function is of finite lower order, then E contains an interval whose length depends on the lower order.

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In [8], the authors considered the case of meromorphic functions with ∞ as their deficient value and, under some additional conditions, they proved the set E has a definitely positive measure.

In this paper, we remove the additional condition in [8, Theorem 1] and prove the following result.

THEOREM 1.1. *Let $f(z)$ be a meromorphic function of lower order $\mu < \infty$ with deficiency $\delta(\infty, f) > 0$. Then*

$$\operatorname{mes} E \geq \min\left\{2\pi, \frac{4}{\mu}\arcsin\sqrt{\frac{\delta(\infty, f)}{2}}\right\}.$$

If ∞ is a Borel exceptional value, then we can prove E contains an interval with a definite length. Let $f(z)$ be a meromorphic function in $\mathbb{C}$ of order $0 < \lambda < \infty$. Recall that $a \in \overline{\mathbb{C}}$ is a *Borel exceptional value of* $f(z)$ if it satisfies

$$\varlimsup_{r\to\infty} \frac{\log n(r, f = a)}{\log r} < \lambda,$$

where $n(r, f = a)$ is the counting function in value distribution theory of meromorphic functions.

In this case,we have the following result.

THEOREM 1.2. *Let $f(z)$ be a transcendental meromorphic function of finite order $\lambda > 0$. Suppose that ∞ is a Borel exceptional value of $f(z)$. Then there exists a closed interval $I \in R$ such that all $\theta \in I$ are limiting directions of $J(f)$ and* $\operatorname{mes} I \geq \pi/\max(1/2, \lambda)$.

The proofs of the theorems depend strongly on the Nevanlinna theory of meromorphic functions. The reader can refer to [4] and [7] for the basic definitions and results in value distribution theory of meromorphic functions, in particular for the symbols such as $T(r, f)$, $N(r, f)$, and so on.

2. Proof of Theorems 1.1 and 1.2

The following lemma, which is a special form of the result proved in [2], is sufficient to prove our theorem.

LEMMA 2.1 ([2]). *Let $f(z)$ be a meromorphic function of finite lower order μ. Suppose ∞ is a deficient value of f with $\delta(\infty, f) > 0$. Let $M_j \to +\infty$ $(j \to \infty)$ and define*

$$E(r) = \left\{\theta : \left|f(re^{i\theta})\right| > r^{M_j}\right\}. \tag{2.1}$$

Then there is a sequence $\{r_j\}$ *with* $r_j \to \infty$ $(j \to \infty)$ *such that*

$$\liminf_{j\to\infty} \operatorname{mes} E(r_j) \geq \min\left\{2\pi, \frac{4}{\mu}\arcsin\sqrt{\frac{\delta(\infty, f)}{2}}\right\}.$$

In the following we denote the angular domain $\{z : \theta - \delta < \arg(z - z_0) < \theta + \delta\}$ by $\Omega(z_0, \theta, \delta)$, where $\theta \in \mathbb{R}$ and $0 < \delta < \pi$. We state Lemma 1 from [6] in the following form.

LEMMA 2.2 ([6]). *Let* $f(z)$ *be analytic in* $\Omega(z_0, \theta, \delta)$. *Suppose that* $f(\Omega(z_0, \theta, \delta))$ *is contained in a simply connected hyperbolic domain in* $\mathbb{C}$. *Then*

$$|f(z)| < O(|z|)^{\pi/\delta}, \quad z \in \Omega(z_0, \theta, \delta')$$

for any $\delta' \in (0, \delta)$.

The proof of Lemma 2.2 is the same as that of [6, Lemma 1]. For meromorphic functions, the form we state in Lemma 2.2 is more convenient for our use.

PROOF OF THEOREM 1.1. Set

$$\sigma = \min\left\{2\pi, \frac{4}{\mu}\arcsin\sqrt{\frac{\delta(\infty, f)}{2}}\right\}.$$

We conversely suppose that $\operatorname{mes} E < \sigma$ and seek a contradiction.

Take a $t > 0$ such that $\sigma - \operatorname{mes} E > t > 0$. Since E is closed, $S = [0, 2\pi) \setminus E$ consists of (at most countably many) open intervals I from which we can find finitely many open intervals I_i $(i = 1, 2, \ldots, m)$ such that $\operatorname{mes}\left(S \setminus \bigcup_{i=1}^{m} I_i\right) < K/2$, where $K = \sigma - \operatorname{mes} E - t > 0$. By the assumption of Theorem 1.1, it follows from Lemma 2.1 that there exists a sequence $\{r_j\}$ of positive numbers such that $\operatorname{mes} E(r_j) > \sigma - t > 0$, where $E(r_j)$ is defined as in (2.1). Obviously we have

$$\operatorname{mes}(E(r_j) \cap S) = \operatorname{mes}\left(E(r_j) \setminus (E \cap E(r_j))\right) \geq \operatorname{mes} E(r_j) - \operatorname{mes} E \geq K > 0.$$

Thus there exists an open interval $I = I_{i_0} \subset S$ such that for infinitely many j

$$\operatorname{mes}(E(r_j) \cap I) > \frac{K}{2m} > 0. \tag{2.2}$$

By passing to a subsequence if it is necessary, we can assume that for each j, (2.2) holds. Write $I = (a, b)$. Take a positive number α such that

$$\operatorname{mes}(E(r_j) \cap I_\alpha) > \frac{K}{3m} > 0, \quad j = 1, 2, \ldots, \tag{2.3}$$

where we denote by I_α the interval $(a+\alpha, b-\alpha)$, $(0 < 8\alpha < b-a)$. It is easy to see from $I \cap E = \emptyset$ that there exists a positive R such that

$$\Omega(R, I_\alpha) = \left\{z \in \mathbb{C} : |z| \geq R \text{ and } \arg z \in I_\alpha\right\} \subset F(f).$$

By choosing a point z_0 on the bisector of I, we see that the angular domain

$$\left\{z : z \in \mathbb{C}; |z - z_0| \geq 0 \text{ and } \arg(z - z_0) \in I_\alpha\right\} \subset F(f).$$

So without loss of generality, we can suppose $\Omega(0, I_\alpha) \subset F(f)$.

In the following we assume that α is a fixed number such that (2.3) holds. Since $\Omega(0, I_\alpha) \subset F(f)$, $f(z)$ has no pole in Ω and also does not take the values in $J(f)$. Take two fixed points $w_j \in J(f)$, $(j = 1, 2)$. Thus f is meromorphic in $\Omega(0, I_\alpha)$ and misses three points including infinity. Therefore the family $\{f \circ \varphi\}$, where φ is a conformal automorphism of $\Omega(0, I_\alpha)$, is normal in $\Omega(0, I_\alpha)$ (compare [5]). So take a sequence of automorphisms $\varphi_j(z)$ of $\Omega(0, I_\alpha)$ such that $\varphi_j(z) = r_j z$, $r_j = |z_j|$. We see that $f \circ \varphi_j$ converges to a function g, which is either analytic or identically ∞ in $\Omega(0, I_\alpha)$. Now f is unbounded on $\{z_j\}$ and hence $g \equiv \infty$. Thus $f \circ \varphi_j$ converges uniformly on $\{z : |z| = 1\} \cap \Omega(0, I_\alpha)$ to ∞. This implies that

$$\lim_{\substack{z \in L_j \\ j \to \infty}} |f(z)| = +\infty, \tag{2.4}$$

where $L_j = \{z : |z| = r_j\} \cap \Omega(0, I_{2\alpha})$.

In the following we prove the number of bounded components of $\mathbb{C} \setminus f(\Omega')$, where $\Omega' = \Omega(0, I_{2\alpha})$ is at most one. If our conclusion is wrong, then we can take two bounded components U_1, U_2 from $\mathbb{C} \setminus f(\Omega')$. Choose two Jordan curves γ_1, γ_2 in $f(\Omega')$ such that γ_1 and γ_2 do not pass through critical values of $f(z)$, $U_1 \subset \operatorname{int}(\gamma_1)$, $U_2 \subset \operatorname{int}(\gamma_2)$, and $\overline{\operatorname{int}(\gamma_1)} \cap \overline{\operatorname{int}(\gamma_2)} = \emptyset$. We choose a branch of f^{-1} such that $f^{-1}(\gamma_1)$, $f^{-1}(\gamma_2) \subset \Omega'$. Then $f^{-1}(\gamma_1) \cap f^{-1}(\gamma_2) = \emptyset$. Take a fixed $R > 0$ such that $\gamma_1, \gamma_2 \subset \{z : |z| < R\}$. Noting that (2.4) holds, we see that every component of $f^{-1}(\gamma_j)$, $j = 1, 2$, is bounded. Since the interior of γ_j contains some points in $J(f)$, it is easy to see that any component of $f^{-1}(\gamma_j)$, $j = 1, 2$, cannot be closed. So it is a Jordan arc. Now we take fixed j_0 such that $|f(z)| > R$ for all $z \in L_j (j > j_0)$ and $f^{-1}(\gamma_j) \cap \Omega' \cap \{|z| < r_{j_0}\} \neq \emptyset$, $j = 1, 2$.

Take a component of $f^{-1}(\gamma_j)$, $j = 1$ or 2, in $\Omega'_{j_0} = \Omega' \cap \{|z| < r_{j_0}\}$. Let σ_j be a component of $f^{-1}(\gamma_j)$ in Ω'_{j_0}, $j = 1, 2$. It is easy to see that σ_1 is homotopic to σ_2. As $f(z)$ is analytic on $\overline{\Omega}'_{j_0}$, we deduce that $\gamma_1 = f(\sigma_1)$ is homotopic to $\gamma_2 = f(\sigma_2)$. This is a contradiction, which proves our claim.

For a transcendental meromorphic function f, its Julia set is an unbounded set in $\mathbb{C}$. If $J(f)$ contains an unbounded component Γ, then $\mathbb{C} \setminus \Gamma$ is a simply connected hyperbolic domain D and $f(\Omega') \subset D$. Otherwise all components of $J(f)$ are bounded and there are infinitely many bounded components in $J(f)$. Using the fact we just proved, it is not hard to find a simply connected hyperbolic domain $D \subset \mathbb{C}$ such that $f(\Omega') \subset D$.

Using Lemma 2.2, there exists a positive number M such that $|f(z)| < |z|^M$ for all sufficiently large $z \in \Omega'$. On the other hand, there are $z_j \in L_j$ such that $|f(z_j)| > |z_j|^{M_j}$ for all sufficiently large j. Noting that $M_j \to \infty$, we get a contradiction and Theorem 1.1 is proved. $\square$

Proof of Theorem 1.2. Let $f(z)$ be a transcendental meromorphic function in the complex domain of order $0 < \lambda < \infty$. If ∞ is the Borel exceptional value of f, then

$$\varlimsup_{r\to\infty} \frac{\log N(r, f)}{\log r} < \lambda.$$

Thus $f(z)$ must have the form $f(z) = G(z)/\Pi(z)$, where $G(z)$ is a transcendental entire function and $\Pi(z)$ is an entire function that is the typical product of the poles of $f(z)$. The functions $G(z)$ and $\Pi(z)$ have the following properties.

$$\varlimsup_{r\to\infty} \frac{\log T(r, \Pi)}{\log r} = \varlimsup_{r\to\infty} \frac{\log m(r, \Pi)}{\log r} = \varlimsup_{r\to\infty} \frac{\log N(r, f)}{\log r} = \sigma < \lambda$$

and

$$\varlimsup_{r\to\infty} \frac{\log T(r, f)}{\log r} = \varlimsup_{r\to\infty} \frac{\log T(r, G)}{\log r} = \varlimsup_{r\to\infty} \frac{\log m(r, G)}{\log r} = \lambda.$$

Since $G(z)$ is a transcendental entire function of finite order λ, it follows from the Phragmén-Lindelöf Theorem that there is an interval (a, b) with $b-a \geq \min(2\pi, \pi/\lambda)$ such that

$$\limsup_{r\to\infty} \frac{\log\log |G(re^{i\theta})|}{\log r} = \lambda$$

for all $\theta \in (a, b)$.

We are now able to prove $[a, b] \subset E$. If it is not true, then there is an subinterval $I \subset (a, b)$ such that the angular domain $\Omega(\{|z| > R, \arg z \in I\}) \subset F(f)$. Let $\arg z = \theta_0$ be the bisector of I. Then we have $\log|\Pi(re^{i\theta_0})| < r^{\sigma+\varepsilon}$, and

$$\begin{aligned}\log|f(r_je^{i\theta_0})| &= \log\left|\frac{G(r_je^{i\theta_0})}{\Pi(r_je^{i\theta_0})}\right| = \log|G(r_je^{i\theta_0})| - \log|\Pi(r_je^{i\theta_0})| \\ &> r_j^{\lambda-\varepsilon} - r_j^{\sigma+\varepsilon} = r_j^{\lambda-\varepsilon'}\end{aligned}$$

for some $\varepsilon' > 0$. Thus we can find a sequences of points $\{z_j\}$ on the bisector such that $\log|f(z_j)| > |z_j|^{\lambda-\varepsilon}$ for some $\varepsilon > 0$.

Therefore, as in the proof of Theorem 1.1, we can find a sequence of

$$L_j = \left\{|z_j|e^{i\theta} : a + \alpha \le \theta \le b - \alpha\right\}, \quad 0 < \alpha < (b-a)/8,$$

such that (2.4) holds.

By the same argument of the proof of Theorem 1.1, we arrive at a contradiction. The proof of Theorem 1.2 is completed. □

REMARK. Theorem 1.2 is also true for meromorphic functions of finite lower order μ with poles having order of growth less than μ. In fact in this case, as in the proof of Theorem 1.2, f can be written as $f(z) = G(z)/\Pi(z)$, where $G(z)$ is an entire function of finite lower order μ, and $\Pi(z)$ is an entire function with order less than μ. So applying a theorem of Baernstein in [1] to $G(z)$, we get a similar result as in Theorem 1.2.

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