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NORMAL CHARACTERIZATION BY ZERO CORRELATIONS

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Abstract

Suppose X_i, $i = 1, \dots, n$ are independent and identically distributed with $E|X_1|^r < \infty$, $r = 1, 2, \dots$. If $\mathrm{Cov}\left((\bar{X} - \mu)^r, S^2\right) = 0$ for $r = 1, 2, \dots,$ where $\mu = EX_1$, $S^2 = \sum_{i=1}^n (X_i - \bar{X})^2/(n-1)$, and $\bar{X} = \sum_{i=1}^n X_i/n$, then we show that $X_1 \sim \mathcal{N}(\mu, \sigma^2)$, where $\sigma^2 = \mathrm{Var}(X_1)$. This covariance zero condition characterizes the normal distribution. It is a moment analogue, by an elementary approach, of the classical characterization of the normal distribution by independence of $\bar{X}$ and S^2 using semi-invariants. More generally, if $\mathrm{Cov}\left((\bar{X} - \mu)^r, S^2\right) = 0$ for $r = 1, \dots, k$, then $E((X_1 - \mu)/\sigma)^{r+2} = EZ^{r+2}$ for $r = 1, \dots, k$, where $Z \sim \mathcal{N}(0, 1)$. Conversely $\mathrm{Corr}((\bar{X} - \mu)^r, S^2)$ may be arbitrarily close to unity in absolute value, but for unimodal X_1, $\mathrm{Corr}^2(\bar{X}, S^2) < 15/16$, and this bound is the best possible.

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1. Introduction

The classical characterization of the normal distribution by independence of $\bar{X}$ and S^2 by Geary [4] depended heavily on a result of Fisher [3], obtained from his introduction of k-statistics, whose sampling cumulants were shown to be obtainable by combinatorial methods (see also David and Barton [1]; Kendall and Stuart [7, Chapter 12]). Our approach is direct, but still depends on the prior finite moments assumption.

In a famous paper Lukacs [9] showed, using characteristic functions and dispensing with the excessive moment conditions providing only $E(X_1^2) < \infty$, that independence of $\bar{X}$ and S^2 is enough to characterize the normal distribution. Kawata and Sakamoto [6] in a paper submitted in 1944 but whose publication was delayed due to World War II, showed (understandably unaware of Lukacs [9]) also using characteristic function technology, that the condition $E(X_1^2) < \infty$ could be removed entirely.

A simpler proof of this is indicated in Quine [11]. These few remarks supplement the reviews of characterizations of the normal distribution within Quine [11] and Quine and Seneta [12] regarding assumptions behind these characterizations.

In this paper, we show that if, instead of the independence of $\bar{X}$ and S^2, we suppose that (all moments are finite and)

$$\mathrm{Cov}\left((\bar{X}-\mu)^r, S^2\right) = 0, \quad r = 1, 2, \ldots,$$

then the underlying distribution is normal. In other words, the *infinitely many zero correlations* condition has the same effect as if we supposed independence. The form of this characterization (by zero correlations) appears to be new. However, if we first express the zero correlations condition in an equivalent form as

$$E\left((\bar{X}-\mu)^r, S^2\right) = \sigma^2 E\left((\bar{X}-\mu)^r\right), \quad r = 0, 1, 2, \ldots,$$

we have, according to Lukacs [10], that this last condition is equivalent to the so-called regression property $E(S^2|\bar{X}) = E(S^2)$. Therefore it is equivalent to the classical differential equation for the characteristic function of the normal probability density function which has received exhaustive treatment in the past 50 years (see Kagan, Linnik and Rao [5, Theorem 6.3.1] and Rao and Shanbhag [13, Chapter 9]). Thus we see an unexpected equivalence of the zero correlations condition and the regression condition.

However, the main new result which emerges from our working is the consequence of assuming only a *fixed finite number of zero correlations*. It may be stated as follows.

THEOREM. *Suppose that the distribution of the X's has its first $k+2$ moments finite. Then the zero correlations condition*

$$\mathrm{Cov}\left((\bar{X}-\mu)^r, S^2\right) = 0, \quad r = 1, 2, \ldots, k,$$

implies that the distribution of the X's has the same j-th central moments, $j = 1, 2, \ldots, k+2$, *as a normal distribution with the same mean and variance.*

For example, if $k = 1$, then the skewness of the underlying distribution of X is zero.

In the proof of the characterization below, the only change needed to see the validity of the theorem, is to replace $r = 1, 2, \ldots$ with $r = 1, 2, \ldots, k$.

In our final section, Section 3, we examine the other extreme: there exist distributions where the same correlations may be as close to $+1$ or to -1 as we like. However, using a theorem of Khinchin on unimodal distributions (of which the normal distribution is one), it is shown that for such distributions the absolute value of correlations is bounded away from 1, and, in fact, we derive a best possible bound.

This paper is in the spirit of work on characterization by Lukacs and Lancaster, who both published in the first volume of the Journal of the Australian Mathematical Society.

2. Proof of the characterization

By putting $X_i - \mu$ in place of X_i, since S^2 is unaffected, we may without loss of generality assume $\mu = EX_1 = 0$ and so our assumption becomes

$$\mathrm{Cov}(\bar{X}^r, S^2) = 0 \quad r = 1, 2, \ldots,$$

that is,

$$\mathrm{Cov}\left(\left(\sum_{i=1}^n X_i\right)^r, \sum_{i=1}^n (X_i - \bar{X})^2\right) = 0, \quad r = 1, 2, \ldots. \tag{1}$$

Let $\mu_r = E(X^r)$, so $\mu_1 = \mu = 0$, $\mu_2 = E(X^2) = \sigma^2$. Recall that

$$\sum_{i=1}^n (X_i - \bar{X})^2 = \sum_{i=1}^n X_i^2 - \frac{\left(\sum_{i=1}^n X_i\right)^2}{n}, \quad \text{and} \quad E(S^2) = \sigma^2. \tag{2}$$

We first consider the case of an odd r. Put $r = 2w - 1$, $w = 1, 2, \ldots$. Then by (1) and (2),

$$\begin{aligned}
&\mathrm{Cov}\left(\left(\sum_{i=1}^n X_i\right)^r, \sum_{i=1}^n (X_i - \bar{X})^2\right) \\
&= E\left\{\left(\sum_{i=1}^n X_i\right)^{2w-1}\left(\sum_{i=1}^n X_i^2\right) - \frac{1}{n}\left(\sum_{i=1}^n X_i\right)^{2w+1} \right. \\
&\qquad \left. - (n-1)\sigma^2\left(\sum_{i=1}^n X_i\right)^{2w-1}\right\}.
\end{aligned} \tag{3}$$

We prove, by induction on w, that if (1) holds for all $r = 2v - 1$, $v \geq 1$, then

$$\mu_{2v-1} = 0, \quad v \geq 1. \tag{4}$$

At $w = 1$, by independence of the X_i's and since $EX_i = 0$, (3) becomes $\sum_{i=1}^n E(X_i^3) = nE(X_1^3)$ since the X_i's are identically distributed. Hence, using (1) at $r = 3$, gives $\mu_3 = 0$.

Now suppose $\mu_{2s-1} = 0$, $s = 1, 2, \ldots, w$. Expanding $\left(\sum_{i=1}^{n} X_i\right)^{2w-1}$ in general gives a sum of products; at least one factor of each summand must be an X_i to an odd power, since all powers must add to $2w-1$. Multiplying $\left(\sum_{i=1}^{n} X_i\right)^{2w-1}$ by $\left(\sum_{i=1}^{n} X_i^2\right)$ makes the odd powers of summand factors at most $2w-1$, except for leading terms which are of form X_i^{2w+1}. Thus, by the inductive assumption,

$$E\left(\left(\sum_{i=1}^{n} X_i\right)^{2w-1}\left(\sum_{i=1}^{n} X_i^2\right)\right) = n\mu_{2w+1}.$$

A similar argument gives

$$E\left(\left(\sum_{i=1}^{n} X_i\right)^{2w+1}\right) = n\mu_{2w+1} \quad \text{and} \quad E\left(\left(\sum_{i=1}^{n} X_i\right)^{2w-1}\right) = 0.$$

Thus (3) becomes $(n-1)\mu_{2w+1}$. Hence, using (1) at $r = 2w+1$ gives $\mu_{2w+1} = 0$, completing the induction. We needed to only assume that (1) held at all $r = 2v-1$, $v = 1, 2, \ldots$.

We next consider the case of even r. For $r = 2w$, $w \geq 1$, using (1) and (2) yields

(5)
$$\begin{aligned}\operatorname{Cov}&\left(\left(\sum_{i=1}^{n} X_i\right)^{r}, \sum_{i=1}^{n}(X_i - \bar{X})^2\right)\\ &= E\left\{\left(\sum_{i=1}^{n} X_i\right)^{2w}\left(\sum_{i=1}^{n} X_i^2\right) - \frac{1}{n}\left(\sum_{i=1}^{n} X_i\right)^{2(w+1)}\right.\\ &\qquad \left. - (n-1)\sigma^2\left(\sum_{i=1}^{n} X_i\right)^{2w}\right\}.\end{aligned}$$

We prove by induction that if (1) holds for all $r \geq 1$, then

(6)
$$\mu_{2v} = \sigma^{2v}\frac{(2v)!}{2^v v!}, \qquad v \geq 1$$

and

(7)
$$E\left\{\left(\sum_{i=1}^{n} X_i\right)^{2v}\right\} = \sigma^{2v}\frac{(2v)!}{2^v v!}n^v, \qquad v \geq 1.$$

Both are clearly true at $v = 1$. Now suppose both are true for $v = 1, \ldots, w$, and consider

(8)
$$E\left(\sum_{i=1}^{n} X_i\right)^{2(w+1)} = \sum \binom{2(w+1)}{j_1, j_2, \ldots, j_n} E\left(X_1^{j_1} X_2^{j_2} \cdots X_n^{j_n}\right),$$

where the sum is over all partitions of $2(w+1)$, $\sum_{i=1}^{n} j_i = 2(w+1)$, where each of $j_1, \ldots, j_n$ is an *even* integer. Contributions from products involving X_i to an odd-power are zero since we have proved $\mu_{2r+1} = 0$, $r = 0, 1, 2, \ldots$. Thus putting $j_i = 2x_i$, the summation is over all partitions of length n and hence expression (8) becomes

$$E\left(\sum_{i=1}^{n} X_i\right)^{2(w+1)} = \sum_{x_1+\cdots+x_n=w+1} \binom{2(w+1)}{2x_1, \ldots, 2x_n} E\left(X_1^{2x_1} \cdots X_n^{2x_n}\right)$$
$$= \sum_{x_1+\cdots+x_n=w+1} \binom{2(w+1)}{2x_1, \ldots, 2x_n} \mu_{2x_1} \cdots \mu_{2x_n}.$$

Taking out the leading terms gives

$$n\mu_{2(w+1)} + \sum_{x_1+\cdots+x_n} \binom{2(w+1)}{2x_1, \ldots, 2x_n} \mu_{2x_1} \cdots \mu_{2x_n},$$

where in the summation $x_i \le w$, $i = 1, \ldots, n$. By induction on w (using (6) for $v \le w$) yields

$$E\left(\sum_{i=1}^{n} X_i\right)^{2(w+1)}$$
$$= n\mu_{2(w+1)} + \sum_{x_1+\cdots+x_n=w+1} \binom{2(w+1)}{2x_1, \ldots, 2x_n} \frac{(2x_1)!}{2^{x_1}x_1!} \cdots \frac{(2x_n)!}{2^{x_n}x_n!} \sigma^{2(w+1)}$$
$$= n\mu_{2(w+1)} + \sum \frac{(2(w+1))!}{2^{w+1}x_1!x_2! \cdots x_n!} \sigma^{2(w+1)}$$
$$= n\mu_{2(w+1)} + \frac{(2(w+1))!}{2^{w+1}(w+1)!} \sigma^{2(w+1)} \sum_{x_i \le w} \frac{(w+1)!}{x_1!x_2! \cdots x_n!}$$
$$= n\mu_{2(w+1)} + \frac{(2(w+1))!}{2^{w+1}(w+1)!} \sigma^{2(w+1)} \left\{ (\overbrace{1 + \cdots + 1}^{n})^{w+1} - n \right\}.$$

Therefore,

$$E\left\{\left(\sum_{i=1}^{n} X_i\right)^{2(w+1)}\right\} = n\mu_{2(w+1)} + \frac{(2(w+1))!}{2^{w+1}(w+1)!} \sigma^{2(w+1)} \{n^{w+1} - n\}. \tag{9}$$

Also, from the inductive hypothesis, (using (7)) we have

$$E\left\{\left(\sum_{i=1}^{n} X_i\right)^{2w}\right\} = \sigma^{2w} \frac{(2w)!}{2^w w!} n^w. \tag{10}$$

Next consider

$$E\left\{\left(\sum_{i=1}^n X_i\right)^{2w}\left(\sum_{i=1}^n X_i^2\right)\right\} = \left(\sum_{i=1}^n X_i^{2w}\right)\left(\sum_{i=1}^n X_i^2\right) + \left(\sum_{j_i\neq 2w}\binom{2w}{j_1,\ldots,j_n}X_1^{j_1}\cdots X_n^{j_n}\right)\left(\sum_{i=1}^n X_i^2\right),$$

where the second sum is taken over $\{j_i\}$ such that $\sum_{i=1}^n j_i = 2w$, but with $0 \le j_i \le 2w-1$, $i = 1, \ldots, n$.

Now

$$E\left(\binom{2w}{j_1,\ldots,j_n}X_1^{j_1}\cdots X_n^{j_n}\sum_{i=1}^n X_i^2\right) = \binom{2w}{j_1,\ldots,j_n}\left(\mu_{j_1+2}\mu_{j_2}\cdots\mu_{j_n} + \mu_{j_1}\mu_{j_2+2}\cdots\mu_{j_n} + \cdots + \mu_{j_1}\mu_{j_2}\cdots\mu_{j_{n-1}}\mu_{j_n+2}\right)$$

with only $j_1, \ldots, j_n$ all *even* integers, $j_i = 2x_i$, since $\mu_{2r-1} = 0$, $r \ge 1$ and hence $0 \le x_i \le w-1$. Then, using (6) at $v \le w$ (according to the inductive hypothesis), this equals

$$\frac{(2w)!\sigma^{2(w+1)}}{(2x_1)!\cdots(2x_n)!}\left\{\frac{(2x_1+2)!(2x_2)!\cdots(2x_n)!}{2^{x_1+1}(x_1+1)!2^{x_2}x_2!\cdots 2x^{x_n}x_n!} + \cdots + \frac{(2x_1)!(2x_2)!\cdots(2x_{n-1})!(2x_n+2)!}{2^{x_1}x_1!2^{x_2}x_2!\cdots 2^{x_n+1}(x_n+1)!}\right\}$$
$$= (2w)!\sigma^{2(w+1)}\left\{\frac{1}{2^w}\frac{(2x_1+1)+\cdots+(2x_n+1)}{x_1!\cdots x_n!}\right\}$$
$$= \frac{(2w)!\sigma^{2(w+1)}}{w!2^w}\left\{(2w+n)\frac{w!}{x_1!\cdots x_n!}\right\}.$$

Thus, summing over the x_i's, we obtain

$$\frac{(2w)!\sigma^{2(w+1)}}{w!2^w}(2w+n)(n^w-n).$$

Therefore

$$E\left\{\left(\sum_{i=1}^n X_i\right)^{2w}\left(\sum_{i=1}^n X_i^2\right)\right\} = \left\{\sum_{i=1}^n X_i^{2(w+1)} + 2\sum_{i\neq j} X_i^{2w}X_j^2 + \left(\sum_{j_2\neq 2w}\binom{2w}{j_1,\ldots,j_n}X_1^{j_1}\cdots X_n^{j_n}\sum_{i=1}^n X_i^2\right)\right\}$$

so that

$$
(11) \qquad E\left\{\left(\sum_{i=1}^{n} X_i\right)^{2w}\left(\sum_{i=1}^{n} X_i^2\right)\right\} = n\mu_{2(w+1)} + n(n-1)\frac{(2w)!}{2^w w!}\sigma^{2(w+1)} + \frac{(2w)!}{2^w w!}(2w+n)\{n^w - n\}\sigma^{2(w+1)},
$$

the expression for the second term following from the inductive hypothesis (with (6) evaluated at $v = w$).

Thus from (5),

$$
\begin{aligned}
\operatorname{Cov}&\left(\left(\sum_{i=1}^{n} X_i\right)^{2w}, \sum_{i=1}^{n}(X_i - \bar{X})^2\right) \\
&= (11) - \frac{1}{n}(9) - (n-1)(10) \\
&= n\mu_{2(w+1)} + n(n-1)\frac{(2w)!}{2^w w!}\sigma^{2(w+1)} + \frac{(2w)!}{2^w w!}(2w+n)\{n^w - n\}\sigma^{2(w+1)} \\
&\quad - \left\{\mu_{2(w+1)} + \frac{(2(w+1))!}{2^{w+1}(w+1)!}\sigma^{2(w+1)}\{n^w - 1\}\right\} - (n-1)\frac{(2w)!}{2^w w!}n^w\sigma^{2(w+1)} \\
&= (n-1)\mu_{2(w+1)} + n(n-1)\frac{(2w)!}{2^w w!}\sigma^{2(w+1)} - \\
&\quad - \frac{(2w)!}{2^w w!}(2w+n)n\sigma^{2(w+1)} + \frac{(2(w+1))!}{2^{w+1}(w+1)!}\sigma^{2(w+1)} \\
&= (n-1)\mu_{2(w+1)} + \frac{(2w)!\sigma^{2(w+1)}}{2^w w!}\{-n - 2wn + (2w+1)\} \\
&= (n-1)\mu_{2(w+1)} + \frac{(2w)!}{2^w w!}\sigma^{2(w+1)}\{(2w+1) - n + (2w+1)\} \\
&= (n-1)\mu_{2(w+1)} - \frac{(2w)!}{2^w w!}\{(n-1)(2w+1)\} \\
&= (n-1)\mu_{2(w+1)} - (n-1)\frac{(2(w+1))!}{2^{w+1}(w+1)!}.
\end{aligned}
$$

Since we assume (5) is zero for all $r \geq 1$, we obtain (6) at $v = w + 1$. Further, substituting $\mu_{2(w+1)}$ into (9), we obtain

$$
\begin{aligned}
E\left\{\left(\sum_{i=1}^{n} X_i\right)^{2(w+1)}\right\} &= n\sigma^{2(w+1)}\frac{(2(w+1))!}{2^{w+1}(w+1)!} + \frac{(2(w+1))!}{2^{w+1}(w+1)!}\sigma^{2(w+1)}\{n^{w+1} - n\} \\
&= \sigma^{2(w+1)}\frac{(2(w+1))!}{2^{w+1}(w+1)!}n^{w+1},
\end{aligned}
$$

which is (7) at $v = (w + 1)$. This completes the inductive proof, and so (6) and (7) hold for all $v \geq 1$, under the blanket assumption that

$$\mathrm{Cov}\left(\left(\sum_{i=1}^{n} X_i\right)^r, \sum_{i=1}^{n}(X_i - \bar{X})^2\right) = 0, \quad r \geq 1.$$

We have used this only for $r \leq 2w$ to infer that (6) holds for $v \leq w + 1$.

As is well known (see, for example, Kendall and Stuart [7, Section 4.23, page 111]) the moment structure

$$\mu_{2v-1} = 0 \quad \text{and} \quad \mu_{2v} = \sigma^{2v}\frac{(2v)!}{2^v v!}, \quad v \geq 1$$

corresponds to the moment structure of a normal distribution about 0.

3. What is the other extreme?

We have just seen that $\mathrm{Cov}(\bar{X}^r, S^2) = 0$, $r = 1, 2, \ldots$, characterizes the set of normal distributions with zero mean. Apart from the degenerate case, this condition is equivalent to the following condition on correlations:

$$\mathrm{Corr}(\bar{X}^r, S^2) = 0, \quad r = 1, 2, \ldots.$$

For concreteness, here is the exact formula for $r = 1$

$$\rho := \mathrm{Corr}(\bar{X}, S^2) = \frac{\mu_3}{\sqrt{\mu_2(\mu_4 - [(n-3)/(n-1)]\mu_2^2)}}, \tag{12}$$

and so

$$\rho^2 \leq \frac{\mu_3^2}{\mu_2(\mu_4 - \mu_2^2)}. \tag{13}$$

In this final section, we show that these correlations can be as close to ± 1 as we want, and for this, it is enough to consider two-point distributions. The random variable that takes values $a \leq 0 \leq b$, $a < b$, has 0 expectation if a is taken with probability $b/(b - a)$ and b is taken with probability $-a/(b - a)$. If a is fixed and $b \to +\infty$, then for $r \geq 2$, $\mu_r \sim -ab^{r-1}$. Thus

$$\mathrm{Var}(\bar{X}^r) = \frac{\left(E\left(\left(\sum_{i=1}^{n} X_i\right)^{2r}\right) - E^2\left(\left(\sum_{i=1}^{n} X_i\right)^{r}\right)\right)}{n^{2r}} \sim -\frac{ab^{2r-1}}{n^{2r-1}},$$

$$\mathrm{Var}(S^2) = \frac{\mu_4}{n} - \frac{(n-3)}{n(n-1)}\mu_2^2 \sim -\frac{ab^3}{n},$$

$$E(\bar{X}^r S^2) \sim -\frac{ab^{r+1}}{n^r}, \quad E(\bar{X}^r)\mu_2 \sim \frac{a^2 b^r}{n^{r-1}}.$$

So, since $E(X) = 0$, $\mathrm{Cov}(\bar{X}^r, S^2) = E(\bar{X}^r S^2) - E(\bar{X}^r)\mu_2 \sim -ab^{r+1}/n^r$. Hence

$$\mathrm{Corr}(\bar{X}^r, S^2) = \frac{\mathrm{Cov}(\bar{X}^r, S^2)}{\sqrt{\mathrm{Var}(\bar{X}^r)\,\mathrm{Var}(S^2)}} \to 1,$$

as $b \to +\infty$. Similarly, if b is fixed and $a \to -\infty$, the correlation approaches -1.

However, for *unimodal* distributions, we show that $\rho^2 < 15/16$, and so $|\rho|$ is bounded away from 1, and we also show that this bound is best possible. Without loss of generality suppose that the mode of our unimodal X is 0 and apply Khinchin's decomposition theorem (see Khinchin [8] or Feller [2, page 155]) which says that X is unimodal at 0 if and only if $X = UV$, where U is Uniform $(0, 1)$ and V is independent of U. Suppose, for $1 \le r \le 4$, $E|X|^r < \infty$, so (by independence) $E|V|^r < \infty$.

We introduce the following notation: $m = E(V)/2$, and $m_r = E(V - 2m)^r$ (the r-th central moment of V). Then

$$3\mu_2 = m_2 + m^2, \tag{14}$$

$$4\mu_3 = m_3 + 2mm_2, \tag{15}$$

$$5\mu_4 = m_4 + 3mm_3 + 4m^2 m_2 + m^4. \tag{16}$$

Now notice that $\mathrm{Corr}^2(V - 2m, (V - 2m)^2) \le 1$ implies

$$m_2(m_4 - m_2^2) - m_3^2 \ge 0. \tag{17}$$

Using (15) and (17), we get $16\mu_3^2 < A + B + C + D$, where

$$\begin{aligned}
A &= [m_3 + 2mm_2]^2 = m_3^2 + 4mm_2m_3 + 4m^2m_2^2;\\
B &= (1/m_2)(m_2 + m^2)[m_2(m_4 - m_2^2) - m_3^2]\\
&= m_2m_4 - m_2^3 - m_3^2 + m^2m_4 - m^2m_2^2 - m^2m_3^2/m_2;\\
C &= (1/m_2)[mm_3 + m_2(3m^2 - m_2)/2]^2\\
&= m^2m_3^2/m_2 + 9m^4m_2/4 + m_2^3/4 - 6m^2m_2^2/4 + 3m^3m_3 - mm_3m_2;\\
D &= (m_2 + m^2)[7m_2^2 + 16m^4 + 23m^2m_2]/36\\
&= (7m_2^3 + 30m^2m_2^2 + 16m^6 + 39m^4m_2)/36
\end{aligned}$$

since B, C, D are clearly non-negative. Now $A + B + C + D = E + F$, where

$$\begin{aligned}
E &= m_2[m_4 + 3mm_3 + 4m^2m_2 + m^4 - (5/9)(m_2^2 + m^4 + 2m_2m^2)];\\
F &= m^2[m_4 + 3mm_3 + 4m^2m_2 + m^4 - (5/9)(m_2^2 + m^4 + 2m_2m^2)].
\end{aligned}$$

Thus,

$$16\mu_3^2 < E + F = (m_2 + m^2)[m_4 + 3mm_3 + 4m^2m_2 + m^4 - (5/9)(m_2 + m^2)^2]$$
$$= 15\mu_2(\mu_4 - \mu_2^2),$$

where the last equality follows from (14) and (16).

By (13) this implies

$$\rho^2 < 15/16. \tag{18}$$

To see that the bound (18) is best possible, let X be unimodal random variable where V has the specific two point distribution (in terms of a and b) discussed above. Then $m = 0$, $m_r \sim -ab^{r-1}$ as $b \to \infty$; and in terms of (14) and (16)

$$\mu_2 \sim -ab/3, \quad \mu_3 \sim -ab^2/4, \quad \mu_4 \sim -ab^3/5.$$

Thus as $b \to \infty$,

$$\frac{16}{15}\frac{\mu_3^2}{\mu_2(\mu_4 - [(n-3)/(n-1)]\mu_2^2)} \sim 1.$$

Hence from (12), as $b \to \infty$, we see that $\rho^2 \to 15/16$.

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