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C^*-ALGEBRAS ASSOCIATED WITH PRESENTATIONS OF SUBSHIFTS II. IDEAL STRUCTURE AND LAMBDA-GRAPH SUBSYSTEMS

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Abstract

A λ-graph system is a labeled Bratteli diagram with shift transformation. It is a generalization of finite labeled graphs and presents a subshift. In *Doc. Math.* **7** (2002) 1–30, the author constructed a C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with a λ-graph system $\mathfrak{L}$ from a graph theoretic view-point. If a λ-graph system comes from a finite labeled graph, the algebra becomes a Cuntz-Krieger algebra. In this paper, we prove that there is a bijective correspondence between the lattice of all saturated hereditary subsets of $\mathfrak{L}$ and the lattice of all ideals of the algebra $\mathcal{O}_{\mathfrak{L}}$, under a certain condition on $\mathfrak{L}$ called (II). As a result, the class of the C^*-algebras associated with λ-graph systems under condition (II) is closed under quotients by its ideals.

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1. Introduction

In [7], Cuntz and Krieger presented a class of C^*-algebras associated with finite square matrices with entries in $\{0, 1\}$. The C^*-algebras are called *Cuntz-Krieger algebras*. They are simple if the matrices are irreducible with condition (I). Cuntz-Krieger observed that the C^*-algebras have a close relationship to topological Markov shifts ([7]). The topological Markov shifts form a subclass of subshifts. For a finite set Σ, a *subshift* (Λ, σ) is a topological dynamical system defined by a closed shift-invariant subset Λ of the compact set $\Sigma^{\mathbb{Z}}$ of all bi-infinite sequences of Σ with shift transformation σ. In [21] (compare [25, 5]), the author generalized the class of the Cuntz-Krieger algebras to a class of C^*-algebras associated with subshifts. He also introduced several topological conjugacy invariants and presentations for subshifts by using K-theory and algebraic structure of the associated C^*-algebras with the subshifts in [23]. For presentation of subshifts, notions of the λ-graph system and symbolic matrix system have been introduced ([23]). They are generalizations of the λ-graph (labeled graph) and the symbolic matrix for sofic subshifts to general subshifts.

We henceforth denote by $\mathbb{Z}_+$ the set of all nonnegative integers. Let Σ be a finite set that is called an alphabet. A λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ consists of a vertex set $V = \bigcup_{l\in\mathbb{Z}_+} V_l$, an edge set $E = \bigcup_{l\in\mathbb{Z}_+} E_{l,l+1}$, a labeling map $\lambda : E \to \Sigma$ and a surjective map $\iota(= \iota_{l,l+1}) : V_{l+1} \to V_l$ for each $l \in \mathbb{Z}_+$ with a certain compatible condition, called the local property. Its matrix presentation $(\mathcal{M}_{l,l+1}, I_{l,l+1}), l \in \mathbb{Z}_+$ is called a symbolic matrix system, denoted by $(\mathcal{M}, I)$. The λ-graph systems give rise to subshifts by gathering label sequences appearing in the labeled Bratteli diagrams of the λ-graph systems. Conversely, there is a canonical method to construct a λ-graph system from an arbitrary subshift [23]. It is called the *canonical λ-graph system* for subshift Λ.

In [24], the author constructed C^*-algebras from λ-graph systems and studied their structure. Let $\mathfrak{L} = (V, E, \lambda, \iota)$ be a λ-graph system over alphabet Σ. Let $\{v_1^l, \ldots, v_{m(l)}^l\}$ be the set of the vertex V_l. We henceforth assume that a λ-graph system $\mathfrak{L}$ is left-resolving, that is, there are no distinct edges with the same label and the same terminal vertex. The C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is realized as a universal unique C^*-algebra subject to certain operator relations among generating partial isometries S_α, corresponding to the symbols $\alpha \in \Sigma$ and projections E_i^l corresponding to the vertices $v_i^l \in V_l$, $i = 1, \ldots, m(l)$, $l \in \mathbb{Z}_+$, encoded by the concatenation rule of $\mathfrak{L}$. Irreducibility and aperiodicity for finite directed graphs have been generalized to λ-graph systems in [24]. If $\mathfrak{L}$ satisfies condition (I), a condition generalizing condition (I) for finite square matrices defined by [7], and is irreducible, then the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is simple. In particular, if $\mathfrak{L}$ is aperiodic, then $\mathcal{O}_{\mathfrak{L}}$ is simple and purely infinite ([24], compare [27]).

In this paper, we investigate ideal structures of the C^*-algebras $\mathcal{O}_{\mathfrak{L}}$. The discussions are based on a line of Cuntz's paper [6] in which the ideal structure of the Cuntz-Krieger algebras were studied (compare [13]). We generalize condition (II) for finite directed graphs, defined in [6], to λ-graph systems. By considering saturated hereditary subsets of $\mathfrak{L}$ with respect to arrows of edges, we show the following theorem.

THEOREM A (Proposition 3.5, Theorem 3.6). *Suppose that $\mathfrak{L}$ satisfies condition* (II). *There is a bijective correspondence between the lattice of all saturated hereditary subsets of $\mathfrak{L}$ and the lattice of all ideals of the algebra $\mathcal{O}_{\mathfrak{L}}$. Furthermore, for any ideal $\mathcal{I}$ of $\mathcal{O}_{\mathfrak{L}}$, the quotient C^*-algebra $\mathcal{O}_{\mathfrak{L}}/\mathcal{I}$ is isomorphic to the C^*-algebra $\mathcal{O}_{\mathfrak{L}^{\setminus C_{\mathcal{I}}}}$ associated with the λ-graph system $\mathfrak{L}^{\setminus C_{\mathcal{I}}}$, obtained by removing the corresponding saturated hereditary subset $C_{\mathcal{I}}$ for $\mathcal{I}$.*

COROLLARY B. *In the λ-graph systems satisfying condition* (II), *the class of the C^*-algebras associated with λ-graph systems is closed under quotients by ideals.*

By Corollary B, it is expected that rich examples of simple purely infinite nuclear C^*-algebras of this class live outside Cuntz-Krieger algebras (compare [24, Theorem 7.7], [16], [26] and [20]). We further study the structure of an ideal of $\mathcal{O}_{\mathfrak{L}}$ in Section 4. We prove that an ideal of $\mathcal{O}_{\mathfrak{L}}$ is stably isomorphic to the C^*-subalgebra of $\mathcal{O}_{\mathfrak{L}}$ associated with the corresponding saturated hereditary subset of V (Theorem 4.3). As a result, the K-theory formulae for ideals of $\mathcal{O}_{\mathfrak{L}}$ are presented in terms of the corresponding saturated hereditary subsets of V (Theorem 4.5).

If a λ-graph system $\mathfrak{L}$ comes from a finite directed graph G, the associated C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ becomes a Cuntz-Krieger algebra $\mathcal{O}_{A_G}$ for its adjacency matrix A_G with entries in $\{0, 1\}$. The results of this paper, Theorem A, Corollary B, Theorem 4.3, Theorem 4.5, and Proposition 4.6 are generalizations of Cuntz's result [6, Theorem 2.5] for Cuntz-Krieger algebras. Other generalizations of Cuntz-Krieger algebras from this graph point of view have been studied by [2, 10, 12, 15, 17, 18, 30, 34] and [35]. Related discussions for C^*-algebras generated by Hilbert C^*-bimodules can be found in [14].

2. Review of the C^*-algebras associated with λ-graph systems

Recall that a λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ over an alphabet Σ is a directed Bratteli diagram with vertex set $V = \bigcup_{l\in\mathbb{Z}_+} V_l$ and edge set $E = \bigcup_{l\in\mathbb{Z}_+} E_{l,l+1}$ that is labeled with symbols in Σ by $\lambda : E \to \Sigma$, and that is supplied with surjective maps $\iota(= \iota_{l,l+1}) : V_{l+1} \to V_l$ for $l \in \mathbb{Z}_+$. Here, both the vertex sets V_l, $l \in \mathbb{Z}_+$ and the edge sets $E_{l,l+1}$, $l \in \mathbb{Z}_+$ are finite disjoint sets. An edge e in $E_{l,l+1}$ has its source vertex $s(e)$ in V_l and its terminal vertex $t(e)$ in V_{l+1} respectively. Every vertex in V has a successor and every vertex in V_l for $l \in \mathbb{N}$ has a predecessor. It is required that there exists a bijective correspondence, which preserves labels, between $\{e \in E_{l,l+1} \mid t(e) = v, \iota(s(e)) = u\}$ and $\{e \in E_{l-1,l} \mid s(e) = u, t(e) = \iota(v)\}$ for all pairs of vertices $u \in V_{l-1}$ and $v \in V_{l+1}$. This property of the λ-graph systems is called the *local property*. We call an edge $e \in E_{l,l+1}$ a λ*-edge* and a connecting finite sequence of λ-edges a λ*-path*. For $u, v \in V$, if $\iota(v) = u$, we say that there exists an ι*-edge* from v to u. Similarly we use the term ι*-path*. We denote by $\{v_1^l, v_2^l, \ldots, v_{m(l)}^l\}$ the vertex set V_l of V at level l. A finite labeled graph (G, λ) over Σ with underlying finite directed graph $G = (V, E)$ and labeling map $\lambda : E \to \Sigma$ yields a λ-graph system $\mathfrak{L}_{(G,\lambda)}$ by setting $V_l = V$, $E_{l,l+1} = E$ for $l \in \mathbb{Z}_+$ and $\iota = \mathrm{id}$ (compare [24, Section 7]).

Let us now briefly review the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ associated with the λ-graph system $\mathfrak{L}$, which was originally constructed in [24] to be a groupoid C^*-algebra of a groupoid of a continuous graph obtained by $\mathfrak{L}$ (compare [8, 9, 31]). The C^*-algebras $\mathcal{O}_{\mathfrak{L}}$ are generalization of the C^*-algebras associated with subshifts. That is, if the λ-graph system is the canonical λ-graph system for a subshift Λ, the constructed C^*-algebra coincides with the C^*-algebra $\mathcal{O}_\Lambda$ associated with the subshift Λ in [26] (compare [5]).

Let $\mathfrak{L} = (V, E, \lambda, \iota)$ be a left-resolving λ-graph system over Σ. We denote by Λ the presented subshift $\Lambda_{\mathfrak{L}}$ by $\mathfrak{L}$. We denote by Λ^k the set of admissible words in Λ of length k. We set $\Lambda^* = \bigcup_{k=0}^{\infty} \Lambda^k$, where Λ^0 denotes the empty word. Define the transition matrices $A_{l,l+1}$, $I_{l,l+1}$ of $\mathfrak{L}$ by setting for $i = 1, 2, \ldots, m(l)$, $j = 1, 2, \ldots, m(l+1)$, $\alpha \in \Sigma$,

$$A_{l,l+1}(i, \alpha, j) = \begin{cases} 1 & \text{if } s(e) = v_i^l, \lambda(e) = \alpha, t(e) = v_j^{l+1} \text{ for some } e \in E_{l,l+1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$I_{l,l+1}(i, j) = \begin{cases} 1 & \text{if } \iota_{l,l+1}(v_j^{l+1}) = v_i^l, \\ 0 & \text{otherwise.} \end{cases}$$

The C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is realized as the universal unital C^*-algebra generated by partial isometries S_α, $\alpha \in \Sigma$ and projections E_i^l, $i = 1, 2, \ldots, m(l)$, $l \in \mathbb{Z}_+$ subject to the following operator relations called $(\mathfrak{L})$

$$\sum_{\alpha \in \Sigma} S_\alpha S_\alpha^* = 1, \tag{2.1}$$

$$\sum_{i=1}^{m(l)} E_i^l = 1, \quad E_i^l = \sum_{j=1}^{m(l+1)} I_{l,l+1}(i, j) E_j^{l+1}, \tag{2.2}$$

$$S_\beta S_\beta^* E_i^l = E_i^l S_\beta S_\beta^*, \tag{2.3}$$

$$S_\beta^* E_i^l S_\beta = \sum_{j=1}^{m(l+1)} A_{l,l+1}(i, \beta, j) E_j^{l+1}, \tag{2.4}$$

for $\beta \in \Sigma$, $i = 1, 2, \ldots, m(l)$, $l \in \mathbb{Z}_+$. It is nuclear ([24, Proposition 5.6]). The relations (2.1), (2.3) and (2.4) yield the relations

$$E_i^l = \sum_{\alpha \in \Sigma} \sum_{j=1}^{m(l+1)} A_{l,l+1}(i, \alpha, j) S_\alpha E_j^{l+1} S_\alpha^*, \tag{2.5}$$

for $i = 1, 2, \ldots, m(l)$, $l \in \mathbb{Z}_+$. For a word $\mu = \mu_1 \cdots \mu_k \in \Lambda^k$, we set $S_\mu = S_{\mu_1} \cdots S_{\mu_k}$. Then the algebra of all finite linear combinations of the elements of the form $S_\mu E_i^l S_\nu^*$, for $\mu, \nu \in \Lambda^*$, $i = 1, \ldots, m(l)$, $l \in \mathbb{Z}_+$, is a dense $*$-subalgebra of $\mathcal{O}_{\mathfrak{L}}$. We define three C^*-subalgebras $\mathcal{F}_k^l$, $(k \le l)$, $\mathcal{F}_k^\infty$ and $\mathcal{F}_{\mathfrak{L}}$ of $\mathcal{O}_{\mathfrak{L}}$. The first one, $\mathcal{F}_k^l$, is generated by $S_\mu E_i^l S_\nu^*$, $\mu, \nu \in \Lambda^k$, $i = 1, \ldots, m(l)$, the second one, $\mathcal{F}_k^\infty$, is generated by $\mathcal{F}_k^l$, $k \le l$, $l \in \mathbb{Z}_+$, and the third one, $\mathcal{F}_{\mathfrak{L}}$, is generated by $\mathcal{F}_k^\infty$, $k \in \mathbb{Z}_+$. There exist two embeddings $\iota_{l,l+1} : \mathcal{F}_k^l \hookrightarrow \mathcal{F}_k^{l+1}$, coming from the second relation of (2.2) and $\lambda_{k,k+1} : \mathcal{F}_k^l \hookrightarrow \mathcal{F}_{k+1}^{l+1}$, coming from (2.5). The latter embeddings induce an embedding of $\mathcal{F}_k^\infty$ into $\mathcal{F}_{k+1}^\infty$ that we also denote by $\lambda_{k,k+1}$. Since the algebra $\mathcal{F}_k^l$ is finite dimensional, the embeddings $\iota_{l,l+1} : \mathcal{F}_k^l \hookrightarrow \mathcal{F}_k^{l+1}$, $l \in \mathbb{N}$ yield the AF-algebra $\mathcal{F}_k^\infty$, and the embeddings $\lambda_{k,k+1} : \mathcal{F}_k^\infty \hookrightarrow \mathcal{F}_{k+1}^\infty$, $k \in \mathbb{N}$ yield the AF-algebra $\mathcal{F}_{\mathfrak{L}}$.

For a vertex $v_i^l \in V_l$, set

$$\Gamma^+(v_i^l) = \left\{ (\alpha_1, \alpha_2, \ldots) \in \Sigma^{\mathbb{N}} \;\middle|\; \begin{array}{l} \text{there exists an edge } e_{n,n+1} \in E_{n,n+1} \text{ for } n \ge l \\ \text{such that } v_i^l = s(e_{l,l+1}),\, t(e_{n,n+1}) = s(e_{n+1,n+2}), \\ \lambda(e_{n,n+1}) = \alpha_{n-l+1} \end{array} \right\},$$

the set of all label sequences in $\mathfrak{L}$ starting at v_i^l. We say that $\mathfrak{L}$ satisfies condition (I) if for each $v_i^l \in V$, the set $\Gamma^+(v_i^l)$ contains at least two distinct sequences. Under condition (I), the algebra $\mathcal{O}_{\mathfrak{L}}$ can be realized as the unique C^*-algebra subject to the relations ($\mathfrak{L}$). This means that if $\widehat{S}_\alpha$, $\alpha \in \Sigma$, and $\widehat{E}_i^l$, $i = 1, \ldots, m(l)$, $l \in \mathbb{Z}_+$, are another family of nonzero partial isometries and nonzero projections satisfying the relations ($\mathfrak{L}$), then the map $S_\alpha \to \widehat{S}_\alpha$, $E_i^l \to \widehat{E}_i^l$ extends to an isomorphism from $\mathcal{O}_{\mathfrak{L}}$ onto the C^*-algebra $\widehat{\mathcal{O}}_{\mathfrak{L}}$ generated by $\widehat{S}_\alpha$, $\alpha \in \Sigma$, and $\widehat{E}_i^l$, $i = 1, \ldots, m(l)$, $l \in \mathbb{Z}_+$ ([24, Theorem 4.3]).

Let $\mathcal{A}_{\mathfrak{L}}$ be the C^*-subalgebra of $\mathcal{O}_{\mathfrak{L}}$ generated by the projections E_i^l, $i = 1, 2, \ldots, m(l)$, $l \in \mathbb{Z}_+$. Let $\Omega_{\mathfrak{L}}$ the projective limit of the system $\iota_{l,l+1} : V_{l+1} \to V_l$, $l \in \mathbb{Z}_+$. We endow $\Omega_{\mathfrak{L}}$ with the projective limit topology so that it is a compact Hausdorff space. An element of $\Omega_{\mathfrak{L}}$ is called an *ι-orbit*. By the universality of the algebra $\mathcal{O}_{\mathfrak{L}}$ the algebra $\mathcal{A}_{\mathfrak{L}}$ is isomorphic to the commutative C^*-algebra $C(\Omega_{\mathfrak{L}})$ of all complex valued continuous functions on $\Omega_{\mathfrak{L}}$. As a corollary of [24, Theorem 4.3], if $\mathfrak{L}$ satisfies condition (I), for a nonzero ideal $\mathcal{I}$ of $\mathcal{O}_{\mathfrak{L}}$, we have $\mathcal{I} \cap \mathcal{A}_{\mathfrak{L}} \ne \{0\}$.

A λ-graph system $\mathfrak{L}$ is said to be *irreducible* if for a vertex $v \in V_l$ and an ι-orbit $x = (x_i)_{i \in \mathbb{Z}_+} \in \Omega_{\mathfrak{L}}$, there exists a λ-path starting at v and terminating at x_{l+N} for some $N \in \mathbb{N}$. Define a positive operator $\lambda_{\mathfrak{L}}$ on $\mathcal{A}_{\mathfrak{L}}$ by $\lambda_{\mathfrak{L}}(X) = \sum_{\alpha \in \Sigma} S_\alpha^* X S_\alpha$ for $X \in \mathcal{A}_{\mathfrak{L}}$. The operator $\lambda_{\mathfrak{L}}$ on $\mathcal{A}_{\mathfrak{L}}$ induces the embedding $\mathcal{F}_k^\infty \subset \mathcal{F}_{k+1}^\infty$, $k \in \mathbb{N}$ so as to define the AF-algebra $\mathcal{F}_{\mathfrak{L}} = \varinjlim \mathcal{F}_k^\infty$. We say that $\lambda_{\mathfrak{L}}$ is *irreducible* if there exists no non-trivial ideal of $\mathcal{A}_{\mathfrak{L}}$ invariant under $\lambda_{\mathfrak{L}}$. Then $\mathfrak{L}$ is irreducible if and only if $\lambda_{\mathfrak{L}}$ is irreducible. If $\mathfrak{L}$ is irreducible with condition (I), the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is simple ([24, Theorem 4.7], compare [27]).

3. Hereditary subsets of the vertices and ideals

This section and the next section are the main parts of this paper. In what follows we assume that a λ-graph system $\mathfrak{L} = (V, E, \lambda, \iota)$ over Σ is left-resolving and satisfies condition (I). We mean by an ideal of a C^*-algebra a closed two-sided ideal. Recall that the vertex set V_l is denoted by $\{v_1^l, \dots, v_{m(l)}^l\}$.

For $v_i^l \in V_l$ and $v_j^{l+1} \in V_{l+1}$, we write $v_i^l \overset{\iota}{\geq} v_j^{l+1}$ if $\iota_{l,l+1}(v_j^{l+1}) = v_i^l$. We also write $v_i^l \overset{\lambda}{\geq} v_j^{l+1}$ if there exists an edge $e \in E_{l,l+1}$ such that $s(e) = v_i^l, t(e) = v_j^{l+1}$. For $v_i^l \in V_l$ and $v_m^{l+k} \in V_{l+k}$, we write $v_i^l \overset{\iota}{\geq} v_m^{l+k}$ (respectively $v_i^l \overset{\lambda}{\geq} v_m^{l+k}$) if there exist $v_{i_1}^{l+1}, \dots, v_{i_{k-1}}^{l+k-1}$ such that

$$v_i^l \overset{\iota}{\geq} v_{i_1}^{l+1} \overset{\iota}{\geq} \cdots \overset{\iota}{\geq} v_{i_{k-1}}^{l+k-1} \overset{\iota}{\geq} v_m^{l+k} \quad \left(\text{respectively } v_i^l \overset{\lambda}{\geq} v_{i_1}^{l+1} \overset{\lambda}{\geq} \cdots \overset{\lambda}{\geq} v_{i_{k-1}}^{l+k-1} \overset{\lambda}{\geq} v_m^{l+k}\right).$$

A subset C of V is said to be *ι-hereditary* (respectively *λ-hereditary*) if for $v_i^l \in C \cap V_l$ the condition $v_i^l \overset{\iota}{\geq} v_j^{l+1}$ (respectively $v_i^l \overset{\lambda}{\geq} v_j^{l+1}$) implies $v_j^{l+1} \in C$. It is said to be *hereditary* if C is both *ι-hereditary* and *λ-hereditary*. It is said to be *ι-saturated* (respectively *λ-saturated*) if it contains every vertex $v_i^l \in C \cap V_l$ for which $v_i^l \overset{\iota}{\geq} v_j^{l+1}$ (respectively $v_i^l \overset{\lambda}{\geq} v_j^{l+1}$) implies $v_j^{l+1} \in C$. If C is both *ι-saturated* and *λ-saturated*, it is said to be *saturated*.

DEFINITION. A λ-graph system $\mathfrak{L}' = (V', E', \lambda', \iota')$ over Σ' is said to be a *λ-graph subsystem* of $\mathfrak{L}$ if it satisfies the following conditions:

$$\begin{aligned} &\emptyset \neq V_l' \subset V_l, \quad \emptyset \neq E_{l,l+1}' \subset E_{l,l+1}, \quad \text{for } l \in \mathbb{Z}_+, \\ &\lambda|_{E'} = \lambda', \quad \iota|_{V'} = \iota', \quad \Sigma' \subset \Sigma, \end{aligned}$$

and an edge $e \in E$ belongs to E' if and only if the both vertices $s(e), t(e)$ belong to V'. Hence a λ-graph subsystem is determined by only its vertex set.

LEMMA 3.1. *For a saturated hereditary subset $C \subset V$, set*

$$\begin{aligned} V^{\setminus C} &= V \setminus C, \\ E^{\setminus C} &= \{e \in E \mid s(e),\ t(e) \in V \setminus C\}, \\ \lambda^{\setminus C} &= \lambda\ |_{E^{\setminus C}}, \quad \iota^{\setminus C} = \iota\ |_{V^{\setminus C}}\ . \end{aligned}$$

Then $(V^{\setminus C}, E^{\setminus C}, \lambda^{\setminus C}, \iota^{\setminus C})$ is a λ-graph subsystem over Σ of $\mathfrak{L}$.

PROOF. For a vertex $u \in V_l^{\setminus C}$, there exists a vertex $w \in V_{l+1}^{\setminus C}$ such that $\iota(w) = u$, because C is ι-saturated. Similarly, there exist an edge $e \in E_{l,l+1}^{\setminus C}$ and a vertex $w' \in V_{l+1}^{\setminus C}$ such that $s(e) = u$, $t(e) = w'$, because C is λ-saturated. Let u, v be vertices with $u \in V_l^{\setminus C}$, $v \in V_{l+2}^{\setminus C}$. Put $v' = \iota(v)$. As C is ι-hereditary, we have that v' belongs to $V_{l+1}^{\setminus C}$. As C is λ-hereditary, if an edge $e \in E_{l,l+1}$ satisfies $t(e) = v$, one sees that $s(e)$ belongs to $V_{l+1}^{\setminus C}$ and hence e belongs to $E_{l,l+1}^{\setminus C}$. Therefore $(V^{\setminus C}, E^{\setminus C}, \lambda^{\setminus C}, \iota^{\setminus C})$ inherits the local property of $\mathfrak{L}$. Thus $(V^{\setminus C}, E^{\setminus C}, \lambda^{\setminus C}, \iota^{\setminus C})$ becomes a λ-graph system. $\square$

We denote by $\mathfrak{L}^{\backslash C}$ the λ-graph system $(V^{\backslash C}, E^{\backslash C}, \lambda^{\backslash C}, \iota^{\backslash C})$ and call it the λ-graph subsystem of $\mathfrak{L}$ obtained by removing C. Let $\mathcal{I}_C$ be the closed ideal of $\mathcal{O}_{\mathfrak{L}}$ generated by the projections E_i^l for $v_i^l \in C$, that is, $\mathcal{I}_C = \overline{\mathcal{O}_{\mathfrak{L}}\{E_i^l \mid v_i^l \in C\}\mathcal{O}_{\mathfrak{L}}}$ the closure of $\mathcal{O}_{\mathfrak{L}}\{E_i^l \mid v_i^l \in C\}\mathcal{O}_{\mathfrak{L}}$.

LEMMA 3.2. *The set of all linear combinations of elements of the form*

$$S_\mu E_i^l S_\nu^*, \quad \text{for } v_i^l \in C, \ \mu, \nu \in \Lambda^* \tag{3.1}$$

is dense in $\mathcal{I}_C$.

PROOF. Since the finite linear combinations of elements of the form $S_\xi E_f^p S_\eta^*$ for $|\xi|, |\eta| \le p$, $f = 1, \ldots, m(p)$ is dense in $\mathcal{O}_{\mathfrak{L}}$, elements of the form

$$S_\xi E_f^p S_\eta^* E_i^l S_\zeta E_g^q S_\gamma^*, \quad \text{for } v_i^l \in C, \ |\xi|, |\eta| \le p, \ |\zeta|, |\gamma| \le q$$

span the ideal $\mathcal{I}_C$. Put $T = S_\xi E_f^p S_\eta^* E_i^l S_\zeta E_g^q S_\gamma^*$ and assume $T \ne 0$. The equality

$$S_\eta^* E_i^l S_\eta = \sum_{j=1}^{m(l+|\eta|)} A_{l,l+|\eta|}(i, \eta, j) E_j^{l+|\eta|}$$

holds, where $A_{l,l+|\eta|}(i, \eta, j) = 1$, if there exists a λ-path from v_i^l to $v_j^{l+|\eta|}$ with label η, otherwise $A_{l,l+|\eta|}(i, \eta, j) = 0$. The vertex $v_j^{l+|\eta|}$ belongs to C if $A_{l,l+|\eta|}(i, \eta, j) = 1$, because $v_i^l \in C$ and C is λ-hereditary. As $T = S_\xi E_f^p S_\eta^* E_i^l S_\eta S_\eta^* S_\zeta E_g^q S_\gamma^*$ and we may assume that l is large enough, T is assumed to be of the form $T = S_\xi E_i^l S_\eta^* S_\zeta E_g^q S_\gamma^*$ for $v_i^l \in C$. As $T \ne 0$, the element $E_i^l S_\eta^* S_\zeta$ is either of the form $E_i^l S_\nu$, or $E_i^l S_\nu^*$ for some word ν. In the former case, we have $T = S_\xi S_\nu S_\nu^* E_i^l S_\nu E_g^q S_\gamma^*$. Since $S_\nu^* E_i^l S_\nu$ is a finite linear combination of $E_j^{l+|\nu|}$ for $v_j^{l+|\nu|} \in C$ and l is large enough, T is a finite linear combinations of elements of the form (3.1), because C is λ-hereditary. In the latter case, we have $T = S_\xi E_i^l S_\nu^* E_g^q S_\nu S_\nu^* S_\gamma^*$. Since $S_\nu^* E_g^q S_\nu$ is a finite linear combinations of $E_j^{q+|\nu|}$ for $v_j^{q+|\nu|} \in V_{q+|\nu|}$ and l is large enough, we have $T = S_\xi E_i^l S_{\gamma\nu}^*$. Hence we get the desired assertion. □

LEMMA 3.3. *If E_i^l belongs to the ideal $\mathcal{I}_C$, the vertex v_i^l belongs to the set C.*

PROOF. For $k \le l$, set

$$E_{k,l} = \sum_{\substack{\mu, j \\ |\mu|=k, v_j^l \in C}} S_\mu E_j^l S_\mu^*$$

belonging to $\mathcal{I}_C$. For an operator $T = S_\xi E_i^l S_\eta^*$ with $v_i^l \in C$, it follows that $T E_{k,l} = E_{k,l} T = T$ for large enough k, l. Lemma 3.2 says that $\{E_{k,l}\}_{k,l}$ is an approximate unit for $\mathcal{I}_C$. Suppose that a vertex $v_J^L \in V$ does not belong to C. It suffices to show that the equality

$$\|E_J^L E_{k,l} - E_J^L\| = 1 \tag{3.2}$$

holds for all large enough k, l. We fix $k \le l$ large enough. We may assume that $E_J^L E_{k,l} \ne 0$ and $L + k \le l$. There exists an admissible word μ of length k such that $S_\mu^* E_J^L S_\mu E_j^l \ne 0$ and hence $S_\mu^* E_J^L S_\mu \ge E_j^l$. On the other hand, C is saturated, so we may find a λ-path π in $E_{L,L+k}$ whose source vertex $s(\pi)$ is v_J^L, and an ι-path from the terminal vertex $t(\pi)$ of π to a vertex v_p^l that does not belong to C. We set $\gamma = \lambda(\pi)$ the label of π so that $S_\gamma^* E_J^L S_\gamma \ge E_p^l$. It then follows that

$$E_J^L \ge S_\mu S_\mu^* E_J^L S_\mu S_\mu^* + S_\gamma S_\gamma^* E_J^L S_\gamma S_\gamma^* \ge S_\mu E_j^l S_\mu^* + S_\gamma E_p^l S_\gamma^*.$$

Since $\sum_{|\nu|=k, v_j^l \in C} S_\nu E_j^l S_\nu^*$ is orthogonal to $S_\gamma E_p^l S_\gamma^*$, one obtains that

$$E_J^L E_{k,l} - E_J^L \ge S_\gamma E_p^l S_\gamma^*$$

so that (3.2) holds. □

LEMMA 3.4. *For any nonzero closed ideal $\mathcal{I}$ of the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$, put*

$$C_{\mathcal{I}} = \{v_i^l \in V \mid E_i^l \in \mathcal{I}\}.$$

Then $C_{\mathcal{I}}$ is a nonempty saturated hereditary subset of V.

PROOF. Since $\mathfrak{L}$ satisfies condition (I), the set $C_{\mathcal{I}}$ is nonempty because of the uniqueness of the algebra $\mathcal{O}_{\mathfrak{L}}$. Take $v_i^l \in C_{\mathcal{I}}$. Suppose that v_j^{l+1} satisfies $v_i^l \overset{\iota}{\ge} v_j^{l+1}$. The inequality $E_i^l \ge E_j^{l+1}$ assures $E_j^{l+1} \in \mathcal{I}$. Suppose next $v_i^l \overset{\lambda}{\ge} v_j^{l+1}$. There exists a symbol $\alpha \in \Sigma$ such that $A_{l,l+1}(i, \alpha, j) = 1$. By (2.4), we have $S_\alpha^* E_i^l S_\alpha \ge E_j^{l+1}$ so that $E_j^{l+1} \in \mathcal{I}$. Hence $C_{\mathcal{I}}$ is hereditary. For v_i^l, suppose that $v_i^l \overset{\iota}{\ge} v_j^{l+1}$ implies $v_j^{l+1} \in C_{\mathcal{I}}$. This means that $I_{l,l+1}(i, j) = 1$ implies $E_j^{l+1} \in \mathcal{I}$. By the second equality of (2.2), we see $E_i^l \in \mathcal{I}$. Suppose next that $v_i^l \overset{\lambda}{\ge} v_j^{l+1}$ implies $v_j^{l+1} \in C_{\mathcal{I}}$. This means that $A_{l,l+1}(i, \alpha, j) = 1$ implies $E_j^{l+1} \in \mathcal{I}$. By (2.4), we have $S_\alpha^* E_i^l S_\alpha \in \mathcal{I}$ for all $\alpha \in \Sigma$, so that $E_i^l = \sum_{\alpha \in \Sigma} S_\alpha S_\alpha^* E_i^l S_\alpha S_\alpha^*$ belongs to $\mathcal{I}$. Thus $\mathcal{I}$ is saturated. □

PROPOSITION 3.5. *Let $\mathfrak{L} = (V, E, \lambda, \iota)$ be a λ-graph system satisfying condition* (I). *Let C be a saturated hereditary subset of V. A vertex v_i^l belongs to C if and only if E_i^l belongs to $\mathcal{I}_C$. Hence there exists a bijective correspondence between the set of all saturated hereditary subsets of V and the set of all ideals in $\mathcal{O}_{\mathfrak{L}}$.*

PROOF. Let C be a saturated hereditary subset of V. For a vertex $v_i^l \in V$, we have $v_i^l \in C$ if and only if $E_i^l \in \mathcal{I}_C$ by Lemma 3.3. For an ideal $\mathcal{I}$ of $\mathcal{O}_{\mathfrak{L}}$, we have $E_i^l \in \mathcal{I}$ if and only if $v_i^l \in C_{\mathcal{I}}$ by definition of $C_{\mathcal{I}}$. Hence we conclude the assertions. □

DEFINITION. A λ-graph system $\mathfrak{L}$ satisfies *condition* (II) if for every saturated hereditary subset $C \subset V$, the λ-graph system $\mathfrak{L}^{\setminus C}$ satisfies condition (I).

Let A be an $n \times n$ square matrix with entries in $\{0, 1\}$. Then A satisfies condition (II) in the sense of Cuntz [6] if and only if the natural λ-graph system $\mathfrak{L}^{\Lambda_A}$ constructed from A satisfies condition (II) in the above sense (compare Section 5).

THEOREM 3.6. *Suppose that a λ-graph system $\mathfrak{L}$ satisfies condition* (II). *For an ideal $\mathcal{I}$ of $\mathcal{O}_{\mathfrak{L}}$, the quotient C^*-algebra $\mathcal{O}_{\mathfrak{L}}/\mathcal{I}$ is isomorphic to the C^*-algebra $\mathcal{O}_{\mathfrak{L}^{\setminus C_{\mathcal{I}}}}$ associated with the λ-graph system $\mathfrak{L}^{\setminus C_{\mathcal{I}}}$ obtained from $\mathfrak{L}$ by removing the saturated hereditary subset $C_{\mathcal{I}}$ for $\mathcal{I}$.*

PROOF. We denote by $\overline{S_\alpha}, \overline{E_i^l}$ the quotient images of S_α, E_i^l in the quotient C^*-algebra $\mathcal{O}_{\mathfrak{L}}/\mathcal{I}$ respectively. Let s_α, e_i^l be the canonical generating partial isometries for $\alpha \in \Sigma$ and the projections corresponding to the vertices v_i^l of $V^{\setminus C_{\mathcal{I}}}$ in $\mathcal{O}_{\mathfrak{L}^{\setminus C_{\mathcal{I}}}}$. Since we have $\overline{E_i^l} \neq 0$ if and only if $v_i^l \in V^{\setminus C_{\mathcal{I}}}$, the relations

$$\overline{S_\alpha^*}\overline{E_i^l}\overline{S_\alpha} = \sum_{k=1}^{m(l+1)} A_{l,l+1}(i, \alpha, k)\overline{E_k^{l+1}}, \quad \text{for } \alpha \in \Sigma$$

hold. By the uniqueness of the algebras $\mathcal{O}_{\mathfrak{L}}$ and $\mathcal{O}_{\mathfrak{L}^{\setminus C_{\mathcal{I}}}}$, subject to the operator relations, the correspondence $\overline{S_\alpha} \leftrightarrow s_\alpha$, $\overline{E_i^l} \leftrightarrow e_i^l$ for $\alpha \in \Sigma$, $v_i^l \in V^{\setminus C_{\mathcal{I}}}$ extends to an isomorphism between $\mathcal{O}_{\mathfrak{L}}/\mathcal{I}$ and $\mathcal{O}_{\mathfrak{L}^{\setminus C_{\mathcal{I}}}}$. □

COROLLARY 3.7. *In the λ-graph sytems satisfying condition* (II), *the class of the C^*-algebras associated with λ-graph systems is closed under quotients by its ideals.*

We say a closed ideal $\mathcal{J}$ of $\mathcal{A}_{\mathfrak{L}}$ to be *saturated* if $\lambda_{\mathfrak{L}}(E_i^l) \in \mathcal{J}$ implies $E_i^l \in \mathcal{J}$. We are assuming that a λ-graph system $\mathfrak{L}$ satisfies condition (I).

LEMMA 3.8. *For an ideal $\mathcal{I}$ of $\mathcal{O}_{\mathfrak{L}}$, set $\mathcal{J} = \mathcal{I} \cap \mathcal{A}_{\mathfrak{L}}$. Then $\mathcal{J}$ is a nonzero $\lambda_{\mathfrak{L}}$-invariant saturated ideal of $\mathcal{A}_{\mathfrak{L}}$.*

PROOF. It suffices to show that $\mathcal{J}$ is saturated. Suppose that $\lambda_{\mathfrak{L}}(E_i^l) \in \mathcal{J}$. We see $S_\alpha^* E_i^l S_\alpha$ belongs to $\mathcal{J}$ for each $\alpha \in \Sigma$. Hence $E_i^l = \sum_{\alpha\in\Sigma} S_\alpha S_\alpha^* E_i^l S_\alpha S_\alpha^*$ belongs to $\mathcal{J}$. □

LEMMA 3.9. *There exists a bijective correspondence between the set of $\lambda_{\mathfrak{L}}$-invariant closed saturated ideals of $\mathcal{A}_{\mathfrak{L}}$ and the set of saturated hereditary subsets of V.*

PROOF. Let $\mathcal{J}$ be a $\lambda_{\mathfrak{L}}$-invariant saturated ideal of $\mathcal{A}_{\mathfrak{L}}$. Put $C_{\mathcal{J}} = \{v_i^l \in V \mid E_i^l \in \mathcal{J}\}$. As $\mathcal{J}$ is $\lambda_{\mathfrak{L}}$-invariant, we have $\sum_{\alpha\in\Sigma} S_\alpha^* E_i^l S_\alpha$ belongs to $\mathcal{J}$ for $v_i^l \in C_{\mathcal{J}}$. Hence

$A_{l,l+1}(i,\alpha,j) = 1$ implies $E_j^{l+1} \in \mathcal{J}$. This means that $C_{\mathcal{J}}$ is λ-hereditary. Suppose that $A_{l,l+1}(i,\alpha,j) = 1$ implies $v_j^{l+1} \in C_{\mathcal{J}}$. It follows that $\lambda_{\mathfrak{L}}(E_i^l) \in \mathcal{J}$ and hence $v_i^l \in C_{\mathcal{J}}$, because $\mathcal{J}$ is saturated. By the second equality of (2.2), we know that $C_{\mathcal{J}}$ is ι-hereditary and ι-saturated.

For a saturated hereditary subset C of V, let $\mathcal{I}_C$ be the ideal of $\mathcal{O}_{\mathfrak{L}}$ generated by E_i^l for $v_i^l \in C$. Put $\mathcal{J}_C = \mathcal{I}_C \cap \mathcal{A}_{\mathfrak{L}}$. By Proposition 3.5, a vertex v_i^l belongs to C if and only if E_i^l belongs to $\mathcal{J}_C$. It is easy to see that $\mathcal{J}_C$ is $\lambda_{\mathfrak{L}}$-invariant because C is λ-hereditary, and $\mathcal{J}_C$ is saturated because C is λ-saturated. □

We remark that $\mathfrak{L}$ is irreducible if and only if there is no nontrivial $\lambda_{\mathfrak{L}}$-invariant ideal of $\mathcal{A}_{\mathfrak{L}}$. The latter property is also equivalent to the condition that there is no proper hereditary and ι-saturated subset of V. Thus we see the following theorem.

THEOREM 3.10. *Consider the following six conditions.*

(i) *$\mathcal{O}_{\mathfrak{L}}$ is simple.*
(ii) *There is no nontrivial $\lambda_{\mathfrak{L}}$-invariant saturated ideal of $\mathcal{A}_{\mathfrak{L}}$.*
(iii) *There is no proper saturated hereditary subset of V.*
(iv) *$\mathfrak{L}$ is irreducible.*
(v) *There is no nontrivial $\lambda_{\mathfrak{L}}$-invariant ideal of $\mathcal{A}_{\mathfrak{L}}$.*
(vi) *There is no proper hereditary and ι-saturated subset of V.*

Conditions (i)–(iii) *are equivalent to each other, and also conditions* (iv)–(vi) *are equivalent to each other. The latter conditions imply the former conditions.*

PROOF. As nontrivial ideals of $\mathcal{O}_{\mathfrak{L}}$ bijectively correspond to saturated hereditary subsets of V, the first three conditions are equivalent each other. It suffices to show that (iv) is equivalent to (vi). Assume that $\mathfrak{L}$ is irreducible. Let C be a nonempty hereditary and ι-saturated subset of V. Take a vertex $v_i^l \in C$. Let $U_N(v_i^l)$ be the set of ι-orbits $u = (u_n)_{n\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}}$ such that there exists a λ-path of length N from v_i^l to the vertex u_{l+N}. Since $\mathfrak{L}$ is irreducible, we have $\Omega_{\mathfrak{L}} = \bigcup_{N=0}^{\infty} U_N(v_i^l)$. Hence there exist $N_1, N_2, \dots, N_n$ such that $\Omega_{\mathfrak{L}} = \bigcup_{j=1}^{n} U_{N_j}(v_i^l)$, because $U_N(v_i^l)$ is open in $\Omega_{\mathfrak{L}}$. We may assume that $0 \le N_1 \le N_2 \le \cdots \le N_n$. We put $N_n = L$. For a vertex $w \in V_{l+L}$, find an ι-orbit $x = (x_n)_{n\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}}$ such that $x_{l+L} = w$. Take N_k such that $x \in U_{N_k}(v_i^l)$. Since C is λ-hereditary and ι-hereditary, we see $x_{l+N_k} \in C$ and hence $w \in C$. This implies $V_{l+N} \subset C$. Now C is ι-saturated, so we conclude that $V = C$. Therefore we get the implication from (iv) to (vi).

Suppose that $\mathfrak{L}$ is not irreducible. There exists an ι-orbit $u = (u_n)_{n\in\mathbb{Z}_+} \in \Omega_{\mathfrak{L}}$ and a vertex v_i^l such that u does not belong to $\cup_{N=0}^{\infty} U_N(v_i^l)$. Let $V^N(v_i^l)$ be the set of all vertices w in V_{l+N} that are terminal vertices of λ-edges whose source vertices are v_i^l. Put $V(v_i^l) = \cup_{N=0}^{\infty} V^N(v_i^l)$ and

$$W(v_i^l) = \left\{ w \in V \mid v \overset{\iota}{\geq} w \ \text{ for some vertex } \ v \in V(v_i^l) \right\} \cup V(v_i^l).$$

By the local property of the λ-graph system, the set $W(v_i^l)$ is λ-hereditary and the vertices u_n do not belong to $W(v_i^l)$ for all $n \in \mathbb{Z}_+$. It is by definition that $W(v_i^l)$ is ι-hereditary. Let C be the saturation of $W(v_i^l)$ with respect to $\overset{\iota}{\geq}$. As $W(v_i^l)$ is λ-hereditary, C is so from the local property of λ-graph system. It is obvious that C is ι-hereditary. We obtain a proper hereditary and ι-saturated subset C of V. $\square$

4. Structure of ideals

In this section, we prove that an ideal of $\mathcal{O}_{\mathfrak{L}}$ is stably isomorphic to the C^*-subalgebra of $\mathcal{O}_{\mathfrak{L}}$ associated with the corresponding saturated hereditary subset of V. As a result, we can present the K-theory formulae for ideals of $\mathcal{O}_{\mathfrak{L}}$ in terms of the corresponding saturated hereditary subsets of V. The notation is as in the previous sections. For a saturated hereditary subset C of V, put for $v_i^l \in C$

$$\Lambda^C(v_i^l) = \left\{ \mu \in \Lambda^* \,\middle|\, \begin{array}{l} \text{there exists a } \lambda\text{-path } \pi \text{ such that } \lambda(\pi) = \mu, \\ s(\pi) \in C, t(\pi) = v_i^l \end{array} \right\},$$

where $s(\pi)$ and $t(\pi)$ are the source vertex and the terminal vertex of π respectively. We denote by $\mathcal{O}_{\mathfrak{L}}(C)$ the C^*-subalgebra of $\mathcal{O}_{\mathfrak{L}}$ generated by elements of the form $S_\mu E_i^l S_\nu^*$, for $\mu, \nu \in \Lambda^C(v_i^l)$, $v_i^l \in C$.

LEMMA 4.1. *The set of all finite linear combinations of elements of the form $S_\mu E_i^l S_\nu^*$, for $\mu, \nu \in \Lambda^C(v_i^l)$, $v_i^l \in C$, is a dense $*$-subalgebra of $\mathcal{O}_{\mathfrak{L}}(C)$.*

PROOF. For $v_i^l, v_j^k \in C$, $\mu, \nu \in \Lambda^C(v_i^l)$, $\xi, \eta \in \Lambda^C(v_j^k)$, suppose that

$$S_\mu E_i^l S_\nu^* S_\xi E_j^k S_\eta^* \neq 0.$$

We may assume $|\nu| > |\xi|$. We then have $\nu = \xi \nu'$ for some ν', so that

$$S_\mu E_i^l S_\nu^* S_\xi E_j^k S_\eta^* = S_\mu E_i^l S_{\nu'}^* E_j^k S_{\nu'} S_{\eta\nu'}^*.$$

If $|\nu'| + k \leq l$, we have that $E_i^l S_{\nu'}^* E_j^k S_{\nu'} = E_i^l$. If $|\nu'| + k \geq l$, we see that $E_i^l S_{\nu'}^* E_j^k S_{\nu'}$ is a finite sum of projections $E_h^{|\nu'|+k}$ with $v_h^{|\nu'|+k} \in C$. In both cases, $S_\mu E_i^l S_\nu^* S_\xi E_j^k S_\eta^*$ is a finite linear combination of $S_\zeta E_h^m S_\delta^*$ with $\zeta, \delta \in \Lambda^C(v_h^m)$, $v_h^m \in C$. $\square$

We prove that the ideal $\mathcal{I}_C$ of $\mathcal{O}_{\mathfrak{L}}$ is stably isomorphic to the C^*-algebra $\mathcal{O}_{\mathfrak{L}}(C)$ under some condition. Put $P_l = \sum_{i, v_i^l \in C} E_i^l$ for $l \in \mathbb{N}$. It belongs to the algebra $\mathcal{O}_{\mathfrak{L}}(C)$ and satisfies $P_l \leq P_{l+1}$. We see then a sequence of natural embeddings $P_l \mathcal{O}_{\mathfrak{L}} P_l \subset P_{l+1} \mathcal{O}_{\mathfrak{L}} P_{l+1} \subset \cdots$.

PROPOSITION 4.2. $\mathcal{O}_{\mathfrak{L}}(C) = \lim_{l \to \infty} P_l \mathcal{O}_{\mathfrak{L}} P_l$.

PROOF. We first prove the inclusion relation $\mathcal{O}_{\mathfrak{L}}(C) \subset \lim_{l\to\infty} P_l \mathcal{O}_{\mathfrak{L}} P_l$. For $v_i^l \in C$ and $\mu \in \Lambda^C(v_i^l)$, take a λ-path π such that $s(\pi) \in C$, $t(\pi) = v_i^l$, and $\lambda(\pi) = \mu$. We put $s(\pi) = v_{j_1}^{l_1}$. The projection $E_{j_1}^{l_1}$ satisfies the inequality $S_\mu^* E_{j_1}^{l_1} S_\mu \geq E_i^l$ so that $E_{j_1}^{l_1} S_\mu E_i^l = S_\mu E_i^l$. As $\mathfrak{L}$ is left-resolving, we know that $S_\mu^* E_{k_1}^{l_1} S_\mu E_i^l = 0$ for $k_1 \neq j_1$. It then follows that $P_{l_1} S_\mu E_i^l = S_\mu E_i^l$. Symmetrically we have that $E_i^l S_\nu^* P_{l_2} = E_i^l S_\nu^*$ for some l_2. Hence we see that $P_{l_1} S_\mu E_i^l S_\nu^* P_{l_2} = S_\mu E_i^l S_\nu^*$. Thus we have proved that for $v_i^l \in C$ and $\mu, \nu \in \Lambda^C(v_i^l)$, there exists $M \in \mathbb{N}$ such that $P_m S_\mu E_i^l S_\nu^* P_m = S_\mu E_i^l S_\nu^*$ for all $m \geq M$. This implies the inclusion relation $\mathcal{O}_{\mathfrak{L}}(C) \subset \lim_{l\to\infty} P_l \mathcal{O}_{\mathfrak{L}} P_l$.

For $v_i^l \in V$, $\mu, \nu \in \Lambda^*$, and $v_{j_1}^{l_1}, v_{j_2}^{l_2} \in C$, we next prove that the element $E_{j_1}^{l_1} S_\mu E_i^l S_\nu^* E_{j_2}^{l_2}$ belongs to the algebra $\mathcal{O}_{\mathfrak{L}}(C)$. We may assume that l is large enough because of the second relation of (2.2). Assume $S_\mu^* E_{j_1}^{l_1} S_\mu E_i^l S_\nu^* E_{j_2}^{l_2} S_\nu \neq 0$ so that $S_\mu^* E_{j_1}^{l_1} S_\mu \geq E_i^l$. Hence there exists a λ-path whose source is $v_{j_1}^{l_1}$ and terminal is connected to v_i^l by an ι-path. By the local property of the λ-graph system, we may find a λ-path π in E such that $\lambda(\pi) = \mu$, $t(\pi) = v_i^l$ and an ι-path that connects between $s(\pi)$ and $v_{j_1}^{l_1}$. Since $v_{j_1}^{l_1}$ belongs to C and C is hereditary, we see that $v_i^l \in C$ and μ belongs to $\Lambda^C(v_i^l)$. Symmetrically one sees that ν belongs to $\Lambda^C(v_i^l)$ from the inequality $S_\nu^* E_{j_2}^{l_2} S_\nu \geq E_i^l$. Hence we have $E_{j_1}^{l_1} S_\mu E_i^l S_\nu^* E_{j_2}^{l_2} = S_\mu E_i^l S_\nu^*$ and it belongs to the algebra $\mathcal{O}_{\mathfrak{L}}(C)$. Thus we have $\lim_{l\to\infty} P_l \mathcal{O}_{\mathfrak{L}} P_l \subset \mathcal{O}_{\mathfrak{L}}(C)$. □

THEOREM 4.3. *The ideal $\mathcal{I}_C$ is stably isomorphic to the algebra $\mathcal{O}_{\mathfrak{L}}(C)$.*

PROOF. Let $X_l = \mathcal{O}_{\mathfrak{L}} P_l$ for $l \in \mathbb{N}$. Then X_l has a Hilbert left $\overline{\mathcal{O}_{\mathfrak{L}} P_l \mathcal{O}_{\mathfrak{L}}}$-module and a Hilbert right $P_l \mathcal{O}_{\mathfrak{L}} P_l$-module structure in a natural way. Its left $\overline{\mathcal{O}_{\mathfrak{L}} P_l \mathcal{O}_{\mathfrak{L}}}$-valued inner product and right $P_l \mathcal{O}_{\mathfrak{L}} P_l$-valued inner product are given by

$$\langle aP_l, bP_l \rangle_L = aP_l b^*, \quad \langle aP_l, bP_l \rangle_R = P_l a^* b P_l,$$

for $a, b \in \mathcal{O}_{\mathfrak{L}}$ respectively. Hence the norms on X_l coming from their respect inner products coincide with the norm on the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$. As $P_l \leq P_{l+1}$, we have a natural embedding $X_l \hookrightarrow X_{l+1}$. Let X_C be the closure of $\bigcup_{l=1}^{\infty} X_l$ in the norm of $\mathcal{O}_{\mathfrak{L}}$, that is regarded as the inductive limit of the inclusions $X_l \hookrightarrow X_{l+1}$, $l \in \mathbb{N}$. The ideal $\mathcal{I}_C$ and the algebra $\mathcal{O}_{\mathfrak{L}}(C)$ are the inductive limits $\lim_{l\to\infty} \overline{\mathcal{O}_{\mathfrak{L}} P_l \mathcal{O}_{\mathfrak{L}}}$ and $\lim_{l\to\infty} P_l \mathcal{O}_{\mathfrak{L}} P_l$ respectively. We then see that the subspace X_C of $\mathcal{O}_{\mathfrak{L}}$ has an induced left $\mathcal{I}_C$-valued inner product and right $\mathcal{O}_{\mathfrak{L}}(C)$-valued inner product such as

$$\langle \xi, \eta \rangle_L = \xi \eta^* \in \mathcal{I}_C, \quad \langle \xi, \eta \rangle_R = \xi^* \eta \in \mathcal{O}_{\mathfrak{L}}(C),$$

for $\xi, \eta \in X_C$ respectively. It also has a natural left $\mathcal{I}_C$-module and right $\mathcal{O}_{\mathfrak{L}}(C)$-module structures respectively. It is easy to see that both the linear spans of $\langle \xi, \eta \rangle_L$, for $\xi, \eta \in X_C$, and $\langle \xi, \eta \rangle_R$, for $\xi, \eta \in X_C$, are dense in $\mathcal{I}_C$ and $\mathcal{O}_{\mathfrak{L}}(C)$ respectively. Hence X_C is a full Hilbert left $\mathcal{I}_C$-module, and a full Hilbert right $\mathcal{O}_{\mathfrak{L}}(C)$-module such that $\langle \xi, \eta \rangle_L \zeta = \xi \langle \eta, \zeta \rangle_R$, for $\xi, \eta, \zeta \in X_C$. This means that X_C is an $\mathcal{I}_C - \mathcal{O}_{\mathfrak{L}}(C)$ imprimitivity bimodule, so that $\mathcal{I}_C$ and $\mathcal{O}_{\mathfrak{L}}(C)$ are Morita equivalent ([32]). By [4], they are stably isomorphic to each other. □

By using the above result, we next compute the K-theory of the ideal $\mathcal{I}_C$. The subalgebra $\mathcal{O}_{\mathfrak{L}}(C)$ is invariant globally under the gauge action $\alpha_{\mathfrak{L}}$ on $\mathcal{O}_{\mathfrak{L}}$. We still denote by $\alpha_{\mathfrak{L}}$ the restriction of $\alpha_{\mathfrak{L}}$ to $\mathcal{O}_{\mathfrak{L}}(C)$. We denote by $\mathcal{F}_{\mathfrak{L}}(C)$ the C^*-subalgebra of $\mathcal{O}_{\mathfrak{L}}(C)$ generated by $S_\mu E_i^l S_\nu^*$, $\mu, \nu \in \Lambda^C(v_i^l)$, $|\mu| = |\nu|$, $v_i^l \in C$. That is, $\mathcal{F}_{\mathfrak{L}}(C) = \mathcal{F}_{\mathfrak{L}} \cap \mathcal{I}_C$. It is direct to see that the fixed point algebra $\mathcal{O}_{\mathfrak{L}}(C)^{\alpha_{\mathfrak{L}}}$ of $\mathcal{O}_{\mathfrak{L}}(C)$ under $\alpha_{\mathfrak{L}}$ is the algebra $\mathcal{F}_{\mathfrak{L}}(C)$. A similar discussion to [22] (compare [24]) assures that the crossed product $\mathcal{O}_{\mathfrak{L}}(C) \rtimes_{\alpha_{\mathfrak{L}}} \mathbb{T}$ is stably isomorphic to $\mathcal{F}_{\mathfrak{L}}(C)$. We can show the following result.

LEMMA 4.4 (compare [24, Lemma 7.5], [22, Lemma 4.3]).

(i) $K_0(\mathcal{O}_{\mathfrak{L}}(C)) \cong K_0(\mathcal{O}_{\mathfrak{L}}(C) \rtimes_{\alpha_{\mathfrak{L}}} \mathbb{T})/(\mathrm{id} - \widehat{\alpha_{\mathfrak{L}}}_*^{-1}) K_0(\mathcal{O}_{\mathfrak{L}}(C) \rtimes_{\alpha_{\mathfrak{L}}} \mathbb{T})$.

(ii) $K_1(\mathcal{O}_{\mathfrak{L}}(C)) \cong \mathrm{Ker}(\mathrm{id} - \widehat{\alpha_{\mathfrak{L}}}_*^{-1})$ *on* $K_0(\mathcal{O}_{\mathfrak{L}}(C) \rtimes_{\alpha_{\mathfrak{L}}} \mathbb{T})$,

where $\widehat{\alpha_{\mathfrak{L}}}$ *is the dual action of* $\alpha_{\mathfrak{L}}$.

Let $\mathcal{F}_k^l(C)$ be the C^*-subalgebra of $\mathcal{F}_{\mathfrak{L}}(C)$ generated by $S_\mu E_i^l S_\nu^*$, $\mu, \nu \in \Lambda^C(v_i^l)$, $|\mu| = |\nu| = k$, $v_i^l \in C \cap V_l$ and $\mathcal{F}_k^\infty(C)$ the C^*-subalgebra of $\mathcal{F}_{\mathfrak{L}}(C)$ generated by $\mathcal{F}_k^l(C)$, $k \le l \in \mathbb{N}$. Hence we see that

$$\mathcal{F}_k^l(C) = \mathcal{F}_k^l \cap \mathcal{O}_{\mathfrak{L}}(C), \quad \mathcal{F}_k^\infty(C) = \mathcal{F}_k^\infty \cap \mathcal{O}_{\mathfrak{L}}(C).$$

The embeddings of $\iota_{l,l+1} : \mathcal{F}_k^l \hookrightarrow \mathcal{F}_k^{l+1}$ and $\lambda_{k,k+1} : \mathcal{F}_k^\infty \hookrightarrow \mathcal{F}_{k+1}^\infty$ of the original AF-algebra $\mathcal{F}_{\mathfrak{L}}$, are inherited in the algebras $\mathcal{F}_k^l(C)$, $\mathcal{F}_k^\infty(C)$, $\mathcal{F}_{\mathfrak{L}}(C)$, so that $\mathcal{F}_{\mathfrak{L}}(C)$ is an AF-algebra. Let $m_C(l)$ be the cardinal number of the vertex set $C \cap V_l$. We put $C \cap V_l = \{u_1^l, u_2^l, \ldots, u_{m_C(l)}^l\}$. Define the following matrices:

$$A(C)_{l,l+1}(i,\alpha,j) = \begin{cases} 1 & \text{if } s(e) = u_i^l, \lambda(e) = \alpha,\ t(e) = u_j^{l+1} \text{ for some } e \in E_{l,l+1} \\ 0 & \text{otherwise,} \end{cases}$$

$$I(C)_{l,l+1}(i,j) = \begin{cases} 1 & \text{if } \iota_{l,l+1}(u_j^{l+1}) = u_i^l \\ 0 & \text{otherwise,} \end{cases}$$

$$A(C)_{l,l+1}(i,j) = \sum_{\alpha \in \Sigma} A(C)_{l,l+1}(i,\alpha,j),$$

for $i = 1, 2, \ldots, m_C(l)$, $j = 1, 2, \ldots, m_C(l+1)$. Let

$$D(C)_{l,l+1} = I(C)_{l,l+1}^t - A(C)_{l,l+1}^t : \mathbb{Z}^{m_C(l)} \to \mathbb{Z}^{m_C(l+1)}, \quad l \in \mathbb{Z}_+.$$

As $I(C)^t_{l+1,l+2}A(C)^t_{l,l+1} = A(C)^t_{l+1,l+2}I(C)^t_{l,l+1}$, the matrix $I(C)^t_{l+1,l+2}$ induces a homomorphism from $\mathbb{Z}^{m_C(l+1)}/D(C)_{l,l+1}\mathbb{Z}^{m_C(l)}$ to $\mathbb{Z}^{m_C(l+2)}/D(C)_{l+1,l+2}\mathbb{Z}^{m_C(l+1)}$ that is denoted by $\overline{I(C)}^t_{l+1,l+2}$. Thanks to Theorem 4.3, we can present the K-theory formulae for ideals of $\mathcal{O}_{\mathfrak{L}}$.

THEOREM 4.5. *Let $\mathfrak{L}$ be a λ-graph system satisfying condition* (II). *Let $\mathcal{I}$ be an ideal of $\mathcal{O}_{\mathfrak{L}}$ and C its corresponding saturated hereditary subset of the vertex set of $\mathfrak{L}$. Then we have*

$$K_0(\mathcal{I}) \cong \varinjlim_{l} \left\{\mathbb{Z}^{m_C(l+1)}/D(C)^t_{l,l+1}\mathbb{Z}^{m_C(l)}; \overline{I(C)}^t_{l+1,l+2}\right\},$$
$$K_1(\mathcal{I}) \cong \varinjlim_{l} \left\{\mathrm{Ker} D(C)_{l,l+1} \text{ in } \mathbb{Z}^{m_C(l)}; I(C)^t_{l,l+1}\right\}.$$

Although the C^*-algebra $\mathcal{O}_{\mathfrak{L}}$ is not necessarily defined by a λ-graph system, in the case when C has a *bounded upper bound*, it is given by a λ-graph system. Let

$$V_C^\iota = C \cup \{v \in V \mid \text{ there exists } u_0 \in C \text{ such that } \iota^m(u_0) = v \text{ for some } m \in \mathbb{N}\}.$$

A saturated hereditary subset C of V is said to have a *bounded upper bound* if the cardinality of the set $V_C^\iota \backslash C$ is finite. It is equivalent to the condition that there exists $L \in \mathbb{N}$ such that $V_n \cap V_C^\iota = V_n \cap C$ for all $n \geq L$. We will assume that C has a bounded upper bound. Take $L \in \mathbb{N}$ as above. Define for $l \in \mathbb{Z}_+$

$$V_l^C = C \cap V_{l+L},$$
$$E_{l,l+1}^C = \left\{e \in E_{l+L,l+L+1} \mid s(e) \in V_l^C,\ t(e) \in V_{l+1}^C\right\},$$
$$\lambda^C = \lambda\mid_{E^C}, \quad \iota^C_{l,l+1} = \iota\mid_{V^C_{l,l+1}}.$$

Since $V_C^\iota \cap V_{l+L} = C \cap V_{l+L}$, one sees that $\iota(u) \in V_l^C$ for $u \in V_{l+1}^C$. It is straightforward to see that $(V_l^C, E_{l,l+1}^C, \lambda^C, \iota^C_{l,l+1})_{l\in\mathbb{Z}_+}$ yields a λ-graph system, denoted by $\mathfrak{L}_C$. We note that C has a bounded upper bound if and only if there exists $L \in \mathbb{N}$ such that $P_l = P_L$ for all $l \geq L$.

PROPOSITION 4.6. *Let $\mathfrak{L}$ be a λ-graph system satisfying condition* (II). *If a saturated hereditary subset C of V has a bounded upper bound, the algebra $\mathcal{O}_{\mathfrak{L}}(C)$ is isomorphic to the C^*-algebra $\mathcal{O}_{\mathfrak{L}_C}$ associated with the λ-graph system $\mathfrak{L}_C$. Hence the ideal $\mathcal{I}_C$ is stably isomorphic to the C^*-algebra $\mathcal{O}_{\mathfrak{L}_C}$.*

PROOF. Take $L \in \mathbb{N}$ such that $V_n \cap V_C^\iota = V_n \cap C$ for all $n \geq L$. As $P_l = P_L$ for all $l \geq L$, one has $\mathcal{O}_{\mathfrak{L}}(C) = P_L\mathcal{O}_{\mathfrak{L}}P_L$ by Proposition 4.2. Let $\mathfrak{L}^{(L)} = (V^{(L)}, E^{(L)}, \lambda^{(L)}, \iota^{(L)})$ be the L-shift λ-graph system of $\mathfrak{L}$ defined by

$$V_l^{(L)} = V_{l+L}, \quad E_{l,l+1}^{(L)} = E_{l+L,l+L+1}, \quad \lambda^{(L)} = \lambda|_{E^{(L)}}, \quad \iota^{(L)}_{l,l+1} = \iota_{l+L,l+L+1}$$

for $l \in \mathbb{Z}_+$. By [28, Proposition 2.3], the algebra $\mathcal{O}_{\mathfrak{L}}$ coincides with the the algebra $\mathcal{O}_{\mathfrak{L}^{(L)}}$. It is direct to see that $P_L \mathcal{O}_{\mathfrak{L}^{(L)}} P_L$ is isomorphic to $\mathcal{O}_{\mathfrak{L}_C}$. Hence $\mathcal{O}_{\mathfrak{L}}(C)$ is isomorphic to $\mathcal{O}_{\mathfrak{L}_C}$. □

5. Examples

Let $G = (V, E)$ be a finite directed graph with finite vertex set V and finite edge set E. Let $\mathcal{G} = (G, \lambda)$ be a labeled graph over an alphabet Σ defined by G and a labeling map $\lambda : E \to \Sigma$. Suppose that it is left-resolving and predecessor-separated (see [19]). Let A_G be the adjacency matrix of G that is defined by

$$A_G(e, f) = \begin{cases} 1 & \text{if } t(e) = s(e), \\ 0 & \text{otherwise}, \end{cases}$$

for $e, f \in E$. The matrix A_G defines a shift of finite type by regarding the edge set E as its alphabet. Since the matrix A_G has entries in $\{0, 1\}$, we have the Cuntz-Krieger algebra $\mathcal{O}_{A_G}$ defined by A_G ([7] compare [18, 33]). By putting $V_l^{\mathcal{G}} = V$, $E_{l,l+1}^{\mathcal{G}} = E$ for $l \in \mathbb{Z}_+$, and $\lambda^{\mathcal{G}} = \lambda, \iota^{\mathcal{G}} = \mathrm{id}$, we have a λ-graph system $\mathfrak{L}_{\mathcal{G}} = (V^{\mathcal{G}}, E^{\mathcal{G}}, \lambda^{\mathcal{G}}, \iota^{\mathcal{G}})$. The C^*-algebra $\mathcal{O}_{\mathfrak{L}_{\mathcal{G}}}$ is isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_{A_G}$ ([24, Proposition 7.1]).

Let us consider the following labeled graph. The vertex set V is $\{v_1, v_2, v_3\}$. The edges labeled α are from v_2 to v_3 and from v_3 to v_2 and a self-loop at v_1. The edges labeled β are self-loops at v_1 and at v_3. The edge labeled γ is from v_1 to v_2. The resulting labeled graph is denoted by $\mathcal{G}$. The λ-graph system $\mathfrak{L}_{\mathcal{G}}$ is left-resolving and satisfies condition (II). In $\mathfrak{L}_{\mathcal{G}}$, let C be the vertex set corresponding to $\{v_2, v_3\}$. It is saturated hereditary. The λ-graph subsystem $\mathfrak{L}_{\mathcal{G}}^{\setminus C}$ of $\mathfrak{L}_{\mathcal{G}}$ obtained by removing C consists of one ι-orbit of the vertex $\{v_1\}$ with two self-loops labeled α and β. Hence we have

$$\mathcal{O}_{\mathfrak{L}_{\mathcal{G}}} \cong \mathcal{O}_{\left[\begin{smallmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{smallmatrix}\right]}, \quad \mathcal{O}_{\mathfrak{L}_{\mathcal{G}}}/\mathcal{I}_C \cong \mathcal{O}_{\mathfrak{L}_{\mathcal{G}}^{\setminus C}} \cong \mathcal{O}_2, \quad \mathcal{I}_C \otimes \mathcal{K} \cong \mathcal{O}_{\left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]} \otimes \mathcal{K}.$$

The second example is the canonical λ-graph system for the Dyck shift D_2, that is not a sofic subshift. The subshift comes from automata theory and language theory (compare [1, 11]). Its alphabet Σ consists of two kinds of four brackets: (,), and [,]. The forbidden words consist of words that do not obey the standard bracket rules. Let $\mathfrak{L}^{D_2}$ be the canonical λ-graph system for D_2. In [29], the K-groups of the symbolic matrix system for $\mathfrak{L}^{D_2}$ have been computed. They are the K-groups for the associated C^*-algebra $\mathcal{O}_{\mathfrak{L}^{D_2}}$, so that we see $K_0(\mathcal{O}_{\mathfrak{L}^{D_2}}) \cong \mathbb{Z}^\infty$, and $K_1(\mathcal{O}_{\mathfrak{L}^{D_2}}) \cong 0$, where $\mathbb{Z}^\infty$ is the countable infinite sum of the group $\mathbb{Z}$. The C^*-algebra $\mathcal{O}_{\mathfrak{L}^{D_2}}$ has a proper ideal.

The λ-graph system $\mathfrak{L}^{D_2}$ satisfies condition (II). Let $\mathfrak{L}^{Ch(D_2)}$ be the λ-graph subsystem of $\mathfrak{L}^{D_2}$, called the Cantor horizon λ-graph system of D_2 (see [16] for details). Then $\mathfrak{L}^{Ch(D_2)}$ is aperiodic and a minimal irreducible component of $\mathfrak{L}^{D_2}$. Hence the associated algebra $\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}}$ is a simple purely infinite C^*-algebra realized as a quotient of $\mathcal{O}_{\mathfrak{L}^{D_2}}$ by an ideal corresponding to a saturated hereditary subset of $\mathfrak{L}^{D_2}$. In [16], its K-groups have been computed to be $K_0(\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}}) \cong \mathbb{Z}/2\mathbb{Z} \oplus C(\mathfrak{C}, \mathbb{Z})$, and $K_1(\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}}) \cong 0$, where $C(\mathfrak{C}, \mathbb{Z})$ denotes the abelian group of all $\mathbb{Z}$-valued continuous functions on a Cantor discontinuum $\mathfrak{C}$. As $\mathfrak{L}^{Ch(D_2)}$ is predecessor-separated, the algebra $\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}}$ is generated by only the four partial isometries S_α, $\alpha = (,), [,]$ corresponding to the brackets $(,), [,]$. Hence $\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}}$ is finitely generated, but its K_0-group is not finitely generated. This means that the algebra $\mathcal{O}_{\mathfrak{L}^{Ch(D_2)}}$ is simple and purely infinite, but not semi-projective (compare [3]). Full details and its generalizations are seen in [16] and [20].

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