J. Aust. Math. Soc. **82** (2007), 59–83

FOURIER ALGEBRA OF A HYPERGROUP. I

VARADHARAJAN MURUGANANDAM

Dedicated to the memory of my father

(Received 6 April 2005; revised 3 September 2005)

Communicated by G. Willis

Abstract

In this article we study the Fourier space of a general hypergroup and its multipliers. The main result of this paper characterizes commutative hypergroups whose Fourier space forms a Banach algebra under pointwise product with an equivalent norm. Among those hypergroups whose Fourier space forms a Banach algebra, we identify a subclass for which the Gelfand spectrum of the Fourier algebra is equal to the underlying hypergroup. This subclass includes for instance, Jacobi hypergroups, Bessel-Kingman hypergroups.

2000 *Mathematics subject classification*: primary 43A62; secondary 43A22, 43A10, 46J10.
Keywords and phrases: hypergroups, Fourier spaces, multipliers, Gelfand spectrum of a Fourier algebra.

Introduction

Fourier algebras (of locally compact groups) which are being extensively studied by harmonic analysts do not attract the same attention when one considers them over hypergroups, since they need not form an algebra under pointwise product. One important reason is that the product of two continuous positive definite functions on a hypergroup is not necessarily positive definite in general. Yet, existence of several classes of examples of hypergroups—prominent ones are Jacobi hypergroups—for which the product of positive definite functions belonging to the support of the Plancherel-Levitan measure is again positive definite, prompts us to ascertain those hypergroups for which the Fourier space forms a Banach algebra.

Apart from the general theory of Fourier spaces, our primary concern in this article is to characterize commutative hypergroups for which the Fourier space forms a Banach algebra. In the subsequent article (Muruganandam [22]), we study the above problem for a class of hypergroups called *spherical hypergroups*. The spherical hypergroups need not be commutative and include for example the double coset hypergroups associated to any pair (G, K), where K is a compact subgroup of a locally compact group G.

This paper contains 5 sections. In Section 1, we present preliminaries. In Section 2, we define and study the Fourier space and Fourier-Stieltjes space of a general hypergroup. Section 3 is devoted to the study of multipliers of Fourier spaces. In Section 4, we give necessary and sufficient conditions for a commutative hypergroup to have an equivalent norm with respect to which the Fourier space forms a Banach algebra under pointwise product. In Section 5, we study some of the basic properties of these Fourier algebras (whenever they form algebras) of general hypergroups.

1. Preliminaries

Let H denote a (locally compact Hausdorff) hypergroup which admits a left Haar measure m. We follow the definition of a hypergroup as given by Jewett [17] wherein he calls them 'convos'. We freely use the results proved therein and adhere to the notation as far as possible. We also refer to the books by Bloom and Heyer [2] and Trimèche [27] for more details.

However, we recapitulate some notation which we use frequently for the convenience of the reader.

Let $C_c(H)$, $C_b(H)$, $M(H)$ denote the space of all complex valued continuous functions with compact support, Banach space of all bounded continuous functions, and Banach space of all bounded Radon measures on H respectively. For every x in H, let δ_x denote the point measure at x. We shall denote the probability measure $\delta_x * \delta_y$ simply by $x * y$. Let $\check{x}$ denote the involution of x in H.

For all x in H, and for all f in $C(H)$, the space of all continuous functions on H, let the (generalized) left translate of f by x be denoted by $\lambda(x)(f)$ or ${}_x f$. That is,

$$\lambda(x)(f)(y) = f(x * y) = \int_H f(z)\, d(\delta_x * \delta_y)(z).$$

The Banach spaces $L^p(H)$, $1 \leq p \leq \infty$, are understood as usual with respect to the fixed left Haar measure m.

Let $\tilde{H}$ (similarly $\widehat{H}$) denote the equivalence classes of all representations (irreducible representations) of H. Let λ denote the left regular representation and $C^\star(H)$ denote the enveloping $C^\star$-algebra of $L^1(H)$. It is also called the *full $C^\star$-algebra* of H. If $(\pi, \mathcal{H})$ is a representation of H, let the associated representations of $L^1(H)$ and $C^\star(H)$ be also denoted by π itself. If $(\pi, \mathcal{H})$ is any representation and if ξ, η are in $\mathcal{H}$, then the matrix coefficient $\pi_{\xi,\eta}$ associated to π given by $\pi_{\xi,\eta}(x) = \langle \pi(x)(\xi), \eta \rangle$ for all $x \in H$ is continuous and bounded by $\|\xi\|\|\eta\|$. Here $\pi(x)$ denotes $\pi(\delta_x)$.

Let $P_b(H)$ denote the set of all bounded, continuous positive definite functions and let $P_1(H) = \{\phi \in P_b(H) : \phi(e) = 1\}$. Let us recall that the bijective association between $P_b(H)$ and the cyclic representations (up to unitary isomorphism) via the Gelfand-Naimark construction, given by $\phi(x) = \langle \pi(x)\zeta, \zeta \rangle$ and $\|\phi\|_\infty = \langle \zeta, \zeta \rangle$, is valid for hypergroups also. We have that $P_1(H)$, with the topology of uniform convergence on compact subsets, is homeomorphic to the state space of $C^\star(H)$, when it is given the weak*-topology.

2. Fourier-Stieltjes space and Fourier space

The contents of this section are in fact, an adaptation of whatever was done for groups in Eymard [10]. We will not repeat proofs, wherever the proof for groups can be applied to the hypergroups with necessary modifications. As far as possible we will adhere to the notations and conventions of Eymard [10].

2.1. Weak containment If Σ is a subset of $\widetilde{H}$, let $N_\Sigma = \{f \in L^1(H) : \pi(f) = 0$ for all $\pi \in \Sigma\}$. Define an operator norm on $L^1(H)/N_\Sigma$ by

$$\|f^\circ\|_\Sigma = \sup\left\{\|\pi(f)\| : \pi \in \Sigma\right\}.$$

Complete this to get a $C^\star$-algebra and denote it by $C^\star_\Sigma(H)$.

The following theorem is exactly as in the case of groups, proved in Eymard [10, Theorem 1.15].

THEOREM 2.1. *Suppose that Σ is a subset of $\widetilde{H}$. Let $\overline{N}_\Sigma = \{T \in C^\star(H) : \pi(T) = 0$ for all $\pi \in \Sigma\}$. Then the map $f \to f^\circ$ extends to a surjective $\star$-homomorphism from $C^\star(H)$ onto $C^\star_\Sigma(H)$, whose kernel is precisely $\overline{N}_\Sigma$. In particular, the $C^\star$-algebras $C^\star(H)/\left(\overline{N}_\Sigma\right)$ and $C^\star_\Sigma(H)$ are isomorphic.*

DEFINITION 2.2. The $C^\star$-algebra corresponding to $\{\lambda\}$, the left regular representation is called the *reduced $C^\star$-algebra* of H and is denoted by $C^\star_\lambda(H)$.

THEOREM 2.3. *Suppose that Σ is subset of $\widetilde{H}$ and ϕ belongs to $P_b(H)$. Then the following are equivalent.*

(i) *ϕ is the limit of sums of positive definite functions associated to the representations belonging to Σ.*

(ii) $\ker \pi_\phi \supseteq \bigcap_{\pi \in \Sigma} \ker(\pi)$.

(iii) *There exists a positive linear form* Φ *on* $C^\star_\Sigma(H)$ *satisfying*

$$\Phi(f^\circ) = \int_H f(x)\phi(x)\,dx$$

for all f° *belonging to* $L^1(H)/N_\Sigma$.

PROOF. The equivalence of (i) and (ii) follows from Dixmier [8, Theorem 3.4.4].

The equivalence of (ii) and (iii) follows by appropriately modifying the proof of Eymard [10, Proposition 1.21]. □

Let $P_\Sigma(H)$ denote the set of all bounded, continuous positive definite functions satisfying any one of the conditions of Theorem 2.3.

DEFINITION 2.4. Let Σ be a subset of $\widetilde{H}$ and let π be any representation of H. We say that π is *weakly contained* in Σ if any one of the conditions of Theorem 2.3 holds.

2.2. Fourier-Stieltjes space If Σ is a subset of $\widetilde{H}$, let $B_\Sigma(H)$ denote the Banach space dual of $C^\star_\Sigma(H)$. Notice that $B_\Sigma(H)$ is contained in $L^\infty(H)$.

PROPOSITION 2.5. *Let* ϕ *be a function on* H. *Then the following are equivalent.*

(i) ϕ *belongs to* $B_\Sigma(H)$.

(ii) ϕ *is a linear combination of elements of* $P_\Sigma(H)$.

(iii) *There exists a representation* $(\pi, \mathcal{H})$ *of* H, *which is weakly contained in* Σ *and* ξ, η *in* $\mathcal{H}$ *satisfying* $\phi = \pi_{\xi,\eta}$.

(iv) ϕ *is continuous, bounded and satisfies*

$$\sup_{f\in L^1(H);\, \|f^\circ\|_\Sigma\le 1} \left| \int_H f(x)\phi(x)\,dx \right| < \infty.$$

PROOF. The proof is exactly as in Eymard [10, Proposition 2.1]. □

The Banach spaces $B_\Sigma(H)$, corresponding to two particular subsets of $\widetilde{H}$ namely, when $\Sigma = \widetilde{H}$ and when $\Sigma = \{\lambda\}$, are of much importance in the sequel.

DEFINITION 2.6. The Banach space dual of the full $C^\star$-algebra $C^\star(H)$ is called the *Fourier-Stieltjes space* and is denoted by $B(H)$.

The Banach space dual of the reduced $C^\star$-algebra $C^\star_\lambda(H)$ is denoted by $B_\lambda(H)$.

REMARKS 2.7. (1) By Proposition 2.5 we observe that $B(H)$ is precisely the space consisting of all matrix coefficients belonging to all representations of H whereas $B_\lambda(H)$ is the space consisting of all matrix coefficients associated to representations of H that are weakly contained in λ, the left regular representation.

(2) If Σ is a subset of $\widetilde{H}$ then since $C^\star_\Sigma(H)$ is a quotient $C^\star$-algebra of $C^\star(H)$ its dual can be identified as a closed subspace of $B(H)$ and thus we see that the norm of ϕ in $B_\Sigma(H)$ does not depend on Σ. That is, for all ϕ in $B_\Sigma(H)$, $\|\phi\|_\Sigma = \|\phi\|_{\widetilde{H}}$, which we denote by $\|\phi\|$.
Moreover, by Proposition 2.5 (iv), we have

$$\|\phi\| = \sup_{f\in L^1(H);\, \|f^\circ\|_{\widetilde{H}}\leq 1} \left| \int_H f(x)\phi(x)\, dx \right|.$$

(3) By the duality of $C^\star_\lambda(H)$ and $B_\lambda(H)$ we have

$$\int_H f(x)u(x)\, dx = \langle \lambda(f),\, u\rangle \tag{2.1}$$

for every u in $B_\lambda(H)$ and for every f in $L^1(H)$. Moreover,

$$\left| \int_H f(x)u(x)\, dx \right| \leq \|u\| \cdot \|f\|_\lambda = \|u\| \cdot \|\lambda(f)\|\,. \tag{2.2}$$

PROPOSITION 2.8. *Let Σ be a subset of $\widetilde{H}$ and let ϕ be in $B_\Sigma(H)$. Then there exists a representation π in $\widetilde{H}$, which is weakly contained in Σ and vectors ξ, η in $\mathcal{H}_\pi$ satisfying $\phi = \pi_{\xi,\eta}$ and $\|\phi\| = \|\xi\| \cdot \|\eta\|$.*

PROOF. The proof uses the polar decomposition of elements belonging to operator algebras and it is exactly as in the case of groups proved in Eymard [10, Lemma 2.14]. □

REMARKS 2.9. (1) $\|\phi\|_\infty \leq \|\phi\|$ for every ϕ in $B(H)$.
(2) If ϕ belongs to $B_\Sigma(H)$ then $\check{\phi}$, $\bar{\phi}$, ${}_x\phi$, ϕ_x, and $\tilde{\phi}$ are all in $B_\Sigma(H)$. Moreover, $\|\check{\phi}\| = \|\bar{\phi}\| = \|\tilde{\phi}\| = \|\phi\|$, whereas $\|{}_x\phi\| \leq \|\phi\|$ and $\|\phi_x\| \leq \|\phi\|$. Here $\check{\phi}(x) = \phi(\check{x})$ and $\tilde{\phi}(x) = \overline{\phi(\check{x})}$ for every x in H.

2.3. Fourier space For every $f,\, g$ in $L^2(H)$ the function $f * \tilde{g}$ belongs to $B_\lambda(H)$ by Remark 2.9 (2), since

$$f * \tilde{g}(x) = \int_H f(x * y)\overline{g(y)}\, dy = (\lambda_{f,g})\check{}(x).$$

DEFINITION 2.10. The closed subspace of $B_\lambda(H)$ spanned by $\{f * \tilde{f} : f \in C_c(H)\}$ is called the *Fourier space* of H and is denoted by $A(H)$.

The following lemma is proved for groups by Godement [13]. See also Pederson [23, Section 7.2].

LEMMA 2.11. *Let ϕ be a positive definite function belonging to $C_c(H)$. Then there exists a ψ in $L^2(H)$ such that $\phi = \psi * \tilde{\psi}$.*

PROOF. The proof is as in Pederson [23, Lemma 7.2.4]. □

COROLLARY 2.12.

$$\{f * \tilde{f} : f \in C_c(H)\} \subseteq (P_\lambda \cap C_c)(H) \subseteq (P \cap C_c)(H) \subseteq \{f * \tilde{f} : f \in L^2(H)\}.$$

Therefore, $A(H)$ is the closure of the span of each set of the above in $B_\lambda(H)$.

PROOF. If $u = g * \tilde{g}$ for some g belonging to $L^2(H)$, then it belongs to the closure of $\{f * \tilde{f} : f \in C_c(H)\}$ in $B_\lambda(H)$. □

COROLLARY 2.13. $A(H) \subseteq C_0(H)$.

PROOF. The result follows by Definition 2.10 and Remark 2.9 (1). □

COROLLARY 2.14. *If H is compact, then $A(H) = B(H)$.*

PROOF. Since H is compact, $P \cap C_c(H) = P_b(H)$. Therefore, $A(H) = B(H)$. □

REMARK 2.15. If H is a compact hypergroup, Vrem [30] defines $A(H)$ as the space of all those elements in $L^1(H)$ that have absolutely convergent Fourier series. By Vrem [30, Theorem 4.7] and by Corollary 2.14 we see that both the definitions coincide.

PROPOSITION 2.16. *If u belongs to $A(H)$, then $\check{u}$, $\bar{u}$, ${}_xu$, u_x, and $\tilde{u}$ all belong to $A(H)$.*

PROOF. If u belongs to the space $\{f * \tilde{g} : f, g \in C_c(H)\}$, then all the functions listed in the hypothesis belong to the above set and so, to $A(H)$. Since the set given above is dense in $A(H)$, the result follows by Remark 2.9 (2). □

Let $BL(\mathcal{H})$ denote, as usual, the space of all bounded operators on a Hilbert space $\mathcal{H}$. If F is any subset of $BL(\mathcal{H})$, let F'' denote the bicommutant of F.

DEFINITION 2.17. The von Neumann algebra $[\lambda(H)]''$ associated to the left regular representation λ of H is called the *von Neumann algebra of H* and is denoted by $VN(H)$.

REMARKS 2.18. (1) $VN(H)$ is the same as $[\lambda(L^1(H))]''$. Observe that $C_c(H)$ is a quasi-Hilbert algebra as given in Dixmier [9, Part 1, Chapter 5, Definition 1], with the scalar product $\langle f, g\rangle = \int_H f(x)\overline{g(x)}\,dx$ and the involutive antiautomorphism given by $f \to \tilde{f}\Delta^{-1/2}$. Using the commutation theorem for quasi-Hilbert algebras as given in Dixmier [9, Part 1, Chapter 5, Theorem 1], we can see that $VN(H) = [\rho(H)]'$, where ρ denotes the right regular representation of H.
(2) Notice that $C^\star_\lambda(H)$ is contained in $VN(H)$, as $VN(H)$ is the σ-weakly closed subalgebra of $BL(L^2(H))$ containing $\{\lambda(f) : f \in C_c(H)\}$.

THEOREM 2.19. *Let H be a hypergroup. For every T in $VN(H)$ there exists a unique continuous linear functional ϕ_T on $A(H)$ satisfying*

$$\phi_T((f * \tilde{g})\check{}) = \langle T(f), g\rangle_{L^2(H)} \quad \text{for all } f, g \in L^2(H). \tag{2.3}$$

The mapping $T \to \phi_T$ is a Banach space isomorphism between $VN(H)$ and $A(H)^\star$.
Moreover, the above mapping is also a homeomorphism when $VN(H)$ is given the σ-weak topology and $A(H)^\star$ is given the weak*-topology.

PROOF. We omit the proof as it can be obtained by following the proof given for groups in Eymard [10, Theorem 3.10] with appropriate modifications. □

The following proposition clarifies the duality between $C^\star_\lambda(H)$ and $B_\lambda(H)$ on the one hand and between $A(H)$ and $VN(H)$ on the other hand.

PROPOSITION 2.20. *For every T in $C^\star_\lambda(H)$ and for every u in $A(H)$ we have*

$$\phi_T(u) = \langle T, u\rangle. \tag{2.4}$$

Here $\langle\, ,\, \rangle$ appearing on the right-hand side is with respect to the duality between $C^\star_\lambda(H)$ and $B_\lambda(H)$.

PROOF. Let T be equal to $\lambda(h)$ for some h in $L^1(H)$. Let $\check{u} = f * \tilde{g}$ be in $A(H)$. Then

$$\phi_T(u) = \langle h * f, g\rangle_{L^2(H)} = \int_H u(x)h(x)\,dx.$$

Therefore, by (2.1) $\phi_T(u) = \langle u, \lambda(h)\rangle = \langle T, u\rangle$. Thus, (2.4) is valid for all u in E where E is the span of elements of the above form in $A(H)$. As E is dense in $A(H)$ we see that (2.4) is valid for all u in $A(H)$ and for all $\lambda(h)$ in $C^\star_\lambda(H)$ with h in $L^1(H)$. By density of $\{\lambda(f) : f \in L^1(H)\}$ in $C^\star_\lambda(H)$, the result follows. □

PROPOSITION 2.21. *For every μ in $M(H)$ and for every u in $A(H)$, we have*

$$\phi_{\lambda(\mu)}(u) = \int_H u(x)\, d\mu(x). \tag{2.5}$$

PROOF. Observe that for any μ in $M(H)$, $\lambda(\mu)$ need not belong to $C^{\star}_{\lambda}(H)$.

Let μ be in $M(H)$. Using a bounded approximate identity in $L^1(H)$ we see that there exists a net $\{f_i\}$ in $L^1(H)$ such that $\lambda(f_i)$ converges to $\lambda(\mu)$ with respect to the strong operator topology. So, $\phi_{\lambda(f_i)}(u)$ converges to $\phi_{\lambda(\mu)}(u)$ for every $u \in A(H)$. Therefore, $\phi_{\lambda(\mu)}(u) = \int_H u(x)\, d\mu(x)$ for all $u \in A(H)$. □

Convention. For the sake of convenience, the duality between $A(H)$ and $VN(H)$ given by Theorem 2.19 will henceforth be denoted by $\langle u, T\rangle$ with u in $A(H)$ and T in $VN(H)$. In particular,

$$\langle \lambda(x), u\rangle = u(x) \quad \text{for all } u \in A(H) \text{ and } x \in H. \tag{2.6}$$

The following proposition, whose proof is found in Eymard [10] in the case of groups, holds for hypergroups also. Vrem [30] has already shown it for compact hypergroups and his proof is applicable to general hypergroups also.

PROPOSITION 2.22. *Let H be a hypergroup. If K is a nonempty compact subset of H and U is a neighborhood of K, then there exists u in $A(H)$ satisfying* $0 \leq u(x) \leq 1$, $u|_K = 1$ *and* supp u *is contained in U.*

3. Multipliers of $A(H)$

DEFINITION 3.1. A complex valued function ϕ on H is called a *multiplier* of $A(H)$, if $\phi \cdot u$ belongs to $A(H)$ whenever u is in $A(H)$. Let $MA(H)$ denote the space of all multipliers of $A(H)$.

It appears to be difficult to find examples of multipliers apart from constant functions for a general hypergroup. In Section 4, we give several classes of examples of hypergroups for which $B_\lambda(H)$ is contained in $MA(H)$. This section is devoted to the general study of the space of multipliers, which is needed in sequel.

PROPOSITION 3.2. *If ϕ is a multiplier of $A(H)$, then it is continuous.*

PROOF. Let x be in H. If V is a compact neighborhood of x, then by Proposition 2.22 there exists a u in $A(H)$ satisfying $u\,|_V = 1$. Since $\phi \cdot u$ belongs to $A(H)$, it is continuous. As ϕ agrees with $\phi \cdot u$ on V, the result follows. □

For every ϕ in $MA(H)$, let $\|\phi\|_{MA(H)}$ denote the operator norm of m_ϕ where m_ϕ is the linear operator on $A(H)$ given by $m_\phi(u) = \phi u$ for every u in $A(H)$.

THEOREM 3.3. *$\|\cdot\|_{MA(H)}$ defines a norm on $MA(H)$ and $\big(MA(H), \|\cdot\|_{MA(H)}\big)$ forms a Banach algebra.*

PROOF. Suppose that ϕ is in $MA(H)$. We use the closed graph theorem, in order to show that the linear map m_ϕ is bounded.

Suppose that $\{u_n\}$ and u, v are in $A(H)$ such that u_n converges to u and $m_\phi(u_n)$ converges to v. As $\|\cdot\|_\infty \leq \|\cdot\|$ in $A(H)$ we see that u_n converges to u and $\phi \cdot u_n$ converges to v uniformly. However $\phi \cdot u_n$ converges to $\phi \cdot v$ uniformly on compact subsets. Therefore, $m_\phi(u) = \phi \cdot u = v$.

If $m_\phi = 0$ for some ϕ in $MA(H)$, then $\phi \cdot u = 0$ for every u in $A(H)$. However, then, by Proposition 2.22, $\phi = 0$. Therefore, $\|\phi\|_{MA(H)}$ is indeed a norm for $MA(H)$.

Let $\{\phi_n\}$ be Cauchy in $MA(H)$. That is, the sequence of operators m_{ϕ_n} is Cauchy in $BL(A(H))$. So, there exists T in $BL(A(H))$ such that m_{ϕ_n} converges to T in $BL(A(H))$.

Fix a compact subset K of H and let $\epsilon > 0$. Let u be as in Proposition 2.22 satisfying $u|_K = 1$. Then, for every x in K, we have

$$\begin{aligned}|(\phi_n - \phi_m)(x)| = |(\phi_n - \phi_m)(x)u(x)| &\leq \|(\phi_n - \phi_m) \cdot u\|_\infty \\ &\leq \|(\phi_n - \phi_m) \cdot u\| \leq \|(\phi_n - \phi_m)\|_{MA(H)} \cdot \|u\| .\end{aligned}$$

Therefore, $\{\phi_n\}$ is uniformly Cauchy on compacta. Let ϕ be its limit. We show that $T(u) = \phi \cdot u$ for every u in $A(H)$, which in turn implies that ϕ belongs to $MA(H)$.

Fix u in $A(H)$. Then $m_{\phi_n}(u)$ converges to $T(u)$ in $A(H)$. In particular, $\phi_n \cdot u$ converges to $T(u)$ uniformly. However, $\phi_n \cdot u$ converges to $\phi \cdot u$ uniformly on compact sets. Therefore, $T(u) = \phi \cdot u$. So, ϕ belongs to $MA(H)$. Hence $MA(H)$ is a Banach algebra. □

The analog of the following theorem for groups was proved by DeCannière and Haagerup [7]. We make the proof brief as it is similar to that for groups.

THEOREM 3.4. *Let H be a hypergroup and ϕ be a bounded complex valued function on H. Then the following are equivalent.*

(i) *ϕ belongs to $MA(H)$.*

(ii) *There exists a σ-weakly continuous operator M_ϕ on $VN(H)$ satisfying*

$$M_\phi(\lambda(f)) = \lambda(\phi f) \quad \textit{for all} \ \ f \in L^1(H). \tag{3.1}$$

Moreover, $\|\phi\|_{MA(H)} = \|M_\phi\|$.

PROOF. Suppose that ϕ is in $MA(H)$. Let M_ϕ denote the transpose of m_ϕ. Then M_ϕ is a σ-weakly continuous operator on $VN(H)$ and $\|M_\phi\| = \|\phi\|_{MA(H)}$. Moreover, for every f in $L^1(H)$ and for every u in $A(H)$, we have

$$\langle u, M_\phi(\lambda(f))\rangle = \langle m_\phi(u), \lambda(f)\rangle = \int_H (\phi \cdot u)(x) f(x)\, dx = \langle u, \lambda(\phi \cdot f)\rangle$$

by (2.5). Therefore, $M_\phi(\lambda(f)) = \lambda(\phi \cdot f)$.

We prove the converse. Suppose that there exists a σ-weakly continuous operator M_ϕ satisfying (3.1) for some bounded function ϕ. As M_ϕ is σ-weakly continuous, there exists a bounded operator, say S, on $A(H)$ satisfying $\langle S(u), T\rangle = \langle u, M_\phi(T)\rangle$ for all $T \in VN(H)$. Then by (3.1) and (2.5) we see that

$$\int_H f(x)\phi(x)u(x)\,dx = \int_H S(u)(x)f(x)\,dx$$

for every f in $L^1(H)$. Therefore, $S(u) = \phi u$ almost everywhere for all u in $A(H)$.

In order to conclude that they are equal everywhere, we show that ϕ is continuous. Observe that the restriction of M_ϕ to $C^\star_\lambda(H)$ is a bounded linear map satisfying (3.1). By taking the transpose we see that ϕ is a multiplier for $B_\lambda(H)$. Now use Proposition 2.22 and the proof of Proposition 3.2 to conclude that ϕ is continuous.

Therefore, for every u in $A(H)$ we have $S(u) = \phi \cdot u$. So, ϕ belongs to $MA(H)$. □

REMARKS 3.5. (1) In the above proof we have also proved that if ϕ is a multiplier of $A(H)$, then ϕ is a multiplier of $B_\lambda(H)$ also.

(2) We do not know whether a multiplier of $A(H)$, H being a hypergroup, is always bounded. However, if G is a group, then we have $\|\phi\|_\infty \le \|\phi\|_{MA(G)}$, whenever ϕ is a multiplier of $A(G)$.

(3) If H is a hypergroup, let $M_bA(H) = \{\phi \in MA(H) : \phi \text{ is bounded}\}$.

4. Fourier spaces of commutative hypergroups

Throughout this section H is assumed to be commutative. The set consisting of all nonzero hermitian characters on H, equipped with the topology of uniform convergence on compact subsets of H, is called the *dual of* H and is denoted by $\widehat{H}$.

For any f in $L^1(H)$ and μ in $M(H)$, let $\mathcal{F}(f)$ and $\mathcal{F}(\mu)$ denote the Fourier transform and Fourier-Stieltjes transform of f and μ respectively. Let $d\pi$ denote the Plancherel-Levitan measure. Let S denote the subset of $\widehat{H}$ given by

$$S = \left\{\chi \in \widehat{H} : |\mathcal{F}(\mu)(\chi)| \le \|\lambda(\mu)\| \text{ for all } \mu \in M(H)\right\}.$$

Then S is a nonempty closed subset of $\widehat{H}$. Moreover, S is precisely the support of $d\pi$. By $L^p(\widehat{H})$ we mean the Banach space $L^p(\widehat{H}, d\pi)$ where $1 \le p \le \infty$.

If f, μ belong to $L^1(\widehat{H})$ and $M(\widehat{H})$ respectively, let $\mathcal{I}(f)$ and $\mathcal{I}(\mu)$ denote the inverse Fourier transform of f and μ respectively.

As H is commutative, any irreducible representation of H is 1-dimensional and is given by a hermitian character. In particular, the full $C^\star$-algebra $C^\star(H)$ is identified with $C_0(\widehat{H})$. So, the Fourier-Stieltjes space $B(H)$ is identified with $M(\widehat{H})$. That is, if u is a complex valued function on H, then u belongs to $B(H)$ if and only if there exists a unique measure μ in $M(\widehat{H})$ satisfying $u(x) = \mathcal{I}(\mu)(x)$ for all $x \in H$ and $\|u\|_{B(H)} = \|\mu\|$.

A hermitian character γ is weakly contained in the left regular representation λ if and only if $|\mathcal{F}(f)(\gamma)| \leq \|\lambda(f)\|$ for all $f \in L^1(H)$. Equivalently, by Jewett [17, Subsections 7.3B and 7.3D], we see that $\{\chi \in \widehat{H} : \chi$ is weakly contained in $\lambda\} = S$. Therefore, the reduced $C^\star$-algebra of a commutative hypergroup H is identified with $C_0(S)$, and for every u in $B_\lambda(H)$ there exists a unique measure μ in $M(S)$ such that $u(x) = \mathcal{I}(\mu)(x)$ for all $x \in H$ and $\|u\|_{B(H)} = \|\mu\|$.

PROPOSITION 4.1. *Let H be a commutative hypergroup. Then there exists a von Neumann algebra isomorphism identifying $VN(H)$ with $L^\infty(S, d\pi)$ which maps $\lambda(f)$ to $\mathcal{F}(f)$ for every f in $L^1(H)$.*

PROOF. Let us use the same notation, namely $\mathcal{F}$, to denote the Plancherel-Levitan transform also. For every ϕ belonging to $L^\infty(S, d\pi)$ define T_ϕ on $L^2(H)$ by

$$\mathcal{F}(T_\phi(f)) = \phi \cdot \mathcal{F}(f).$$

Then $\|T_\phi\| \leq \|\phi\|_\infty$ and T_ϕ belongs to $VN(H)$; it can be easily seen that it commutes with all (right) generalized translations $\rho(x)$ with x in H. Moreover, one can verify that the above map from $L^\infty(S, d\pi)$ into $VN(H)$ is injective and σ-weakly bicontinuous satisfying $T_{\mathcal{F}(f)} = \lambda(f)$ for every f in $L^1(H)$.

Since the image is a σ-weakly closed subspace containing $\{\lambda(f) : f \in L^1(H)\}$, it includes $VN(H)$ also. Therefore, it is surjective. Thus it is a von Neumann algebra isomorphism by Takesaki [26, Chapter II, Corollary 3.10]. □

In the following proposition, we get back the classical definition of a Fourier space when H is a commutative hypergroup.

PROPOSITION 4.2. *Let H be a commutative hypergroup. Then*

(i) $A(H) = \mathcal{I}\big(L^1(S, d\pi)\big)$. *That is, for every u in $A(H)$ there exists a unique f in $L^1(S, d\pi)$ such that $u(x) = \mathcal{I}(f)(x)$ for all $x \in H$ and $\|u\|_{A(H)} = \|f\|_1$.*

(ii) $A(H) = \{f * \tilde{g} : f, g \in L^2(H)\}$.

(iii) $A(H)$ *is equal to the closure of the subspace* $[B(H) \cap C_c(H)]$ *in* $B(H)$.

(iv) $A(H)$ *is equal to the closure of the subspace* $[B_\lambda(H) \cap C_c(H)]$ *in* $B_\lambda(H)$.

PROOF. (i) By the above proposition and the uniqueness of the preduals of von Neumann algebras, the Banach spaces $L^1(S, d\pi)$ and $A(H)$ are isometrically isomorphic.

We show that if $\Phi : L^1(S, d\pi) \to A(H)$ denotes the above isomorphism, then it is given by the inverse Fourier transform. In fact, for every f in $L^1(S, d\pi)$ and for every g in $L^1(H)$

$$\langle \lambda(g),\ \Phi(f)\rangle = \langle \Phi^{\star}(\lambda(g)),\ f\rangle = \langle \mathcal{F}(g),\ f\rangle = \langle g,\ \mathcal{I}(f)\rangle = \langle \lambda(g),\ \mathcal{I}(f)\rangle.$$

(ii) We already know by Corollary 2.12 that $\{f * \tilde{g} : f, g \in L^2(H)\}$ is dense in $A(H)$.

Let u be in $A(H)$. By (i) there exists h in $L^1(S, d\pi)$ such that $u = \mathcal{I}(h)$. Since any element in $L^1(S, d\pi)$ can be written as a product of two elements of $L^2(S, d\pi)$, we have $u = f * \tilde{g}$ for some f, g in $L^2(H)$.

(iii) Let u be in $B(H) \cap C_c(H)$ Then there exists a unique measure μ in $M(\widehat{H})$ such that $u = \mathcal{I}(\mu)$. By Jewett [17, Lemma 12.2B], $\mu = \mathcal{F}(u)\, d\pi$. In particular, μ belongs to $L^1(S, d\pi)$. By the Fourier inversion theorem, we have $u = \mathcal{I}(\mathcal{F}(u))$ and so u belongs to $A(H)$ by (i). That is, $B(H) \cap C_c(H)$ is contained in $A(H)$.

Since the linear span of $P \cap C_c(H)$ is contained in $B(H) \cap C_c(H)$, and since it is dense in $A(H)$ by Corollary 2.12, we see that $B(H) \cap C_c(H)$ is dense in $A(H)$.

(iv) The proof uses Corollary 2.12 and is similar to the proof of (iii). □

COROLLARY 4.3. *Let H be a commutative hypergroup and ϕ be a bounded function on H. Then ϕ is a multiplier of $A(H)$ if and only if ϕ is a multiplier of $B_\lambda(H)$. Moreover, the respective norms coincide.*

PROOF. We have already seen in Remark 3.5 (1) that if ϕ is a multiplier of $A(H)$, then it is a multiplier of $B_\lambda(H)$ also.

Conversely, suppose that ϕ is a multiplier of $B_\lambda(H)$. If u belongs to $B_\lambda(H) \cap C_c(H)$, then $\phi \cdot u$ belongs to $B_\lambda(H) \cap C_c(H)$. By Proposition 4.2(iv), $\phi \cdot u$ belongs to $A(H)$. Now we use the density of $B_\lambda(H) \cap C_c(H)$ in $A(H)$ to see that ϕ is a multiplier of $A(H)$. □

REMARK 4.4. As in the case of groups we want to know whether $A(H)$ forms a Banach algebra. If the dual space $\widehat{H}$ of a commutative hypergroup H forms a hypergroup with the convolution given by $\mathcal{I}(\mu * \nu) = \mathcal{I}(\mu) \cdot \mathcal{I}(\nu)$ for all $\mu, \nu \in M(\widehat{H})$ and with the involution given by $\check{\chi}$ for every χ in $\widehat{H}$ (see Jewett [17, Subsection 12.4] for details), then, by Proposition 4.2, $A(H)$ turns out to be a Banach algebra under pointwise product.

It is worthwhile to remark that the dual hypergroup exists only in very few cases, even within the classical hypergroups such as the classes of polynomial hypergroups, hypergroups on compact intervals and Chébli-Trimèche hypergroups on the halfline. In fact Zeuner proved that polynomial Jacobi hypergroups, compact Jacobi hypergroups and Bessel-Kingman hypergroups are the only hypergroups, among the classes specified above, for which the dual space forms a hypergroup. See Zeuner [32, Corollary 5.5, Theorem 6.4 and Theorem 7.4].

The main results of this section and the next section give necessary and sufficient conditions for a commutative hypergroup so that its Fourier space forms a Banach algebra with an equivalent norm under pointwise product. We give several classes of examples whose dual space does not form a hypergroup, yet the Fourier space of them forms a Banach algebra.

DEFINITION 4.5. (1) Let $M_bA(H)$ be given a new norm

$$\|\phi\|^\circ = \max\left\{\|\phi\|_{MA(H)},\ \|\phi\|_\infty\right\} \quad \text{for all } \phi \in M_bA(H). \tag{4.1}$$

(2) Let $Q(H)$ be the vector space $L^1(H)$ equipped with the norm given by

$$\|f\|_{Q(H)} = \sup\left\{|\langle f, \phi\rangle| : \phi \in M_bA(H) \text{ and } \|\phi\|^\circ \le 1\right\} \tag{4.2}$$

for every f belonging to $L^1(H)$.

Here $\langle f, \phi\rangle = \int_H f(x)\phi(x)\,dx$ for every f in $Q(H)$ and for every ϕ in $M_bA(H)$.

LEMMA 4.6. (i) *$(M_bA(H), \|\cdot\|^\circ)$ is a Banach space.*
(ii) *Assume that S, the support of $d\pi$, is contained in $MA(H)$. Then $Q(H)$ is a normed linear space and is contained in the dual space of $(M_bA(H), \|\cdot\|^\circ)$.*

PROOF. (i) Only completeness needs proof. If $\{\phi_i\}$ is Cauchy in $(M_bA(H), \|\cdot\|^\circ)$, then it is Cauchy in $MA(H)$ and in $C_b(H)$ also. If ϕ is the limit of the above sequence in $MA(H)$, then by the proof of Theorem 3.3 we see that $\{\phi_i\}$ converges to ϕ uniformly on compact sets. So, ϕ is the limit of the sequence in $C_b(H)$ also. Therefore, $\{\phi_i\}$ converges to ϕ in $(M_bA(H), \|\cdot\|^\circ)$.
(ii) The norm given in (4.2) is finite as

$$\|f\|_{Q(H)} \le \sup\{|\langle f, \phi\rangle| : \forall\phi \in C_b(H) \text{ and } \|\phi\|_\infty \le 1\} \le \|f\|_1 < \infty,$$

since $\|\phi\|_\infty \le \|\phi\|^\circ$.

Assume that there is a f in $L^1(H)$ satisfying $\langle f, \phi\rangle = 0$ for every ϕ in $M_bA(H)$. In particular, $\langle f, \gamma\rangle = 0$ for all γ in S. That is, $\mathcal{F}(f) = 0$. So, $\lambda(f) = 0$ by Jewett [17, Subsection 7.3D]. Therefore, $f = 0$ as λ is faithful. Thus $Q(H)$ is a normed linear space.

Any f in $Q(H)$ defines a linear functional τ_f on $(M_bA(H), \|\cdot\|^\circ)$ given by

$$\tau_f(\phi) = \int_H f(x)\phi(x)\,dx.$$

Moreover, $\left|\tau_f(\phi)\right| \le \|f\|_1\|\phi\|_\infty \le \|f\|_1\,\|\phi\|^\circ$. Therefore, τ_f belongs to the dual of $(M_bA(H), \|\cdot\|^\circ)$ and $\left\|\tau_f\right\| \le \|f\|_1$. □

DEFINITION 4.7. Let H be a commutative hypergroup. We say that H satisfies condition (F) if there exists a constant $M > 0$ satisfying the following:

(F) For every pair τ and γ in S, $\tau\gamma$ belongs to $B_\lambda(H)$ and $\|\tau\gamma\| \le M$.

Equivalently, for every τ and γ in S, there exists a measure μ in $M(S)$, (not necessarily positive) such that, $\|\mu\| \le M$ and $\tau(x)\gamma(x) = \int_{\widehat{H}} \chi(x)\, d\mu(\chi)$ for all $x \in H$.

LEMMA 4.8. *Suppose that H is a commutative hypergroup satisfying condition (F) for some $M > 0$. Then S is contained in $M_bA(H)$. Moreover, $\|\tau\|_{MA(H)} = \|\tau\|^\circ \le M$ for all $\tau \in S$.*

PROOF. For every τ, γ in S, let $\mu_{\tau,\gamma}$ be in $M(S)$ such that $\tau \cdot \gamma = \mathcal{I}(\mu_{\tau,\gamma})$.

If τ belongs to S, then for every f in $L^1(H)$, we have

$$\mathcal{F}(\tau f)(\bar{\gamma}) = \int_H \mathcal{I}(\mu_{\tau,\gamma})(x) f(x)\, dx = \int_{\widehat{H}} \mathcal{F}(f)(\bar{\omega})\, d\mu_{\tau,\gamma}(\omega).$$

Therefore, $|\mathcal{F}(\tau f)(\bar{\gamma})| \le \int_{\widehat{H}} |\mathcal{F}(f)(\bar{\omega})||d\mu_{\tau,\gamma}(\omega)| \le M\|\mathcal{F}(f)\|_\infty$. Hence, the map $\mathcal{F}(f) \to \mathcal{F}(\tau f)$ extends to a bounded linear operator on $C_0(S)$. That is, $\lambda(f) \to \lambda(\tau f)$ is a bounded linear operator on $C^\star_\lambda(H)$. By duality, as given in (2.1), we infer that $u \to \tau u$ is a multiplier on $B_\lambda(H)$ and $\|\tau\|_{MB_\lambda(H)} \le M$. Therefore, by Corollary 4.3, τ is a multiplier on $A(H)$ and $\|\tau\|_{MA(H)} \le M$.

If u belongs to $A(H)$ then $\langle u, \tau(e)\lambda(e)\rangle = \tau(e)\langle u, \lambda(e)\rangle = (\tau \cdot u)(e)$. So,

$$\begin{aligned}|\tau(e)| = \|\tau(e)\lambda(e)\|_{VN(H)} &= \sup_{\|u\|\le 1} |\langle u,\ \tau(e)\lambda(e)\rangle| \\ &\le \sup_{\|u\|\le 1} |\tau \cdot u(e)| \le \sup_{\|u\|\le 1} \|\tau \cdot u\| \le \|\tau\|_{MA(H)}.\end{aligned}$$

Since τ belongs to $\widehat{H}$, $\|\tau\|_\infty = \tau(e)$ by Jewett [17, Subsection 6.3D]. That is, $\|\tau\|_\infty \le \|\tau\|_{MA(H)}$. Therefore, $\|\tau\|^\circ = \|\tau\|_{MA(H)} \le M$. □

REMARK 4.9. Let S be contained in $M_bA(H)$. Then the normed linear spaces $\{(M_bA(H), \|\cdot\|^\circ), Q(H)\}$ form a dual system in the following sense:

(i) If $\langle f, \phi\rangle = 0$ for every ϕ in $M_bA(H)$, then $f = 0$ by the proof of Lemma 4.6 (ii).

(ii) If on the other hand, $\langle f, \phi\rangle = 0$ for every f in $Q(H)$, then $\phi = 0$ as ϕ belongs to $C_b(H)$.

LEMMA 4.10. *Let S be contained in $M_bA(H)$ and the dual system given in the above remark be denoted by $(X, X_\star)$ with $X_\star = Q(H)$ and $X = (M_bA(H), \|\cdot\|^\circ)$. Then the following are true*:

(i) $\|\phi\|^{\circ} = \sup\{|\langle\phi, f\rangle| : f \in Q(H)$ *and* $\|f\|_{Q(H)} \le 1\}$ *for every* ϕ *in* $M_b A(H)$.

(ii) *The* $\sigma(X, X_\star)$*-closed convex hull of a* $\sigma(X, X_\star)$*-compact subset of* X *is* $\sigma(X, X_\star)$*-compact.*

PROOF. (i) By duality we see that if f belongs to $Q(H)$ and $\|f\|_{Q(H)} \le 1$, then we have $|\langle\phi, f\rangle| \le \|\phi\|^{\circ} \cdot \|f\|_{Q(H)} \le \|\phi\|^{\circ}$.

Now, we show the reverse inequality. Let $\epsilon > 0$ and let ϕ be in $M_b A(H)$.

If $\|\phi\|_{MA(H)} \le \|\phi\|_\infty$, then $\|\phi\|^{\circ} = \|\phi\|_\infty$. If f in $L^1(H)$ satisfies $\|f\|_1 \le 1$ and $|\langle f, \phi\rangle| + \epsilon \ge \|\phi\|_\infty$, then

$$\|f\|_{Q(H)} \le 1 \quad \text{and} \quad \|\phi\|^{\circ} \le |\langle f, \phi\rangle| + \epsilon. \tag{4.3}$$

Suppose that on the other hand, $\|\phi\|_\infty \le \|\phi\|_{MA(H)}$. Let u be in $A(H)$ satisfying

$$\|u\| \le 1 \quad \text{and} \quad \|\phi\|_{MA(H)} \le \|\phi u\| + \epsilon. \tag{4.4}$$

Then there exists f in $L^1(H)$ satisfying $\|\lambda(f)\| \le 1$ and $\|\phi u\| = |\langle\phi \cdot u, \lambda(f)\rangle| + \epsilon$. As $\langle\phi \cdot u, \lambda(f)\rangle = \int_H f(x)\phi(x)u(x)\,dx = \langle fu, \phi\rangle$, we have, by (4.4)

$$\|\phi\|_{MA(H)} \le |\langle fu, \phi\rangle| + 2\epsilon. \tag{4.5}$$

Now, fu belongs to $Q(H)$ and $\|fu\|_{Q(H)} \le 1$, since

$$\begin{aligned}
\|fu\|_{Q(H)} &= \sup\{|\langle fu, \psi\rangle| : \|\psi\|^{\circ} \le 1\} \\
&= \sup\left\{\left|\int_H f(x)u(x)\psi(x)\,dx\right| : \|\psi\|^{\circ} \le 1\right\} \\
&= \sup\{|\langle\psi u, \lambda(f)\rangle| : \|\psi\|^{\circ} \le 1\} \le \sup\{\|\psi u\|\|\lambda(f)\| : \|\psi\|^{\circ} \le 1\} \\
&\le \sup\{\|u\|\|\psi\|_{MA(H)}\|\lambda(f)\| : \|\psi\|^{\circ} \le 1\} \le 1.
\end{aligned}$$

By (4.3) and (4.5) we have

$$\|\phi\|^{\circ} \le \sup\left\{|\langle\phi, f\rangle| : f \in Q(H) \text{ and } \|f\|_{Q(H)} \le 1\right\}.$$

This proves (i).

(ii) Let E be a $\sigma(X, X_\star)$-compact subset of X. Let $\tilde{E}$ denote the convex hull of E in X. We shall prove that $\tilde{E}$ is pre-compact in the $\sigma(X, X_\star)$-topology.

By Grothendieck [14, Chapter 2, Proposition 15], it is sufficient to show that $\tilde{E}$ is bounded in the $\sigma(X, X_\star)$-topology. That is, for every f in $Q(H)$ the set $\{\langle\phi, f\rangle : \phi \in \tilde{E}\}$ is bounded in $\mathbb{C}$.

It can be seen that E as a subset of $L^\infty(H)$ is $\sigma(L^\infty(H), L^1(H))$-compact. So, $\tilde{E}$ is $\sigma(L^\infty(H), L^1(H))$-bounded which in turn implies that, it is $\sigma(X, X_\star)$-bounded also. Hence, (ii) follows. □

REMARK 4.11. We recall Arveson's theorem [1, Proposition 1.2]. Suppose that X is a Banach space and $X_\star$ is a subspace of the dual space $X^\star$ of X satisfying the following:

(i) $\|x\| = \sup\{|\langle x, f\rangle| : f \in X_\star, \text{ and } \|f\| \leq 1\}$.

(ii) The $\sigma(X, X_\star)$-closed convex hull of any $\sigma(X, X_\star)$-compact set in X is $\sigma(X, X_\star)$-compact.

Then the following result holds. Let Q be a locally compact space and let $x : Q \to X$ be norm bounded and $\sigma(X, X_\star)$-continuous function. Then, for every μ belonging to $M(Q)$, there exists a vector x_μ in X satisfying $\langle x_\mu, f\rangle = \int_Q \langle x(s), f\rangle \, d\mu(s)$ for all $f \in X_\star$.

THEOREM 4.12. *Let H be a commutative hypergroup. Suppose that H satisfies condition (F), for some $M > 0$. Then $B_\lambda(H)$ is contained in $M_bA(H)$. Moreover, $\|u\|_{MA(H)} \leq M \|u\|$ for all $u \in B_\lambda(H)$.*

PROOF. We show that if μ belongs to $M(S)$, then $\mathcal{I}(\mu)$ belongs to $M_bA(H)$ and $\|\mathcal{I}(\mu)\|_{MA(H)} \leq M \|\mu\|$.

Take $Q = S$, $X = (M_bA(H), \|\cdot\|^\circ)$ and $X_\star = Q(H)$. From Lemma 4.10, the conditions given in Arveson's theorem are satisfied. By Lemma 4.8, S is contained in $M_bA(H)$ and $\|\gamma\|^\circ \leq M$ for every γ in S.

As the topology in S is the Gelfand topology and

$$\langle f, \gamma\rangle = \overline{\mathcal{F}(\bar{f})(\gamma)} \quad \text{for all } \ f \in Q(H),$$

we see that the inclusion map from S into $(M_bA(H), \|\cdot\|^\circ)$ is $\sigma(X, X_\star)$-continuous. By Arveson's theorem we see that for every μ in $M(S)$, there exists a vector ϕ_μ in $M_bA(H)$ satisfying $\langle f, \phi_\mu\rangle = \int_S \langle f, \gamma\rangle \, d\mu(\gamma)$. Since,

$$\begin{aligned}\int_S \langle f, \gamma\rangle \, d\mu(\gamma) &= \int_S \left[\int_H f(x)\gamma(x)\, dx\right] d\mu(\gamma)\\ &= \int_H f(x)\mathcal{I}(\mu)(x)\, dx,\end{aligned}$$

we see that $\phi_\mu = \mathcal{I}(\mu)$. Thus, $\mathcal{I}(\mu)$ belongs to $M_bA(H)$.

Now,

$$\begin{aligned}|\langle f, \mathcal{I}(\mu)\rangle| &\leq \int_{\widehat{H}} |\langle f, \gamma\rangle| \; d|\mu|(\gamma)\\ &\leq \int_{\widehat{H}} \|f\|_{Q(H)} \, \|\gamma\|^\circ \; d\,|\mu|\,(\gamma)\\ &\leq M \,\|\mu\| \,\|f\|_{Q(H)}\,.\end{aligned}$$

Therefore, $\|\mathcal{I}(\mu)\|^\circ \leq M \|\mu\|$. In particular, $\|\mathcal{I}(\mu)\|_{MA(H)} \leq M \|\mu\|$. □

COROLLARY 4.13. *Let H be a commutative hypergroup satisfying condition(F). Then the Fourier space $A(H)$ is an algebra under pointwise product. Moreover,*

$$\|u \cdot v\| \leq M \, \|u\| \, \|v\| \quad \textit{for all } \; u, v \in A(H). \tag{4.6}$$

In particular, if $M = 1$, then $A(H)$ forms a Banach algebra. Similar results hold for $B_\lambda(H)$ also.

PROOF. Let u belong to $A(H)$. Then, by the above theorem, we see that u belongs to $M_b A(H)$ and $\|u\|_{MA(H)} \leq M \, \|u\|$. In particular, $A(H)$ is an algebra under pointwise product. Moreover, for any u, v in $A(H)$, we have

$$\|uv\| \leq \|u\|_{MA(H)} \, \|v\| \leq M \, \|u\| \, \|v\| \, .$$

Thus $A(H)$ is a Banach algebra if $M = 1$.

Now if $u = \mathcal{I}(\mu)$ is in $B_\lambda(H)$ for some μ, then, by the above theorem, u is a multiplier of $A(H)$ and so, by Corollary 4.3 it is a multiplier of $B_\lambda(H)$ also. The rest follows as above. □

COROLLARY 4.14. *Suppose that H is a commutative hypergroup satisfying (F). If the trivial character 1 belongs to S, then $A(H)$ is a Banach algebra under the multiplier norm which is equivalent to the original norm.*

PROOF. By Corollary 4.13, $B_\lambda(H) \subseteq M_b(B_\lambda(H))$. Since 1 belongs to S, it belongs to $B_\lambda(H)$. Therefore, $M_b(B_\lambda(H)) = B_\lambda(H)$ which in turn equals $M_b A(H)$ by Corollary 4.3. The norms in $MA(H)$ and $B_\lambda(H)$ are equivalent since

$$\|\phi\| = \|\phi \cdot 1\| \leq \|\phi\|_{MB_\lambda(H)} \leq M \, \|\phi\| \, ,$$

and since $\|\phi\|_{MA(H)} = \|\phi\|_{MB_\lambda(H)}$.

As $A(H)$ is closed in $B_\lambda(H)$, it is closed with respect to the multiplier norm also. Therefore, $A(H)$ is a Banach algebra under the multiplier norm, and the original norm is equivalent to the multiplier norm. □

The following corollary is quite useful in our further study of multipliers of $A(H)$, which we take up in future.

COROLLARY 4.15. *Suppose that S' is a locally compact set such that $S \subseteq S' \subseteq \widehat{H}$ satisfying the following: there exists $M > 0$ such that for every τ in S' and γ in S, $\tau\gamma$ belongs to $B_\lambda(H)$ and $\|\tau\gamma\| \leq M$.*

Then for any μ in $M(\widehat{H})$ whose support is contained in S', $\mathcal{I}(\mu)$ belongs to $M_b A(H)$. Moreover, $\|\mathcal{I}(\mu)\|_{MA(H)} \leq M \, \|\mu\|$.

PROOF. The proof follows exactly as in the proofs of Lemma 4.8 and Theorem 4.12. □

The following theorem discusses the converse of Theorem 4.12.

THEOREM 4.16. *Suppose that $A(H)$ is an algebra under pointwise product and there exists $M > 0$ satisfying $\|u \cdot v\| \leq M \|u\| \|v\|$ for all $u, v \in A(H)$. Then for every γ, γ' in S, $\gamma \cdot \gamma'$ belongs to $B_\lambda(H)$ and $\|\gamma \cdot \gamma'\| \leq M$.*

PROOF. Let γ belong to S. We are done if we show that γ belongs to $MB_\lambda(H)$ and $\|\gamma\|_{MB_\lambda(H)} \leq M$. By Corollary 4.3, it is sufficient to show that γ belongs to $MA(H)$ and $\|\gamma\|_{MA(H)} \leq M$. However, as $C_c(H) \cap A(H)$ is dense in $A(H)$, we need to show only that for every u in $C_c(H) \cap A(H)$, $\gamma \cdot u$ belongs to $A(H)$ and $\|\gamma \cdot u\| \leq M \|u\|$.

If u belongs to $C_c(H) \cap A(H)$, then by hypothesis u belongs to $MA(H)$ and so to $MB_\lambda(H)$. Therefore, $u \cdot \gamma$ belongs to $B_\lambda(H) \cap C_c(H)$ which is contained in $A(H)$ and $\|u \cdot \gamma\| \leq M \|u\|$. Hence the result follows. □

4.1. Examples of commutative hypergroups satisfying condition (F)

EXAMPLE 4.17. It is easy to see that any finite commutative hypergroup satisfies condition (F) of Definition 4.7. We observe that the constant M is strictly greater than 1 for the following example.

Let $H = \{e, a, b\}$ be the hypergroup given in Jewett [17, Example 9.1C], whose dual space $\widehat{H}$ is equal to $\{1, \chi, \psi\}$, where the character χ is given by $\chi(e) = 1$, $\chi(a) = -3/4$ and $\chi(b) = 1/2$. If f is defined on $\widehat{H}$ by $f(1) = 17/36$, $f(\chi) = (-3/68)(17/4)$ and $f(\psi) = (153/100)(175/306)$, then $\mathcal{I}(f) = \chi^2$ and

$$\left\|\chi^2\right\| = \|f\|_1 = \frac{666}{612}.$$

See *op. cit* for the computation of Plancherel-Levitan measure of this hypergroup and other unexplained details. Vrem has already computed $\left\|\chi^2\right\|$, see [30].

EXAMPLE 4.18 (cosh hypergroup). Let H denote the cosh hypergroup studied by Zeuner [31]. It is a 1-dimensional hypergroup on $\mathbb{R}_+$ defined by the following convolution formula.

$$\delta_x * \delta_y = \frac{\cosh(x-y)}{2\cosh(x)\cosh(y)}\,\delta_{|x-y|} + \frac{\cosh(x+y)}{2\cosh(x)\cosh(y)}\,\delta_{x+y} \quad x, y > 0.$$

It is shown in [31] that $\widehat{H} = \{\phi_\lambda : \lambda \in \mathbb{R}_+ \cup \imath[0, 1]\}$, whereas $S = \{\phi_\lambda : \lambda \in \mathbb{R}_+\}$, where $\phi_\lambda = (\cos \lambda x)/\cosh x$.

Moreover, by [31, Proposition 4.2 (a)–(b)] we see that the following holds: for every τ in $\widehat{H}$ and for every γ in S, there exists a probability measure $\mu_{\tau,\gamma}$ such that $\mathcal{I}(\mu_{\tau,\gamma}) = \gamma\tau$. In particular, Condition (F) holds for the cosh hypergroup for $M = 1$.

EXAMPLE 4.19 (Jacobi hypergroups). Fix $\alpha \geq \beta \geq -1/2$. Let us exclude the case $\alpha = \beta = -1/2$. Let $H_{\alpha,\beta}$ denote the Jacobi hypergroup (it is also known as Chébli-Trimèche hypergroup of Jacobi type). It is a one-dimensional hypergroup whose characters are Jacobi functions given as follows: if

$$\phi_\lambda(x) = {}_2F_1\left(\frac{\rho - \imath\lambda}{2}, \frac{\rho + \imath\lambda}{2}, \alpha + 1, -\sinh^2 x\right),$$

where ${}_2F_1$ is the Gauss hypergeometric function, then $\widehat{H_{\alpha,\beta}} = \{\phi_\lambda : \lambda \in \mathbb{R}_+ \cup \imath[0, \rho]\}$ and $S = \{\phi_\lambda : \lambda \in \mathbb{R}_+\}$, where $\rho = \alpha + \beta + 1$.

It was proved by Flensted-Jensen and Koornwinder in [11, Section 4] that, for every λ_1 and λ_2 in $\mathbb{R}_+$, the function $\phi_{\lambda_1} \cdot \phi_{\lambda_2}$ is continuous positive definite and therefore condition (F) is satisfied with $M = 1$. See also Koornwinder [18, Section 8.3].

EXAMPLE 4.20 (Bessel-Kingman hypergroups). These are one-dimensional hypergroups $H_\alpha : \alpha > -1/2$, whose characters are given by $\phi_\lambda(x) = j_\alpha(\lambda x)$, where j_α denote (complex valued) modified Bessel functions of type α. These hypergroups are self-dual. That is, $H = \widehat{H}$. So, H satisfies condition (F) with $M = 1$. For more details, see Bloom and Heyer [2, Section 3.5.61],

EXAMPLE 4.21. Suppose that G is a locally compact group. Let B be a subgroup of the group of all topological automorphisms of G containing the inner automorphisms of G, which is relatively compact with respect to the Birkhoff topology. That is to say, G is a $[FIA]_B^-$ group with $I(G) \subseteq B$. If H denotes the space of all $\overline{B}$ orbits in G, then H is a commutative hypergroup and $\widehat{H}$ is also a hypergroup (see Hartmann *et al.* [15] and Ross [25]). In particular, H satisfies condition (F) with $M = 1$.

EXAMPLE 4.22 (Jacobi polynomial hypergroups). Let $\alpha \geq \beta > -1$ and $\alpha + \beta + 1 \geq 0$. Consider the polynomial hypergroup $H_{\alpha,\beta}$ on $\mathbb{N}_0$ corresponding to Jacobi polynomials $R_n^{\alpha,\beta}(x)$. Then $d\pi^{\alpha,\beta}(x) = (1-x)^\alpha(1+x)^\beta dx$ and $S = [-1, 1]$. The characters $\{\chi_x\}$ are given by $\chi_x(n) = R_n^{\alpha,\beta}(x)$. By Gasper [12, Section 1], there exists a constant $M > 0$ satisfying the following:

for every x, y in $[-1, 1]$ there exists a unique, real valued bounded measure, $\mu_{x,y}$ on $[-1, 1]$, independent of n, such that

$$R_n^{\alpha,\beta}(x) R_n^{\alpha,\beta}(y) = \int_{-1}^{1} R_n^{\alpha,\beta}(z)\, d\mu_{x,y}(z) \quad \text{for all } n \in \mathbb{N}_0,$$

and $\|\mu_{x,y}\| \leq M$. Thus, the hypergroup $H_{\alpha,\beta}$ satisfies condition (F).

If furthermore, $\beta \geq -1/2$ or $\alpha + \beta \geq 0$, then $M = 1$. For further details see Lasser [21].

EXAMPLE 4.23. Consider the generalized Chebyshev polynomials $\left\{T_n^{\alpha,\beta}\right\}_{n\in\mathbb{N}_0}$ with $\alpha, \beta > -1$ normalized by $T_n^{\alpha,\beta}(1) = 1$. They are orthogonal on $[-1,\ 1]$ with respect to $d\pi = (1-x^2)^\alpha\,|x|^{2\beta+1}$ (see Chihara [6]).

If $\alpha \geq \beta + 1$, then it gives rise to a polynomial hypergroup say $H_{\alpha,\beta}$ on $\mathbb{N}_0$ (see Lasser [21]). Moreover, S is identified with $[-1,\ 1]$. By Laine [19, Theorem 1], we observe that $H_{\alpha,\beta}$ satisfies the condition (F). Moreover, $M = 1$ if we further assume that $\beta \geq -1/2$ (refer to Laine [19, Theorem 1] again).

REMARK 4.24. We say that a commutative hypergroup H has a dual convolution structure on a subset E of $\widehat{H}$ if, for every γ_1, γ_2 belonging to E, the product $\gamma_1 \cdot \gamma_2$ is positive definite on H. If a hypergroup has the dual convolution structure on S, then it satisfies the condition (F) with $M = 1$. For instance, Jacobi hypergroups, and Bessel-Kingman hypergroups have dual convolution structure. Voit has studied hypergroups having dual convolution structure, along with many examples [28]. For instance, in his recent article Voit [29] discusses dual convolutions on the hypergroups associated with infinite distance transitive graphs.

5. Fourier hypergroups

DEFINITION 5.1. A hypergroup H is called a *Fourier hypergroup* if

(1) The Fourier space $A(H)$ forms an algebra with pointwise product.

(2) There exists a norm on $A(H)$ which is equivalent to the original norm with respect to which $A(H)$ forms a Banach algebra.

A hypergroup is called a *regular Fourier hypergroup* if $A(H)$ is a Banach algebra with its original norm and pointwise product.

REMARK 5.2. We will show in Muruganandam [22] that a double coset hypergroup is a regular Fourier hypergroup.

PROPOSITION 5.3. *If H is a commutative hypergroup satisfying condition (F) for some $M > 0$, then H is a Fourier hypergroup. If $M = 1$, then H is actually a regular Fourier hypergroup.*

PROOF. By Corollary 4.13, $A(H)$ is an algebra under pointwise product.

If we define a new norm on $A(H)$ by $\|u\|_{\text{new}} = M\,\|u\|$ for every u in $A(H)$, then the two norms are equivalent. Moreover, by (4.6) we observe that $(A(H), \|\cdot\|_{\text{new}})$ is a normed algebra. Thus $(A(H), \|\cdot\|_{\text{new}})$ is a Banach algebra. That is, H is a Fourier hypergroup. If $M = 1$, then it is of course regular also. □

REMARK 5.4. The examples enumerated in Subsection 4.1 are all Fourier hypergroups. Whenever $M = 1$ they are regular.

Another classical Banach algebra found in the context of harmonic analysis on the locally compact abelian group G, namely, the Banach algebra $AP(G)$, of all almost periodic functions on G does not form an algebra when considered over commutative hypergroups. Moreover, the Bohr compactification does not carry any hypergroup structure extending that of H. Yet, Lasser in [20] made a systematic study of $AP(H)$ extending many classical results to hypergroups by imposing appropriate assumptions on the commutative hypergroups. See also Blower [5, Section 4.11] for some results on almost periodic functions.

If H is a Fourier hypergroup, let $\Omega(A(H))$ denote the Gelfand spectrum consisting of all nonzero complex homomorphisms of $A(H)$.

REMARK 5.5. If x belongs to H, then the functional ω_x given by $\omega_x(f) = f(x)$ is a nonzero complex homomorphism by Proposition 2.22 and thus provides a map $x \to \omega_x$ from H into $\Omega(A(H))$. In the remaining part of this section, we identify some hypergroups for which $\Omega(A(H))$ is equal to H.

As in the case of groups (see Eymard [10]), we define the support of an element in the Banach space dual $A(H)^\star$ of $A(H)$ as follows.

DEFINITION 5.6. Let T be in $A(H)^\star$. We say that x belongs to the support of T if the following holds. For every neighborhood V of x there exists u in $A(H)$ such that the support of u is contained in V and $\langle u, T\rangle \neq 0$.

We denote the support of T by $\operatorname{supp} T$.

PROPOSITION 5.7. *Suppose that H is a Fourier hypergroup and T belongs to $A(H)^\star$. Then* $\operatorname{supp} T \neq \emptyset$ *if and only if* $T \neq 0$.

PROOF. Use Proposition 2.22 to observe that if K is a compact subset of H, then for every open cover of K there exists a partition of unity in $A(H)$ subordinated to it.

Suppose that T is in $A(H)^\star$ such that $\operatorname{supp} T = \emptyset$. Let u be in $C_c(H) \cap A(H)$ and let K denote $\operatorname{supp} u$.

Since $\operatorname{supp} T = \emptyset$, we see that for every x in K there exists a compact neighborhood U_x such that $\langle v, T\rangle = 0$ for every v belonging to $A(H)$ with $\operatorname{supp} v \subseteq U_x$.

Let $\{u_i\}$ denote the partition of unity in $C_c(H) \cap A(H)$ subordinated to $\{U_x\}_{x\in K}$. Then

$$\langle u, T\rangle = \left\langle \sum_i u \cdot u_i, T \right\rangle = \sum_i \langle u \cdot u_i, T\rangle = 0.$$

That is, $\langle u, T\rangle = 0$ for every u in $C_c(H) \cap A(H)$. As $C_c(H) \cap A(H)$ is dense in $A(H)$, we conclude that $T = 0$. □

REMARK 5.8. For every u in $A(H)$ and for every T in $A(H)^\star$, define $u \cdot T$ by $\langle v, u \cdot T\rangle = \langle u \cdot v, T\rangle$ for all $v \in A(H)$. Then $u \cdot T$ belongs to $A(H)^\star$.

Since the proof of the following proposition is exactly as in the case of groups (see Eymard [10, Proposition 4.8]), we will not prove it here.

PROPOSITION 5.9. *Let H be a Fourier hypergroup. Then*

$$\operatorname{supp}(u \cdot T) \subseteq \operatorname{supp} u \cap \operatorname{supp} T$$

for every u in $A(H)$ and for every T in $A(H)^\star$.

Let $\mathcal{D}_\star(\mathbb{R}_+) = \left\{f : \text{there exists } g \in \mathcal{C}^\infty_{\text{even}}(\mathbb{R}) \cap C_c(\mathbb{R}) \text{ such that } f = g|_{\mathbb{R}_+}\right\}$.

PROPOSITION 5.10. *Let H be either a Jacobi hypergroup or a Bessel-Kingman hypergroup. Then $\mathcal{D}_\star(\mathbb{R}_+)$ is dense in $A(H)$.*

PROOF. By Bloom and Xu [3, Lemma 3.24] or by Koornwinder [18, Theorem 2.3], and by Proposition 4.2(i), we have that $\mathcal{D}_\star(\mathbb{R}_+) \subseteq A(H)$.

Let $\epsilon > 0$ and let $u = f * \check{g}$ be in $A(H)$ with f, g belonging to $L^2(H)$. As $\mathcal{D}_\star(\mathbb{R}_+)$ is dense in $L^2(H)$ (see, for example, Bloom and Zu [4, Lemma 4.12]), there exist f_1, g_1 in $\mathcal{D}_\star(\mathbb{R}_+)$ satisfying $\|f - f_1\|_2 < \epsilon$ and $\|g - g_1\|_2 < \epsilon$.

Then $v = f_1 * \check{g_1}$ belongs to $\mathcal{D}_\star(\mathbb{R}_+)$ by, for instance, Trimèche [27, Proposition 6.11.12], and

$$\begin{aligned}\|u - v\| &\le \|f - f_1\|_2 \, \|g\|_2 + \|f_1\|_2 \, \|g - g_1\|_2 \\ &\le \epsilon \, \|g\|_2 + (\epsilon + \|f\|_2)\epsilon.\end{aligned}$$

Therefore, $\mathcal{D}_\star(\mathbb{R}_+)$ is dense in $A(H)$. □

PROPOSITION 5.11. *Let H be either a Jacobi hypergroup or a Bessel-Kingman hypergroup. If ω belongs to $\Omega(A(H))$ and if $\operatorname{supp} \omega \subseteq \{x\}$ for some x in H, then $\omega = \omega_x$.*

PROOF. As the support of ω is not empty, $\operatorname{supp} \omega = \{x\}$. We will first show that for any u in $\mathcal{D}_\star(\mathbb{R}_+)$ if $\omega(u) = 0$, then $\omega_x(u) = 0$.

Suppose on the contrary that there is a u in $\mathcal{D}_\star(\mathbb{R}_+)$ satisfying $\omega(u) = 0$, but $u(x) \neq 0$. Let V be a compact neighborhood of x such that $|u(y)| \ge \delta$ for some $\delta > 0$ and for every y in V. Let u' be a function in $\mathcal{D}_\star(\mathbb{R}_+)$ with $u'(y) = 1/u(y)$ for every y in V. Then u' belongs to $A(H)$.

Since x belongs to the support of ω there exists v in $A(H)$ with support of v contained in V satisfying $\omega(v) \neq 0$. Then $v = v \cdot u' \cdot u$ and

$$\omega(u)\omega(v \cdot u') = \omega(v \cdot u' \cdot u) = \omega(v) \neq 0,$$

a contradiction to the assumption that $\omega(u) = 0$. Therefore, $\omega_x(u) = 0$.

Let u_0 in $\mathcal{D}_\star(\mathbb{R}_+)$ be fixed so that $\omega(u_0) = 1$. If u belongs to $\mathcal{D}_\star(\mathbb{R}_+)$, then $u = v + \omega(u)u_0$, where v belongs to $\mathcal{D}_\star(\mathbb{R}_+)$ with $\omega(v) = 0$. Therefore, if we fix $\alpha = u_0(x)$, then by the above $\omega_x = \alpha\omega$ on $\mathcal{D}_\star(\mathbb{R}_+)$. As $\mathcal{D}_\star(\mathbb{R}_+)$ is dense in $A(H)$, by the above proposition, we have $\omega_x = \alpha\omega$. Since ω and ω_x are nonzero complex homomorphisms $\alpha = 1$. □

REMARKS 5.12. (1) From the above proof we also observe that if H is a discrete Fourier hypergroup, then the conclusion of the above proposition holds.
(2) The proof Theorem 5.13 is an adaptation of the proof given for groups by Herz in [16].

THEOREM 5.13. *Suppose that H is a Fourier hypergroup satisfying any one of the following*:

(i) *H is discrete.*
(ii) *H is either a Jacobi hypergroup or a Bessel-Kingman hypergroup.*

Then the map $x \to \omega_x$ defines a homeomorphism from H onto $\Omega(A(H))$. Moreover, the Banach algebra $A(H)$ is regular, semisimple and Tauberian.

PROOF. We first show the surjectivity of the above map in both cases. Let ω be in $\Omega(A(H))$. Then the support is not empty by Proposition 5.7. Let x belong to the support of ω.

We show that $\operatorname{supp}\omega \subseteq \{x\}$. If V is any arbitrary compact neighborhood of x, then there exists u in $A(H)$ such that $\operatorname{supp} u \subseteq V$ and $\omega(u) \neq 0$. If v is in $A(H)$ such that $v(y) = 1$ for every y in $\operatorname{supp} u$ and if $\operatorname{supp} v$ is contained in V, which exists by Proposition 2.22, then $u = u \cdot v$. Now $\omega(u) = \omega(u \cdot v) = \omega(u)\omega(v)$. Therefore, $\omega(v) = 1$. In particular, $\omega = v \cdot \omega$ since $v \cdot \omega = \omega(v)\omega$. However, then by Proposition 5.9, we have $\operatorname{supp}(\omega) = \operatorname{supp}(v \cdot \omega) \subseteq \operatorname{supp}(v) \cap \operatorname{supp}(\omega) \subseteq V$. As V is arbitrary, $\operatorname{supp}(\omega) \subseteq \{x\}$. We use the above remark for case (i) and Proposition 5.11 for case (ii) to conclude that $\omega = \omega_x$.

In particular, $A(H)$ is semisimple. As $C_c(H) \cap A(H)$ is dense in $A(H)$, the Banach algebra $A(H)$ is Tauberian. By Proposition 2.22, we see that $A(H)$ is regular. By Rickart [24, Theorem 3.2.4], the mapping $x \to \omega_x$ is indeed a homeomorphism from H onto $\Omega(A(H))$. □

Acknowledgment

The author expresses his deep gratitude to Eymard for suggesting the problem and Faraut for many useful discussions concerning the earlier version of this article. He is very much indebted to Cowling, Faraut, Ross and Voit for their valuable suggestions, which improved the article substantially. The author is thankful to the referee for his helpful suggestions and for introducing the papers of Zeuner [32], Lasser [20] and Blower [5] to him.

References

[1] W. B. Arveson, 'On groups of automorphisms of operator algebras', *J. Funct. Analysis* **15** (1974), 217–243.

[2] W. R. Bloom and H. Heyer, *Harmonic analysis of probability measures on hypergroups*, de Gruyter Studies in Math. 20 (Walter de Gruyter, Berlin, 1995).

[3] W. R. Bloom and Z. Xu, 'The Hardy-Littlewood maximal function for Chébli-Trimèche hypergroups', *Contemp. Math.* **183** (1995), 45–70.

[4] ———, 'Fourier transforms of Schwartz functions on Chébli-Trimèche hypergroups', *Monatsh. Math.* **125** (1998), 89–109.

[5] G. Blower, 'Stationary processes for translation operators', *Proc. London. Math. Soc.* **72** (1996), 697–720.

[6] T. S. Chihara, *An introduction to orthogonal polynomials* (Gordon and Breach, New York, 1978).

[7] J. DeCannière and U. Haagerup, 'Multipliers of the Fourier algebra of some simple Lie groups and their discrete subgroups', *Amer. J. Math.* **107** (1985), 455–500.

[8] J. Dixmier, *$C^\star$-algebras*, volume 15 (North-Holland, Amsterdam, 1977).

[9] ———, *Von Neumann algebras*, volume 27 (North-Holland, Amsterdam, 1981).

[10] P. Eymard, 'L'algèbre de Fourier d'un groupe localement compact', *Bull. Soc. Math. France* **92** (1964), 181–236.

[11] M. Flensted-Jensen and T. H. Koornwinder, 'Jacobi functions: the addition formula and the positivity of the dual convolution structure', *Ark. Math.* **17** (1979), 139–151.

[12] G. Gasper, 'Banach algebras for Jacobi series and positivity of a kernel', *Ann. of Math. (1)* **95** (1972), 261–280.

[13] R. Godement, 'Les fonctions de type positif et la théorie des groupes', *Trans. Amer. Math. Soc.* **63** (1948), 1–84.

[14] A. Grothendieck, *Topological vector spaces*, Notes on Mathematics and its Applications (Gordon and Breach, New York, 1973).

[15] K. Hartmann, R. W. Henrichs and R. Lasser, 'Duals of orbit spaces in groups with relatively compact inner automorphism groups are hypergroups', *Monatsh. Math.* **88** (1979), 229–238.

[16] C. S. Herz, 'Harmonic synthesis for subgroups', *Ann. Inst. Fourier (Grenoble)* **23** (1973), 91 – 123.

[17] R. I. Jewett, 'Spaces with an abstract convolution of measures', *Adv. Math.* **18** (1975), 1–101.

[18] T. H. Koornwinder, 'Jacobi functions and analysis of noncompact semisimple Lie groups', in: *Special functions: Group theoretical aspects and applications* (ed. R. A. Askey *et al.*) (D. Reidel, Dordrecht, 1984) pp. 1–85.

[19] T. P. Laine, 'The product formula and convolution structure for the generalized Chebyshev polynomials', *SIAM. J. Math. Anal* **11** (1980), 133–146.

[20] R. Lasser, 'Almost periodic functions on hypergroups', *Math. Ann.* **252** (1980), 183–196.

[21] ———, 'Orthogonal polynomials and hypergroups', *Rend. Math. Appl.* **3** (1983), 185–209.

[22] V. Muruganandam, 'Fourier algebra of a hypergroup. II. Spherical hypergroups', *Math. Nach.*, to appear.

[23] G. K. Pederson, *$C^\star$-algebras and their automorphism groups*, London Math. Soc. Monographs 14 (Academic Press, London, 1979).
[24] C. E. Rickart, *General theory of Banach algebras*, The University Series in Higher Math. (D. Van Nostrand, Princeton, N.J., 1960).
[25] K. A. Ross, 'Centers of hypergroups', *Trans. Amer. Math. Soc.* **213** (1978), 251–269.
[26] M. Takesaki, *Theory of operator algebras I* (Springer, New York, 1979).
[27] K. Trimèche, *Generalized wavelets and hypergroups* (Gordon and Breach, Amsterdam, 1997).
[28] M. Voit, 'On the Fourier transformation of positive, positive definite measures on commutative hypergroups, and dual convolution structures', *Manuscripta Math.* **72** (1991), 141–153.
[29] ———, 'A product formula for orthogonal polynomials associated with infinite distance-transitive graphs', *J. Approx. Theory* **120** (2003), 337–354.
[30] R. C. Vrem, 'Harmonic analysis on compact hypergroups', *Pacific. J. Math.* **85** (1979), 239–251.
[31] H. Zeuner, 'Properties of the cosh hypergroup', in: *Probability Measures on Groups IX, Proc. Conf. Oberwolfach, 1988*, Springer Lecture Notes in Math. 1379 (Springer, Berlin, 1989) pp. 425–434.
[32] ———, 'Duality of commutative hypergroups', in: *Probability Measures on Groups X, Proc. Conf. Oberwolfach* (Plenum Press, New York, 1990) pp. 467–488.

Department of Mathematics
Pondicherry University
Pondicherry 605 014
India
e-mail: vmuruganandam@gmail.com