RESOLUTION IN HÖLDER SPACES OF AN ELLIPTIC PROBLEM IN AN UNBOUNDED DOMAIN

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Abstract

In this paper we give new results concerning the maximal regularity of the strict solution of an abstract second-order differential equation, with non-homogeneous boundary conditions of Dirichlet type, and set in an unbounded interval. The right-hand term of the equation is a Hölder continuous function.

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1. Introduction

This work is devoted to the study of the second-order abstract differential equation

$$(1.1) \qquad u''(t) + Au(t) = f(t), \quad t \in (0, +\infty),$$

under non-homogeneous boundary conditions of Dirichlet type given by

$$(1.2) \qquad u(0) = \varphi, \quad u(+\infty) = 0,$$

where φ and $f(t)$ belong to a complex Banach space E and A is a closed linear operator with domain $D(A)$ not necessarily dense in E.

For $l \in \mathbb{N}$, we denote by $BUC^l([0, +\infty[\, ; E)$ the space of vector-valued functions with uniformly continuous and bounded derivatives up to order l in $[0, +\infty[$ and by $C^\sigma([0, +\infty[\, ; E)$ for $\sigma \in]0, 1[$, the space of bounded and σ-Hölder continuous functions $f : [0, +\infty[\to E$, such that $\sup_{t \in [0, +\infty[} \|f(t)\|_E < \infty$ and there exists $C > 0$ such that for all $t, \tau \in [0, +\infty[$,

$$\|f(t) - f(\tau)\|_E \leqslant C|t - \tau|^\sigma,$$

endowed with the norm

$$\|f\|_{C^\sigma([0,+\infty[;E)} = \sup_{t\in[0,+\infty[}\|f(t)\|_E + \sup_{t\neq\tau}\frac{\|f(t)-f(\tau)\|_E}{|t-\tau|^\sigma}$$

$$= \|f\|_\infty + [f]_{C^\sigma([0,+\infty[;E)}.$$

For simplicity, we shall write $C^\sigma(E)$ instead of $C^\sigma([0,+\infty[;E)$.

In the present study, we are interested in the existence, the uniqueness and the maximal regularity of the strict solution u in the Banach space $X = BUC([0,+\infty[;E)$, when the right-hand term f is regular (Hölder continuous function).

We recall that $u \in BUC([0,+\infty[;E)$ is a *strict solution* of Problem (1.1)–(1.2) if

$$u \in BUC([0,+\infty[;E) \cap BUC([0,+\infty[;D(A)),$$

and u satisfies (1.1) and (1.2).

Throughout this paper we assume that the resolvent of A verifies the hypothesis that there exists $K > 0$ such that for all $\lambda \geqslant 0$

$$(1.3) \qquad\qquad \left\|(A-\lambda I)^{-1}\right\|_{L(E)} \leqslant \frac{K}{1+\lambda}.$$

Equations (1.1)–(1.2) can be illustrated by the following example of a Laplacian problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) + \dfrac{\partial^2 u}{\partial x^2}(t,x) = f(t,x), & (t,x) \in (0,+\infty) \times (0,1), \\ u(0,x) = \varphi(x), \quad u(+\infty,x) = 0, & x \in (0,1), \\ u(t,0) = u(t,1) = 0. \end{cases}$$

Indeed, in $E = C([0,1])$, we can choose the operator A as follows:

$$D(A) = \{v \in C^2([0,1]) : v(0) = v(1) = 0\} \subset E,$$

$$(Av)(x) = v''(x), \quad v \in D(A).$$

Several authors have studied equation (1.1) under various (homogeneous or nonhomogeneous) boundary conditions, as well as in the case of variable operators, but on a finite interval. See, for instance, Krein [5], Sobolevskii [11], Véron [13], Kuyazyuk [6], Da Prato-Grisvard [9], Labbas [7].

Our approach is based on the direct use of the operational calculus and of Dunford's integrals as in Da Prato-Grisvard [9]. We make use of the real Banach interpolation spaces $D_A(\theta, +\infty)$, between $D(A)$ and E, which are well-known in many concrete cases and can be characterized by

$$D_A(\theta, +\infty) = \left\{\xi \in E : \sup_{t>0}\left\|t^\theta A(A-tI)^{-1}\xi\right\|_E < \infty\right\},$$

where $\theta \in (0, 1)$ (see Grisvard [4]).

Assumption (1.3) does not imply that A is an infinitesimal generator of an analytical semigroup. However it allows us to define $(-A)^{1/2}$, (for details, see Balakrishnan [1]). We do not use this fractional power of the operator nor the techniques of semigroups estimates generated by them as in Krein [5].

Our main results are the following.

THEOREM 1.1. *Let $\varphi \in D(A)$ and $f \in C^{2\theta}(E)$, with $\theta \in (0, 1/2)$ such that*

$$f(0) - A\varphi \in \overline{D(A)}.$$

Then Problem (1.1)–(1.2) admits a unique strict solution.

THEOREM 1.2. *Let $\varphi \in D(A)$ and $f \in C^{2\theta}(E)$, with $\theta \in (0, 1/2)$ such that*

$$f(0) - A\varphi \in D_A(\theta, +\infty).$$

Then the unique strict solution of (1.1)–(1.2) satisfies the property of maximal regularity $Au(\cdot)$, $u''(\cdot) \in C^{2\theta}(E)$.

If f is in a L^p-Lebesgue space, we can prove that the same representation given in (2.1) leads to the existence of a strict solution, because Lebesgue spaces satisfy the so-called UMD property. This no longer holds true for Hölder spaces; so the function needs more regularity (see, for instance, Favini *et al.* [3]).

The paper is organized as follows. In Section 2, we give the natural representation of the solution u to Problem (1.1)–(1.2) by using the operational calculus and Dunford's integral. In Section 3, we give necessary and sufficient conditions on φ and $A\varphi - f(0)$ to obtain the maximal smoothness of the solution u given by Dunford's integral when f is Hölderian. In Section 4, we present an example, to which our abstract results can be applied.

2. Construction of the solution

If A is a complex scalar z such that $\operatorname{Re} \sqrt{-z}$ is positive, then the solution of (1.1)–(1.2) is given by

$$u(t) = e^{-\sqrt{-z}\,t}\varphi - \int_0^{+\infty} k(t, s) f(s)\, ds,$$

where

$$k(t, s) = \begin{cases} \dfrac{e^{-\sqrt{-z}\,t} \sinh \sqrt{-z}\,s}{\sqrt{-z}} & 0 \leqslant s \leqslant t, \\[3mm] \dfrac{e^{-\sqrt{-z}\,s} \sinh \sqrt{-z}\,t}{\sqrt{-z}} & s \geqslant t. \end{cases}$$

Here $\sqrt{-z}$ is the analytic determination defined by $\operatorname{Re} \sqrt{-z} > 0$. In the general case, it is well known that Hypothesis (1.3) implies the existence of $\delta_0 \in (0, \pi/2)$ and $\epsilon_0 > 0$ such that the resolvent set of A contains a sectorial domain of the form

$$S_{\epsilon_0, \delta_0} = S = \{\lambda \in \mathbb{C}^* : |\arg \lambda| \leqslant \delta_0\} \cup \overline{B(O, \epsilon_0)},$$

where $B(O, \epsilon_0)$ is an open ball of radius ϵ_0. If γ denotes the sectorial boundary curve of S_{ϵ_0, δ_0} oriented positively, remaining in $\rho(A) \backslash \mathbb{R}_+$, and defined by

$$\gamma = \{z \in \mathbb{C} : |z| \geqslant \epsilon_0 \text{ and } |\arg z| = \delta_0\} \cup \{z = \epsilon_0 e^{i\upsilon} : \delta_0 \leqslant \upsilon \leqslant 2\pi - \delta_0\},$$

then the natural representation of the solution of (1.1)–(1.2), in the abstract case, is given by Dunford's integral

$$(2.1) \qquad u(t) = \frac{1}{2i\pi} \int_\gamma e^{-\sqrt{-z}t} (A - zI)^{-1} \varphi \, dz$$

$$- \frac{1}{2i\pi} \int_\gamma \int_0^{+\infty} k(t, s)(A - zI)^{-1} f(s) \, ds \, dz.$$

These integrals converge absolutely for every $t \in (0, +\infty)$. Indeed, we have for $z \in \gamma$, $\left| e^{-\sqrt{-z}t} \right| \leqslant e^{-\operatorname{Re}\sqrt{-z}t} = e^{-c_0|z|^{1/2}t}$, where $c_0 = \cos((\pi - \delta_0)/2)$. On the other hand, for any $f \in X$, we see that

$$\left\| \int_0^{+\infty} k(t, s)(A - zI)^{-1} f(s) \, ds \right\|_E \leqslant \frac{1}{c_0|z|^2} \|f\|_X.$$

Set, for $\varphi \in E$ and $t \in (0, +\infty)$,

$$B(t, A)\varphi = \frac{1}{2i\pi} \int_\gamma e^{-\sqrt{-z}t} (A - zI)^{-1} \varphi \, dz.$$

Then we have the following result, which allows us to study the properties of the solution u.

PROPOSITION 2.1. *Under Assumption* (1.3) *we have*

(1) *that there exists $K > 0$ depending only on γ such that for all $\varphi \in E$ and for all $t > 0$, $\|B(t, A)\varphi\|_E \leqslant K \|\varphi\|_E$;*

(2) *for all $\varphi \in E$ and for all $t > 0$, $B(t, A)\varphi \in D(A)$;*

(3) *$B(\cdot, A)\varphi \in X$ if and only if $\varphi \in \overline{D(A)}$.*

PROOF. (1) For $t > 0$, we can write

$$B(t, A)\varphi = \frac{1}{2i\pi} \int_{\gamma_t^+} e^{-\sqrt{-z}t} (A - zI)^{-1} \varphi \, dz + \frac{1}{2i\pi} \int_{\gamma_t^-} e^{-\sqrt{-z}t} (A - zI)^{-1} \varphi \, dz$$

$$= I_+ + I_-,$$

where

(2.2) $$\gamma_t^+ = \{z \in \gamma : |z| \geqslant 1/t^2\}, \quad \gamma_t^- = \{z \in \gamma : |z| \leqslant 1/t^2\}.$$

Then

$$\|I_+\|_E \leqslant K \int_{\gamma_t^+} e^{-(\operatorname{Re}\sqrt{-z})t} \frac{|dz|}{|z|} \|\varphi\|_E \leqslant K \int_1^{+\infty} \frac{e^{-c_0\sigma}}{\sigma^2/t^2} \frac{2\sigma}{t^2} d\sigma \|\varphi\|_E \leqslant K\|\varphi\|_E,$$

and

$$I_- = \frac{1}{2i\pi} \int_{\gamma_t^-} (e^{-\sqrt{-z}t} - 1)(A - zI)^{-1}\varphi \, dz - \frac{1}{2i\pi} \int_{C_t} (A - zI)^{-1}\varphi \, dz = I'_- + I''_-,$$

where $C_t = \{z = e^{i\upsilon}/t^2 : -\delta_0 \leqslant \upsilon \leqslant \delta_0\}$. For I'_-, we write

$$I'_- = \frac{1}{2i\pi} \int_{z \in \gamma_t^-, \, \epsilon_0 \leqslant |z| \leqslant 1/t^2} (e^{-\sqrt{-z}t} - 1)(A - zI)^{-1}\varphi \, dz$$
$$+ \frac{1}{2i\pi} \int_{z \in \gamma_t^-, \, z = \epsilon_0 e^{i\upsilon}} (e^{-\sqrt{-z}t} - 1)(A - zI)^{-1}\varphi \, dz = J_-^1 + J_-^2,$$

then

$$\|J_-^1\|_E \leqslant K \int_{\epsilon_0}^{1/t^2} \frac{|e^{-\sqrt{-z}t} - 1|}{|z|} d|z| \|\varphi\|_E \leqslant K \int_0^{1/t^2} |z|^{1/2} t \frac{d|z|}{|z|} \|\varphi\|_E \leqslant K|\varphi\|_E,$$

$$\|J_-^2\|_E \leqslant \frac{2\epsilon_0}{2\pi} \int_{\delta_0}^{2\pi-\delta_0} \left\| (A - \epsilon_0 e^{i\upsilon} I)^{-1}\varphi \right\|_E d\upsilon \leqslant K\|\varphi\|_E.$$

For I''_-, we have

$$\|I''_-\|_E \leqslant K \int_{-\delta_0}^{+\delta_0} \left\| \left(A - \frac{1}{t^2}e^{i\upsilon}\right)^{-1}\varphi \right\|_E \frac{1}{t^2} d\upsilon \leqslant K\|\varphi\|_E.$$

(2)   This rises from the convergence of the integral

$$\frac{1}{2i\pi} \int_\gamma e^{-\sqrt{-z}t} A(A - zI)^{-1}\varphi \, dz,$$

and from the fact that $e^{-\sqrt{-z}t}(A - zI)^{-1}\varphi \in D(A)$ for all $\varphi \in E$ and $t > 0$.

(3)   Fix $\epsilon > 0$ and let $\varphi \in \overline{D(A)}$, then there exists $\psi \in D(A)$ such that

(2.3) $$\|\varphi - \psi\|_E \leqslant \epsilon.$$

Using the resolvent's identity $(A - zI)^{-1}A\psi = \psi + z(A - zI)^{-1}\psi$, we obtain

$$B(t, A)\psi = \frac{1}{2i\pi} \int_\gamma \frac{e^{-\sqrt{-z}t}}{z}(A - zI)^{-1}A\psi \, dz.$$

Thanks to the inequality

$$\left\| \frac{e^{-\sqrt{-z}t}}{z}(A - zI)^{-1}A\psi \right\|_E \leqslant \frac{K}{|z|^2} \|A\psi\|_E,$$

Lebesgue's and Cauchy's theorems give us

$$\lim_{t \to 0^+} B(t, A)\psi = \frac{1}{2i\pi} \int_\gamma \frac{(A - zI)^{-1}}{z} A\psi \, dz = \psi.$$

Now from the equality

$$B(t, A)\varphi - \varphi = (B(t, A)\varphi - B(t, A)\psi) + (B(t, A)\psi - \psi) + (\psi - \varphi),$$

and from the estimate (2.3), we deduce that $B(t, A)\varphi - \varphi \to 0$ as $t \to 0^+$. The uniform continuity in $t > 0$ is easily verified.

Conversely, if $B(\cdot, A)\varphi \in X$, then for $z \in \gamma$ one has

$$(A - zI)^{-1} \lim_{t \to 0^+} B(t, A)\varphi = \lim_{t \to 0^+} (A - zI)^{-1} B(t, A)\varphi$$
$$= \lim_{t \to 0^+} B(t, A)(A - zI)^{-1}\varphi$$
$$= (A - zI)^{-1}\varphi,$$

wich implies that $\varphi = \lim_{t \to 0^+} B(t, A)\varphi \in \overline{D(A)}$.   □

PROPOSITION 2.2. *Under Assumption* (1.3) *and for* $\theta \in (0, 1/2)$ *we have*

$$B(\cdot, A)\varphi \in C^{2\theta}(E) \quad \text{if and only if} \quad \varphi \in D_A(\theta, +\infty).$$

PROOF. Let $\varphi \in D_A(\theta, +\infty)$ and $0 \leqslant \tau < t$. Thus

$$(\tau, A)\varphi - B(t, A)\varphi = \frac{1}{2i\pi} \int_\gamma \left(e^{-\sqrt{-z}\tau} - e^{-\sqrt{-z}t} \right) \frac{A(A - zI)^{-1}}{z} \varphi \, dz$$
$$= \frac{1}{2i\pi} \int_{z \in \gamma, |z| \geqslant (t-\tau)^{-2}} \left(e^{-\sqrt{-z}\tau} - e^{-\sqrt{-z}t} \right) \frac{A(A - zI)^{-1}}{z} \varphi \, dz$$
$$+ \frac{1}{2i\pi} \int_{z \in \gamma, |z| \leqslant (t-\tau)^{-2}} \left(e^{-\sqrt{-z}\tau} - e^{-\sqrt{-z}t} \right) \frac{A(A - zI)^{-1}}{z} \varphi \, dz,$$

and

$$\big\| B(\tau, A)\varphi - B(t, A)\varphi \big\|_E$$

$$\leqslant K \int_{z\in\gamma,\, |z|\geqslant(t-\tau)^{-2}} \frac{|dz|}{|z|^{\theta+1}} \|\varphi\|_{D_A(\theta,+\infty)}$$

$$+ K' \int_{z\in\gamma,\, |z|\leqslant(t-\tau)^{-2}} \frac{e^{-(\mathrm{Re}\sqrt{-z})\tau}|z|^{1/2}(t-\tau)}{|z|^{\theta+1}} |dz| \|\varphi\|_{D_A(\theta,+\infty)}$$

$$\leqslant K \int_{z\in\gamma,\, |z|\geqslant(t-\tau)^{-2}} \frac{|dz|}{|z|^{\theta+1}} \|\varphi\|_{D_A(\theta,+\infty)}$$

$$+ K' \int_{z\in\gamma,\, |z|\leqslant(t-\tau)^{-2}} \frac{|z|^{1/2}(t-\tau)}{|z|^{\theta+1}} |dz| \|\varphi\|_{D_A(\theta,+\infty)}$$

$$\leqslant \max(K, K')(t-\tau)^{2\theta} \|\varphi\|_{D_A(\theta,+\infty)}.$$

For the proof of the direct sense, we know (see Sinestrari [10]) that if B is the infinitesimal generator of an analytic semigroup $V(t)$, then $V(t)\varphi - \varphi = O(t^\alpha)$, as $t \to 0^+$, if and only if $\varphi \in D_B(\alpha, +\infty)$. Observe that $-(-A)^{1/2}$ is the infinitesimal generator of the analytic semigroup $V(t) = B(t, A)$ (see Krein [5]). Therefore,

$$\varphi \in D_{(-A)^{1/2}}(2\theta, +\infty) = D_A(\theta, +\infty).$$

The last equality holds by using the reiteration theorem in interpolation theory (see Lions-Peetre [8]). □

3. Smoothness of the solution

Now we can state some regularity properties of u and Au.

PROPOSITION 3.1. *Let $\varphi \in D(A)$ and $f \in C^{2\theta}(E)$, with $\theta \in (0, 1/2)$. Then*

(1) *for all $t \geqslant 0$, $u(t) \in D(A)$;*
(2) *$u(\cdot)$ and $u'(\cdot) \in X$;*
(3) *$S(\cdot) = Au(\cdot) - B(\cdot, A)(A\varphi - f(0)) \in C^{2\theta}(E)$;*
(4) *$Au(\cdot) \in X$ if and only if $A\varphi - f(0) \in \overline{D(A)}$;*
(5) *$Au(\cdot) \in C^{2\theta}(E)$ if and only if $A\varphi - f(0) \in D_A(\theta, +\infty)$.*

PROOF. (1) In the second integral in (2.1) we write $f(s) = (f(s) - f(t)) + f(t)$. Then, after a calculation of the integral in $f(t)$, we get

$$u(t) = B(t, A)\varphi - \frac{1}{2i\pi} \int_\gamma \int_0^{+\infty} k(t, s)(A - zI)^{-1}\big(f(s) - f(t)\big)\, ds\, dz$$

$$-\frac{1}{2i\pi}\int_\gamma \frac{e^{-\sqrt{-z}\,t}}{z}(A-zI)^{-1}f(t)\,dz$$

$$+\frac{1}{2i\pi}\int_\gamma \frac{1}{z}(A-zI)^{-1}f(t)\,dz$$

$$=I_1+I_2+I_3+I_4.$$

From Proposition 2.1, we deduce that $I_1 \in D(A)$. The convergence of the integral

$$\frac{1}{2i\pi}\int_\gamma \frac{e^{-\sqrt{-z}\,t}}{z}A(A-zI)^{-1}f(t)\,dz$$

implies that $I_3 \in D(A)$. For I_4 we use the Cauchy theorem, which gives

$$I_4 = A^{-1}f(t) \in D(A).$$

Finally for I_2, it is sufficient to prove the convergence of the integral

$$\frac{1}{2i\pi}\int_\gamma \int_0^{+\infty} k(t,s)A(A-zI)^{-1}(f(s)-f(t))\,ds\,dz.$$

In fact, we have

$$\left\| \frac{1}{2i\pi}\int_\gamma \int_0^{+\infty} k(t,s)A(A-zI)^{-1}(f(s)-f(t))\,ds\,dz \right\|_E$$

$$\leqslant K \int_{\epsilon_0}^{+\infty} \left(\sup_{t>0} \int_0^{+\infty} |k(t,s)||t-s|^{2\theta}\,ds \right) d|z| \|f\|_{C^{2\theta}(E)}$$

$$\leqslant K \int_{\epsilon_0}^{+\infty} \frac{d|z|}{|z|^{1+\theta}} \|f\|_{C^{2\theta}(E)},$$

where the last estimate holds by Hölder's inequality.

(2)   Let us prove that $u(\cdot) \in X$ (the same techniques give the result $u'(\cdot) \in X$). For $0 \leqslant \tau < t$, we get $u(t)-u(\tau) = I + \Delta_1 + \Delta_2 + J$, where

$$I = B(t,A)\varphi - B(\tau,A)\varphi, \qquad J = A^{-1}(f(t)-f(\tau)),$$

$$\Delta_1 = -\frac{1}{2i\pi}\int_\gamma \int_0^{+\infty} k(t,s)(A-zI)^{-1}(f(s)-f(t))\,ds\,dz$$

$$+\frac{1}{2i\pi}\int_\gamma \int_0^{+\infty} k(\tau,s)(A-zI)^{-1}(f(s)-f(\tau))\,ds\,dz,$$

$$\Delta_2 = -\frac{1}{2i\pi}\int_\gamma \frac{e^{-\sqrt{-z}\,t}}{z}(A-zI)^{-1}f(t)\,dz + \frac{1}{2i\pi}\int_\gamma \frac{e^{-\sqrt{-z}\,\tau}}{z}(A-zI)^{-1}f(\tau)\,dz.$$

The result is obvious for I, since $\varphi \in D(A)$. For J we use Assumption (1.3).

On the other hand, we have

$$
\Delta_2 = \frac{1}{2i\pi} \int_\gamma \frac{e^{-\sqrt{-z}t}}{z} (A - zI)^{-1} (f(\tau) - f(t)) \, dz
$$
$$
- \frac{1}{2i\pi} \int_\gamma \frac{\left(e^{-\sqrt{-z}t} - e^{-\sqrt{-z}\tau} \right)}{z} (A - zI)^{-1} f(\tau) \, dz
$$
$$
= \Delta_2' + \Delta_2'',
$$

where

$$
\|\Delta_2'\|_E \leqslant \left(\frac{1}{2\pi} \int_{\epsilon_0}^{+\infty} \frac{|dz|}{|z|^2} \right) (t - \tau)^{2\theta} \|f\|_{C^{2\theta}(E)} \leqslant K (t - \tau)^{2\theta} \|f\|_{C^{2\theta}(E)},
$$

and writing Δ_2'' as

$$
\Delta_2'' = \frac{1}{2i\pi} \int_\gamma \frac{1}{z} \left(\int_\tau^t \sqrt{-z} e^{-\sqrt{-z}\xi} \, d\xi \right) (A - zI)^{-1} f(\tau) \, dz,
$$

we obtain

$$
\|\Delta_2''\|_E \leqslant \left(\frac{1}{2\pi} \int_{\epsilon_0}^{+\infty} \frac{|dz|}{|z|^{3/2}} \right) (t - \tau) \|f\|_X \leqslant K (t - \tau) \|f\|_X.
$$

Now for the quantity Δ_1, one has

$$
\Delta_1 = -\frac{1}{2i\pi} \int_\gamma \int_\tau^t \frac{e^{-\sqrt{-z}t} \sinh \sqrt{-z}\, s}{\sqrt{-z}} (A - zI)^{-1} (f(s) - f(t)) \, ds \, dz
$$
$$
+ \frac{1}{2i\pi} \int_\gamma \int_\tau^t \frac{e^{-\sqrt{-z}s} \sinh \sqrt{-z}\, \tau}{\sqrt{-z}} (A - zI)^{-1} (f(s) - f(\tau)) \, ds \, dz
$$
$$
- \frac{1}{2i\pi} \int_\gamma \int_0^\tau \frac{e^{-\sqrt{-z}t} \sinh \sqrt{-z}\, s}{\sqrt{-z}} (A - zI)^{-1} (f(\tau) - f(t)) \, ds \, dz
$$
$$
- \frac{1}{2i\pi} \int_\gamma \int_0^\tau \frac{e^{-\sqrt{-z}t} - e^{-\sqrt{-z}\tau}}{\sqrt{-z}} \sinh \sqrt{-z}\, s (A - zI)^{-1} (f(s) - f(\tau)) \, ds \, dz
$$
$$
- \frac{1}{2i\pi} \int_\gamma \int_t^{+\infty} \frac{e^{-\sqrt{-z}s} \sinh \sqrt{-z}\, \tau}{\sqrt{-z}} (A - zI)^{-1} (f(\tau) - f(t)) \, ds \, dz
$$
$$
- \frac{1}{2i\pi} \int_\gamma \int_t^{+\infty} \frac{\left(\sinh \sqrt{-z}\, t - \sinh \sqrt{-z}\, \tau \right)}{e^{\sqrt{-z}s} \sqrt{-z}} (A - zI)^{-1} (f(s) - f(t)) \, ds \, dz
$$
$$
= \sum_{i=1}^{i=6} I_i.
$$

Then

$$\|I_1\|_E \leqslant K \int_\tau^t \left(\int_{\epsilon_0}^{+\infty} \frac{|dz|}{|z|^{3/2}} \right) (t-s)^{2\theta} \, ds \|f\|_{C^{2\theta}(E)} \leqslant K(t-\tau)^{2\theta+1} \|f\|_{C^{2\theta}(E)},$$

and

$$\|I_2\|_E \leqslant K \int_\tau^t \left(\int_{\epsilon_0}^{+\infty} \frac{|dz|}{|z|^{3/2}} \right) (s-\tau)^{2\theta} \, ds \|f\|_{C^{2\theta}(E)} \leqslant K(t-\tau)^{2\theta+1} \|f\|_{C^{2\theta}(E)}.$$

Writing I_4 and I_6 as

$$I_4 = \frac{1}{2i\pi} \int_\gamma \int_0^\tau \left(\int_\tau^t e^{-\sqrt{-z}\xi} \, d\xi \right) \sinh\sqrt{-z}s(A-zI)^{-1}(f(s)-f(\tau)) \, ds \, dz,$$

$$I_6 = -\frac{1}{2i\pi} \int_\gamma \int_t^{+\infty} e^{-\sqrt{-z}s} \left(\int_\tau^t \cosh\sqrt{-z}\xi \, d\xi \right) (A-zI)^{-1}(f(s)-f(t)) \, ds \, dz,$$

we get

$$\|I_4\|_E \leqslant \frac{1}{2\pi} \int_{\epsilon_0}^{+\infty} \left(\int_0^\tau \int_\tau^t e^{-\operatorname{Re}\sqrt{-z}(\xi-s)} (\tau-s)^{2\theta} \, d\xi \, ds \right) \frac{|dz|}{1+|z|} \|f\|_{C^{2\theta}(E)}$$

$$\leqslant K \int_0^\tau \int_{\tau-s}^{t-s} \frac{(\tau-s)^{2\theta}}{\eta^2} \left(\int_0^{+\infty} e^{-c_0\sigma} 2\sigma \, d\sigma \right) d\eta \, ds \|f\|_{C^{2\theta}(E)}$$

$$\leqslant K \int_0^\tau (\tau-s)^{2\theta} \left(\frac{1}{\tau-s} - \frac{1}{t-s} \right) ds \|f\|_{C^{2\theta}(E)}$$

$$\leqslant K(t-\tau) \int_0^\tau \frac{(\tau-s)^{2\theta-1}}{(t-\tau+\tau-s)} \, ds \|f\|_{C^{2\theta}(E)},$$

by making the change of variable $(\tau - s) = \xi(t - \tau)$, we obtain

$$\|I_4\|_E \leqslant K(t-\tau)^{2\theta} \left(\int_0^{+\infty} \frac{\xi^{2\theta-1}}{(1+\xi)} \, d\xi \right) \|f\|_{C^{2\theta}(E)} \leqslant K(t-\tau)^{2\theta} \|f\|_{C^{2\theta}(E)}.$$

For I_6, we have

$$\|I_6\|_E \leqslant K \int_{\epsilon_0}^{+\infty} \int_t^{+\infty} \int_\tau^t e^{-\operatorname{Re}\sqrt{-z}(s-\xi)} (s-t)^{2\theta} d\xi \, ds \frac{|dz|}{1+|z|} \|f\|_{C^{2\theta}(E)}$$

$$\leqslant K \int_t^{+\infty} \int_\tau^t \left(\int_0^{+\infty} e^{-c_0\sigma} \sigma \, d\sigma \right) \frac{(s-t)^{2\theta}}{(s-\xi)^2} d\xi \, ds \|f\|_{C^{2\theta}(E)}$$

$$\leqslant K \int_t^{+\infty} (s-t)^{2\theta} \left(\int_{s-t}^{s-\tau} \frac{d\eta}{\eta^2} \right) ds \|f\|_{C^{2\theta}(E)}$$

$$\leqslant K(t-\tau) \int_t^{+\infty} \frac{(s-t)^{2\theta-1}}{(s-t+t-\tau)} \, ds \|f\|_{C^{2\theta}(E)}$$

$$\leqslant K(t-\tau)^{2\theta}\left(\int_0^{+\infty}\frac{\rho^{2\theta-1}}{(1+\rho)}\,d\rho\right)\|f\|_{C^{2\theta}(E)}$$

$$\leqslant K(t-\tau)^{2\theta}\|f\|_{C^{2\theta}(E)}.$$

A direct calculation of the integrals in $(f(\tau)-f(t))$ implies that

$$I_3+I_5=\frac{1}{2i\pi}\int_\gamma\frac{e^{-\sqrt{-z}(t-\tau)}}{z}(A-zI)^{-1}(f(\tau)-f(t))\,dz$$

$$-\frac{1}{2i\pi}\int_\gamma\frac{e^{-\sqrt{-z}t}}{z}(A-zI)^{-1}(f(\tau)-f(t))\,dz$$

$$=J-\Delta_2'.$$

We write J as

$$J=\frac{1}{2i\pi}\int_{\gamma_{(t-\tau)}^+}\frac{e^{-\sqrt{-z}(t-\tau)}}{z}(A-zI)^{-1}(f(\tau)-f(t))\,dz$$

$$+\frac{1}{2i\pi}\int_{\gamma_{(t-\tau)}^-}\frac{e^{-\sqrt{-z}(t-\tau)}}{z}(A-zI)^{-1}(f(\tau)-f(t))\,dz$$

$$=J^++J^-,$$

(where $\gamma_{(t-\tau)}^+$ and $\gamma_{(t-\tau)}^-$ are defined in (2.2)), then

$$\|J^+\|_E\leqslant K\int_1^{+\infty}\frac{e^{-\operatorname{Re}\sqrt{-z}(t-\tau)}}{|z|}\frac{(t-\tau)^{2\theta}}{1+|z|}\,|dz|\|f\|_{C^{2\theta}(E)}$$

$$\leqslant K(t-\tau)^{2\theta}\left(\int_1^{+\infty}\frac{e^{-\operatorname{Re}\sqrt{-z}(t-\tau)}}{|z|}\,|dz|\right)\|f\|_{C^{2\theta}(E)}$$

$$\leqslant K(t-\tau)^{2\theta}\left(\int_1^{+\infty}\frac{e^{-c_0\sigma}}{\sigma}\,d\sigma\right)\|f\|_{C^{2\theta}(E)}$$

$$\leqslant K(t-\tau)^{2\theta}\|f\|_{C^{2\theta}(E)},$$

and for J^- we have

$$J^-=\frac{1}{2i\pi}\int_{\gamma_{(t-\tau)}^-}\left(e^{-\sqrt{-z}(t-\tau)}-1\right)\frac{(A-zI)^{-1}(f(\tau)-f(t))}{z}\,dz$$

$$+A^{-1}(f(\tau)-f(t))-\frac{1}{2i\pi}\int_{C_{(t-\tau)}}\frac{(A-zI)^{-1}(f(\tau)-f(t))}{z}\,dz$$

$$=J_1^-+A^{-1}(f(\tau)-f(t))+J_2^-,$$

with $C_{(t-\tau)}=\left\{z=(t-\tau)^{-2}e^{i\nu}:-\delta_0\leqslant\nu\leqslant\delta_0\right\}$.

So we obtain

$$\|J_1^-\|_E \leqslant K \int_0^{1/(t-\tau)^2} \frac{|z|^{1/2}(t-\tau)}{|z|} \frac{(t-\tau)^{2\theta}}{1+|z|} |dz| \|f\|_{C^{2\theta}(E)}$$

$$\leqslant K(t-\tau)^{2\theta+1} \int_0^{1/(t-\tau)^2} \frac{|z|^{1/2}}{|z|} |dz| \|f\|_{C^{2\theta}(E)} \leqslant K(t-\tau)^{2\theta} \|f\|_{C^{2\theta}(E)},$$

and

$$\|J_2^-\|_E \leqslant K \frac{(t-\tau)^{2\theta}}{1+(t-\tau)^{-2}} \int_{-\delta_0}^{\delta_0} (t-\tau)^2 \frac{dv\|f\|_{C^{2\theta}(E)}}{(t-\tau)^2} \leqslant K(t-\tau)^{2\theta} \|f\|_{C^{2\theta}(E)}.$$

(3)   We have

$$S(t) = \frac{1}{2i\pi} \int_\gamma \frac{e^{-\sqrt{-z}t}}{z} A(A-zI)^{-1} A\varphi\, dz$$

$$- \frac{1}{2i\pi} \int_\gamma \int_0^{+\infty} k(t,s) A(A-zI)^{-1}(f(s)-f(t))\, ds\, dz$$

$$- \frac{1}{2i\pi} \int_\gamma \frac{e^{-\sqrt{-z}t}}{z} A(A-zI)^{-1} f(t)\, dz + f(t)$$

$$- \frac{1}{2i\pi} \int_\gamma \frac{e^{-\sqrt{-z}t}}{z} A(A-zI)^{-1}(A\varphi - f(0))\, dz,$$

thus

$$S(t) = \frac{1}{2i\pi} \int_\gamma \frac{e^{-\sqrt{-z}t}}{z} A(A-zI)^{-1} f(0)\, dz$$

$$- \frac{1}{2i\pi} \int_\gamma \int_0^{+\infty} k(t,s) A(A-zI)^{-1}(f(s)-f(t))\, ds\, dz$$

$$- \frac{1}{2i\pi} \int_\gamma \frac{e^{-\sqrt{-z}t}}{z} A(A-zI)^{-1} f(t)\, dz + f(t).$$

Let $0 \leqslant \tau < t$, then $S(t) - S(\tau) = f(t) - f(\tau) + \Lambda + \Pi$, where

$$\Lambda = \frac{1}{2i\pi} \int_\gamma \int_0^{+\infty} k(\tau,s) A(A-zI)^{-1} (f(s)-f(\tau))\, ds\, dz$$

$$- \frac{1}{2i\pi} \int_\gamma \int_0^{+\infty} k(t,s) A(A-zI)^{-1} (f(s)-f(t))\, ds\, dz,$$

and

$$\Pi = \frac{1}{2i\pi} \left(\int_\gamma \frac{e^{-\sqrt{-z}t}}{z} A(A-zI)^{-1} f(0)\, dz - \int_\gamma \frac{e^{-\sqrt{-z}t}}{z} A(A-zI)^{-1} f(t)\, dz \right)$$

$$- \frac{1}{2i\pi} \left(\int_\gamma \frac{e^{-\sqrt{-z}\,\tau}}{z} A(A - zI)^{-1} f(0)\, dz - \int_\gamma \frac{e^{-\sqrt{-z}\,\tau}}{z} A(A - zI)^{-1} f(\tau)\, dz \right).$$

Regarding the quantity Π, we have

$$
\begin{aligned}
\Pi &= \frac{1}{2i\pi} \int_\gamma \frac{\left(e^{-\sqrt{-z}\,t} - e^{-\sqrt{-z}\,\tau} \right)}{z} A(A - zI)^{-1} f(0)\, dz \\
&\quad + \frac{1}{2i\pi} \int_\gamma \frac{e^{-\sqrt{-z}\,\tau}}{z} A(A - zI)^{-1} f(\tau)\, dz \\
&\quad + \frac{1}{2i\pi} \int_\gamma \frac{e^{-\sqrt{-z}\,t}}{z} A(A - zI)^{-1} f(\tau)\, dz \\
&\quad - \frac{1}{2i\pi} \int_\gamma \frac{e^{-\sqrt{-z}\,t}}{z} A(A - zI)^{-1} f(\tau)\, dz \\
&\quad - \frac{1}{2i\pi} \int_\gamma \frac{e^{-\sqrt{-z}\,t}}{z} A(A - zI)^{-1} f(t)\, dz \\
&= -\frac{1}{2i\pi} \int_\gamma \frac{\left(e^{-\sqrt{-z}\,t} - e^{-\sqrt{-z}\,\tau} \right)}{z} A(A - zI)^{-1} \left(f(\tau) - f(0) \right) dz \\
&\quad + \frac{1}{2i\pi} \int_\gamma \frac{e^{-\sqrt{-z}\,t}}{z} A(A - zI)^{-1} \left(f(\tau) - f(t) \right) dz \\
&= \Pi_1 + \Pi_2.
\end{aligned}
$$

From Proposition 2.1, we deduce that

$$\|\Pi_2\|_E = \|B(t, A)(f(\tau) - f(t))\|_E \leqslant K(t - \tau)^{2\theta} \|f\|_{C^{2\theta}(E)}.$$

For Π_1, we get

$$\|\Pi_1\|_E \leqslant K \int_\tau^t \int_{\epsilon_0}^{+\infty} \frac{|z|^{1/2} e^{-c_0 |z|^{1/2} s} \tau^{2\theta}}{|z|} \, |dz|\, ds \|f\|_{C^{2\theta}(E)},$$

and setting $|z|^{1/2} s = \sigma$ in this last inequality, we obtain

$$
\begin{aligned}
\|\Pi_1\|_E &\leqslant K \int_\tau^t \frac{\tau^{2\theta}}{s} \left(\int_0^{+\infty} e^{-c_0 \sigma} d\sigma \right) ds \|f\|_{C^{2\theta}(E)} \leqslant K \left(\int_\tau^t s^{2\theta - 1}\, ds \right) \|f\|_{C^{2\theta}(E)} \\
&\leqslant K \left(t^{2\theta} - \tau^{2\theta} \right) \|f\|_{C^{2\theta}(E)} \leqslant K(t - \tau)^{2\theta} \|f\|_{C^{2\theta}(E)}.
\end{aligned}
$$

Λ may be written as

$$\Lambda = -\frac{1}{2i\pi} \int_\gamma \int_\tau^t \frac{e^{-\sqrt{-z}\,t} \sinh \sqrt{-z}\, s}{\sqrt{-z}} A(A - zI)^{-1}(f(s) - f(t))\, ds\, dz$$

$$+ \frac{1}{2i\pi} \int_\gamma \int_\tau^t \frac{e^{-\sqrt{-z}s} \sinh \sqrt{-z}\,\tau}{\sqrt{-z}} A(A - zI)^{-1}(f(s) - f(\tau))\, ds\, dz$$

$$- \frac{1}{2i\pi} \int_\gamma \int_0^\tau \frac{e^{-\sqrt{-z}t} \sinh \sqrt{-z}\,s}{\sqrt{-z}} A(A - zI)^{-1}(f(\tau) - f(t))\, ds\, dz$$

$$- \frac{1}{2i\pi} \int_\gamma \int_0^\tau \frac{\left(e^{-\sqrt{-z}t} - e^{-\sqrt{-z}\tau}\right)}{\sqrt{-z}} \sinh \sqrt{-z}\,s$$

$$\times A(A - zI)^{-1}(f(s) - f(\tau))\, ds\, dz$$

$$- \frac{1}{2i\pi} \int_\gamma \int_t^{+\infty} \frac{e^{-\sqrt{-z}s} \sinh \sqrt{-z}\,\tau}{\sqrt{-z}} A(A - zI)^{-1}(f(\tau) - f(t))\, ds\, dz$$

$$- \frac{1}{2i\pi} \int_\gamma \int_t^{+\infty} \frac{e^{-\sqrt{-z}s}\left(\sinh \sqrt{-z}\,t - \sinh \sqrt{-z}\,\tau\right)}{\sqrt{-z}}$$

$$\times A(A - zI)^{-1}(f(s) - f(t))\, ds\, dz$$

$$= \sum_{i=1}^{i=6} \Lambda_i.$$

For Λ_1 and Λ_2, we prove that

$$\|\Lambda_1\|_E \leqslant K \int_{\epsilon_0}^{+\infty} \int_\tau^t \frac{e^{-\operatorname{Re}\sqrt{-z}(t-s)}}{|z|^{1/2}} |t - s|^{2\theta}\, ds|dz| \|f\|_{C^{2\theta}(E)}$$

$$\leqslant K \int_\tau^t (t - s)^{2\theta-1} \left(\int_0^{+\infty} e^{-c_0\sigma}\, d\sigma\right) ds \|f\|_{C^{2\theta}(E)}$$

$$\leqslant K(t - \tau)^{2\theta} \|f\|_{C^{2\theta}(E)},$$

$$\|\Lambda_2\|_E \leqslant K \int_{\epsilon_0}^{+\infty} \int_\tau^t \frac{e^{-\operatorname{Re}\sqrt{-z}(s-\tau)}}{|z|^{1/2}} |s - \tau|^{2\theta}\, ds|dz| \|f\|_{C^{2\theta}(E)}$$

$$\leqslant K \int_\tau^t (s - \tau)^{2\theta-1} \left(\int_0^{+\infty} e^{-c_0\sigma}\, d\sigma\right) ds \|f\|_{C^{2\theta}(E)}$$

$$\leqslant K(t - \tau)^{2\theta} \|f\|_{C^{2\theta}(E)}.$$

Writing Λ_4 and Λ_6 as

$$\Lambda_4 = \frac{1}{2i\pi} \int_\gamma \int_0^\tau \left(\int_\tau^t e^{-\sqrt{-z}\xi} d\xi\right) \sinh \sqrt{-z}\,s\, A(A - zI)^{-1}(f(s) - f(\tau))\, ds\, dz,$$

$$\Lambda_6 = -\frac{1}{2i\pi} \int_\gamma \int_t^{+\infty} e^{-\sqrt{-z}s} \left(\int_\tau^t \cosh \sqrt{-z}\xi\, d\xi\right) A(A - zI)^{-1}(f(s) - f(t))\, ds\, dz,$$

we obtain

$$\|\Lambda_4\|_E \leqslant K \int_{\epsilon_0}^{+\infty} \int_0^\tau \int_\tau^t e^{-\operatorname{Re}\sqrt{-z}(\xi-s)} (\tau - s)^{2\theta}\, d\xi\, ds|dz| \|f\|_{C^{2\theta}(E)}$$

$$\leqslant K \int_0^\tau \int_{\tau-s}^{t-s} \frac{(\tau-s)^{2\theta}}{\eta^2} \left(\int_0^{+\infty} e^{-c_0\sigma} 2\sigma \, d\sigma \right) d\eta \, ds \|f\|_{C^{2\theta}(E)}$$

$$\leqslant K(t-\tau)^{2\theta} \|f\|_{C^{2\theta}(E)},$$

and

$$\|\Lambda_6\|_E \leqslant \frac{1}{2\pi} \int_{\epsilon_0}^{+\infty} \int_t^{+\infty} \int_\tau^t e^{-\operatorname{Re}\sqrt{-z}(s-\xi)} (s-t)^{2\theta} \, d\xi \, ds \, |dz| \|f\|_{C^{2\theta}(E)}$$

$$\leqslant K \int_t^{+\infty} \int_\tau^t \left(\int_0^{+\infty} e^{-c_0\sigma} \sigma \, d\sigma \right) \frac{(s-t)^{2\theta}}{(s-\xi)^2} \, d\xi \, ds \|f\|_{C^{2\theta}(E)}$$

$$\leqslant K \int_t^{+\infty} (s-t)^{2\theta} \left(\int_{s-t}^{s-\tau} \frac{d\eta}{\eta^2} \right) ds \|f\|_{C^{2\theta}(E)}$$

$$\leqslant K(t-\tau)^{2\theta} \|f\|_{C^{2\theta}(E)}.$$

By calculating the integrals in $(f(\tau) - f(t))$, we get

$$\Lambda_3 + \Lambda_5 = \frac{1}{2i\pi} \int_\gamma \frac{e^{-\sqrt{-z}(t-\tau)}}{z} A(A-zI)^{-1}(f(\tau)-f(t)) \, dz$$

$$- B(t,A)(f(\tau)-f(t))$$

$$= Q - \Pi_2,$$

where $Q = B(t-\tau, A)(f(\tau) - f(t))$. Proposition 2.1 implies that

$$\|Q\|_E \leqslant K(t-\tau)^{2\theta} \|f\|_{C^{2\theta}(E)}.$$

(4) Assume that $A\varphi - f(0) \in \overline{D(A)}$. Then, by Proposition 2.1

$$Au(\cdot) = S(\cdot) + B(\cdot, A)(A\varphi - f(0)) \in X.$$

Conversely, if $Au(\cdot) \in X$, the function $B(\cdot, A)(A\varphi - f(0)) = Au(\cdot) - S(\cdot)$, is in X. Now using Proposition 2.1, we obtain $A\varphi - f(0) \in \overline{D(A)}$.

(5) Let us suppose that $A\varphi - f(0) \in D_A(\theta, +\infty)$. By Proposition 2.1, $Au(\cdot) \in C^{2\theta}(E)$. Conversely, if $Au(\cdot) \in C^{2\theta}(E)$, then $B(\cdot, A)(A\varphi - f(0)) \in C^{2\theta}(E)$, and $A\varphi - f(0) \in D_A(\theta, +\infty)$. □

REMARK. By using the same methods and techniques of calculation, we have a similar result to Proposition 3.1, when the right-hand term of the equation has a spatial smoothness, that is, for all $t \geqslant 0$, $f(t) \in D_A(\theta, +\infty)$, $\sup_{t\geqslant 0} \|f(t)\|_{D_A(\theta, +\infty)} < \infty$, with $\theta \in (0, 1/2)$. See [2] for details.

Now, we can deduce our main results concerning the regularity of u.

THEOREM 3.2. *Let $\varphi \in D(A)$ and $f \in C^{2\theta}(E)$, with $\theta \in (0, 1/2)$ such that*

$$f(0) - A\varphi \in \overline{D(A)}.$$

Then u, given in (2.1), *is a strict solution of* (1.1)–(1.2).

By the continuity of $B(\cdot, A)\varphi$ and Lebesgue's theorem we can verify that $u(0) = \varphi$ and $u(+\infty) = 0$. On the other hand, we prove that u verifies (1.1) by using Dunford's operational calculus.

Finally, by Proposition 3.1, the solution u belongs to

$$BUC^2([0, +\infty[\, ; E) \cap BUC([0, +\infty[\, ; D(A)).$$

Furthermore, if $f(0) - A\varphi \in D_A(\theta, +\infty)$, then we have more regularity on $Au(\cdot)$ and $u''(\cdot)$.

THEOREM 3.3. *Let $\varphi \in D(A)$ and $f \in C^{2\theta}(E)$, with $\theta \in (0, 1/2)$ such that*

$$f(0) - A\varphi \in D_A(\theta, +\infty).$$

Then the unique strict solution of (1.1)–(1.2) *satisfies the property of maximal regularity $Au(\cdot), u''(\cdot) \in C^{2\theta}(E)$.*

PROOF. It suffices to apply the previous results, using the fact that $D_A(\theta, +\infty) \subset \overline{D(A)}$. □

4. Example

We give an example governed by (1.1)–(1.2). Consider $E = C([0, 1])$ and

$$D(A) = \{v \in C^2([0, 1]) : v(0) = v(1) = 0\},$$
$$Av = v''.$$

It is easy to check that A satisfies Assumption (1.3). We can thus apply our results to Laplacian problem in an infinite interval, given by

(4.1)
$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t, x) + \dfrac{\partial^2 u}{\partial x^2}(t, x) = f(t, x), & (t, x) \in (0, +\infty) \times (0, 1), \\ u(0, x) = \varphi(x), & x \in (0, 1), \\ u(+\infty, x) = 0, & x \in (0, 1), \\ u(t, 0) = u(t, 1) = 0. \end{cases}$$

Observe that conditions $\varphi \in D(A)$ and $f(0) - A\varphi \in \overline{D(A)}$, become

$$
\text{(4.2)} \quad
\begin{aligned}
& \varphi \in C^2([0, 1]) : \varphi(0) = \varphi(1) = 0, \\
& f(0, \cdot) - \varphi''(\cdot) \in C([0, 1]) \quad \text{and} \\
& f(0, 0) - \varphi''(0) = f(0, 1) - \varphi''(1) = 0.
\end{aligned}
$$

The interpolation space $D_A(\theta, +\infty)$ is given by

$$
D_A(\theta, +\infty) =
\begin{cases}
\{v \in C^{2\theta}([0, 1]) : v(0) = v(1) = 0\} & \text{if } 2\theta < 1, \\
C^{1,*}([0, 1]), & \text{if } 2\theta = 1, \\
\{v \in C^{1,2\theta-1}([0, 1]) : v(0) = v(1) = 0\} & \text{if } 2\theta > 1,
\end{cases}
$$

where $C^{1,*}([0, 1])$ is a Zigmund space (see, for example, [12]). Applying Theorem 3.2 and Theorem 3.3 we have the following results.

THEOREM 4.1. *Let $f \in C^{2\theta}([0, +\infty[\,; C([0, 1]))$, with $2\theta \in (0, 1)$, be such that the conditions* (4.2) *are satisfied. Then Problem* (4.1) *has a unique solution u satisfying $u \in BUC^2([0, +\infty[\,; C([0, 1])) \cap BUC([0, +\infty[\,; C^2([0, 1]))$.*

THEOREM 4.2. *Let $f \in C^{2\theta}([0, +\infty[\,; C([0, 1]))$, with $2\theta \in (0, 1)$, be such that*

$$
\begin{cases}
\varphi \in C^2([0, 1]) : \varphi(0) = \varphi(1) = 0, \\
f(0, \cdot) - \varphi''(\cdot) \in C^{2\theta}([0, 1]) \quad \text{and} \\
f(0, 0) - \varphi''(0) = f(0, 1) - \varphi''(1) = 0.
\end{cases}
$$

Then Problem (4.1) *has a unique solution*

$$
u \in BUC^2([0, +\infty[\,; C([0, 1])) \cap BUC([0, +\infty[\,; C^2([0, 1])).
$$

Moreover, u satisfies the maximal regularity

$$
\partial_x^2 u(\cdot, x), \, \partial_t^2 u(\cdot, x) \in C^{2\theta}([0, +\infty[\,; C([0, 1])).
$$

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