DERIVATIVE-FREE CHARACTERIZATIONS OF Q_K SPACES

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Abstract

We give two characterizations of the Möbius invariant Q_K spaces, one in terms of a double integral and the other in terms of the mean oscillation in the Bergman metric. Both characterizations avoid the use of derivatives. Our results are new even in the case of Q_p spaces.

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1. Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$. For any $0 \le p < \infty$ we consider the space Q_p consisting of all analytic functions f in $\mathbb{D}$ such that

$$\|f\|_{Q_p}^2 = \sup_{\varphi} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi(z)|^2)^p \, dA(z) < \infty,$$

where dA is the area measure on $\mathbb{D}$ normalized so that $A(\mathbb{D}) = 1$, and the supremum is taken over $\varphi \in \mathrm{Aut}(\mathbb{D})$, the group of Möbius maps of $\mathbb{D}$. The space Q_p is Möbius invariant in the sense that $\|f \circ \varphi\|_{Q_p} = \|f\|_{Q_p}$ for every $f \in Q_p$ and $\varphi \in \mathrm{Aut}(\mathbb{D})$.

Since every Möbius map φ can be written as $\varphi(z) = e^{i\theta} \varphi_a(z)$, where θ is real, and $\varphi_a(z) = (a - z)/(1 - \bar{a}z)$ is the Möbius map of the unit disk that interchanges the points 0 and a, we can also write

$$\|f\|_{Q_p}^2 = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 (1 - |\varphi_a(z)|^2)^p \, dA(z).$$

It is well-known that for $p > 1$, we have $Q_p = \mathscr{B}$, the Bloch space of analytic functions f in $\mathbb{D}$ such that $\|f\|_{\mathscr{B}} = \sup_{z \in \mathbb{D}}(1 - |z|^2)|f'(z)| < \infty$. It is also well known that $Q_p = $ BMOA when $p = 1$. When $p = 0$, the space Q_p degenerates to the Dirichlet space, see [9] for a summary of recent research about Q_p spaces.

More generally, for any nonnegative and Lebesgue measurable function K on $(0, 1]$, we consider the space Q_K consisting of all analytic functions f in the unit disk such that

$$\|f\|_{Q_K}^2 = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(1 - |\varphi_a(z)|^2)\, dA(z) < \infty.$$

Clearly, if $K(t) = t^p$, then $Q_K = Q_p$. It is also clear that Q_K is Möbius invariant, that is, $\|f \circ \varphi\|_{Q_K} = \|f\|_{Q_K}$ whenever $f \in Q_K$ and $\varphi \in \mathrm{Aut}(\mathbb{D})$. A great deal of function theory was worked out in [3] for the spaces Q_K. The paper [8] characterized Q_K spaces in terms of higher order derivatives. More research on Q_K spaces can be found in [2], [7].

The purpose of this paper is to give two characterizations of Q_K, which are free of the use of derivatives. The first characterization, Theorem 3.3, is in terms of a double integral involving the area measure on $\mathbb{D}$. The other characterization, Theorem 4.2, is based on the mean oscillation of a function in the Bergman metric. Our results are new even for Q_p spaces, although the corresponding results for the Bloch space are well-known, see [1] and [10].

As a consequence of our main results, the Bloch space and the space BMOA are characterized by the same type of conditions involving the area measure. For example, BMOA can now be characterized in terms of the mean oscillation in the Bergman metric. This seems to be something that has not been noticed before.

2. Preliminaries

We need an elementary, but somewhat non-standard tool, the so-called oro-coordinates on the unit disk. Recall that, for any $r > 0$, the equation

$$\frac{1 - |z|^2}{|1 - z|^2} = r$$

defines a circle C_r that is internally tangent to the unit circle $|z| = 1$ at the point $z = 1$. These are called *oro-cycles* at $z = 1$, see [4]. It is easy to check that C_r can be rewritten as

$$\left| z - \frac{r}{1 + r} \right| = \frac{1}{1 + r}.$$

It follows that the unit disk $\mathbb{D}$ can be parametrized by

$$z = \frac{r}{1+r} + \frac{1}{1+r}\, e^{i\theta}, \quad 0 \le r < \infty, \; 0 \le \theta \le 2\pi.$$

This reparametrization of the unit disk will be called *oro-coordinates* on $\mathbb{D}$. The following lemma gives the right form of the area integral in terms of oro-coordinates.

LEMMA 2.1. *Suppose f is Lebesgue measurable on $\mathbb{D}$. If f is nonnegative or belongs to $L^1(\mathbb{D}, dA)$, then*

$$\int_{\mathbb{D}} f(z)\, dA(z) = \frac{1}{\pi} \int_0^\infty \int_0^{2\pi} f(r, \theta) \frac{1 - \cos\theta}{(1+r)^3}\, dr\, d\theta,$$

where $f(r, \theta)$ is the function f in oro-coordinates.

PROOF. See [6].   □

If the function K is only defined on $(0, 1]$, then we extend it to $(0, \infty)$ by setting $K(t) = K(1)$ for $t > 1$. We can then define an auxiliary function as

$$\varphi_K(s) = \sup_{0 < t \le 1} \frac{K(st)}{K(t)}, \quad 0 < s < \infty.$$

We further assume that K is continuous and nondecreasing on $(0, 1]$. This ensures that the function φ_K is continuous and nondecreasing on $(0, \infty)$.

We will also need to use the Berezin transform. More specifically, for any function $f \in L^1(\mathbb{D}, dA)$, we define a function Bf by

$$Bf(z) = \int_{\mathbb{D}} \frac{(1 - |z|^2)^2}{|1 - z\overline{w}|^4}\, f(w)\, dA(w), \quad z \in \mathbb{D}.$$

It is standard terminology to call Bf the *Berezin transform* of f. By a change of variables, we can also write

$$Bf(z) = \int_{\mathbb{D}} f \circ \varphi_z(w)\, dA(w), \quad z \in \mathbb{D}.$$

See [5] and [10] for basic properties of the Berezin transform.

The following estimate is the key to the main results of this paper.

LEMMA 2.2. *Let K be any nonnegative and Lebesgue measurable function on $(0, \infty)$ and $f(z) = K(1 - |z|^2)$. If*

$$(2.1) \qquad\qquad \int_0^\infty \frac{\varphi_K(x)}{(1+x)^3}\, dx < \infty,$$

then there exists a positive constant C such that $Bf(z) \le Cf(z)$ for all $z \in \mathbb{D}$.

PROOF. Recall that $Bf(z) = \int_{\mathbb{D}} K(1 - |\varphi_z(w)|^2)\, dA(w)$. Since

$$1 - |\varphi_z(w)|^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - z\overline{w}|^2},$$

it follows from the definition of φ_K that

$$K(1 - |\varphi_z(w)|^2) \le K(1 - |z|^2)\varphi_K\left(\frac{1 - |w|^2}{|1 - z\overline{w}|^2}\right),$$

so $Bf(z) \le f(z)\Phi(z)$, $z \in \mathbb{D}$, where

$$\Phi(z) = \int_{\mathbb{D}} \varphi_K\left(\frac{1 - |w|^2}{|1 - z\overline{w}|^2}\right) dA(w).$$

The function Φ is continuous on $\mathbb{D}$, so it is bounded on any compact subset of $\mathbb{D}$. On the other hand, if $z \in \mathbb{D}$ is nonzero, then we use the monotonicity of φ_K (together with the obvious inequality $1 - |w|^2 \le 1 - |zw|^2$) and a change of variables to obtain

$$\begin{aligned}
\Phi(z) &\le \int_{\mathbb{D}} \varphi_K\left(\frac{1 - |z\overline{w}|^2}{|1 - z\overline{w}|^2}\right) dA(w) \\
&= \frac{1}{|z|^2} \int_{|w|<|z|} \varphi_K\left(\frac{1 - |w|^2}{|1 - w|^2}\right) dA(w) \\
&\le \frac{1}{|z|^2} \int_{\mathbb{D}} \varphi_K\left(\frac{1 - |w|^2}{|1 - w|^2}\right) dA(w).
\end{aligned}$$

If the condition in (2.1) is satisfied, then an application of Lemma 2.1 shows that

$$\int_{\mathbb{D}} \varphi_K\left(\frac{1 - |w|^2}{|1 - w|^2}\right) dA(w) = 2\int_0^\infty \frac{\varphi_K(r)}{(1 + r)^3}\, dr < \infty.$$

Combining this with the estimates in the previous paragraphs, we conclude that the function $\Phi(z)$ is bounded in $\mathbb{D}$. This completes the proof. $\qquad\square$

If $K(t) = t^p$ and $f(z) = K(1 - |z|^2)$, then the Berezin transform of f is

$$\begin{aligned}
Bf(z) &= \int_{\mathbb{D}} \left(1 - |\varphi_z(w)|^2\right)^p dA(w) \\
&= (1 - |z|^2)^p \int_{\mathbb{D}} \frac{(1 - |w|^2)^p\, dA(w)}{|1 - z\overline{w}|^{2p}} \\
&= f(z) \int_{\mathbb{D}} \frac{(1 - |w|^2)^p\, dA(w)}{|1 - z\overline{w}|^{2p}}.
\end{aligned}$$

By a well-known estimate (see [10, Lemma 4.2.2], for example), the last integral above is a bounded function of z if and only if $p < 2$.

On the other hand, if $K(t) = t^p$, then it is easy to see that $\varphi_K(t) = t^p$ as well. In this case, it is clear that condition (2.1) holds if and only if $p < 2$. Therefore, at least in the case of Q_p spaces, the condition in (2.1) is best possible.

Let $\beta(z, w)$ denote the Bergman metric between two points z and w in $\mathbb{D}$. It is well-known that

$$\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}.$$

For $z \in \mathbb{D}$ and $R > 0$ we use $D(z, R) = \{w \in \mathbb{D} : \beta(z, w) < R\}$ to denote the Bergman metric ball at z with radius R. If R is fixed, then it can be checked that the area of $D(z, R)$, denoted by $|D(z, R)|$, is comparable to $(1 - |z|^2)^2$ as z approaches the unit circle, see [10].

LEMMA 2.3. *For any $R > 0$ there exists a positive constant C (depending on R) such that*

$$|f(z)|^2 \le \frac{C}{|D(z, R)|} \int_{D(z,R)} |f(w)|^2 \, dA(w)$$

for all $z \in \mathbb{D}$ and all analytic functions f in $\mathbb{D}$.

PROOF. See [10, Lemma 4.3.8], for example. □

3. A double integral characterization of Q_K

In this section we characterize the spaces Q_K in terms of a double integral that does not involve the use of derivatives. We begin with the following estimate of Dirichlet type integrals.

LEMMA 3.1. *There exists a constant $C > 0$ (independent of K) such that*

$$\int_{\mathbb{D}} |f'(z)|^2 K(1 - |z|^2) \, dA(z) \le CI(f)$$

for all analytic functions f in $\mathbb{D}$, where

$$I(f) = \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z) - f(w)|^2}{|1 - z\overline{w}|^4} K(1 - |z|^2) \, dA(z) \, dA(w).$$

PROOF. We write the double integral $I(f)$ as an iterated integral

$$I(f) = \int_{\mathbb{D}} \frac{K(1-|z|^2)}{(1-|z|^2)^2} \, dA(z) \int_{\mathbb{D}} \frac{(1-|z|^2)^2}{|1-z\overline{w}|^4} |f(z) - f(w)|^2 \, dA(w).$$

Making a change of variables in the inner integral, we obtain

(3.1) $$I(f) = \int_{\mathbb{D}} \frac{K(1-|z|^2)}{(1-|z|^2)^2} \, dA(z) \int_{\mathbb{D}} |f(\varphi_z(w)) - f(z)|^2 \, dA(w).$$

It is well-known (verify using Taylor expansion or see [10, Theorem 4.27], for example) that

(3.2) $$\int_{\mathbb{D}} |g(w) - g(0)|^2 \, dA(w) \sim \int_{\mathbb{D}} |g'(w)|^2 (1-|w|^2)^2 \, dA(w)$$

for analytic functions g in $\mathbb{D}$. Applying (3.2) to the inner integral in (3.1) with the function $g(w) = f(\varphi_z(w))$, we see that

$$I(f) \sim \int_{\mathbb{D}} \frac{K(1-|z|^2)}{(1-|z|^2)^2} \, dA(z) \int_{\mathbb{D}} |(f \circ \varphi_z)'(w)|^2 (1-|w|^2)^2 \, dA(w).$$

By the chain rule and a change of variables, we get

(3.3) $$I(f) \sim \int_{\mathbb{D}} K(1-|z|^2) \, dA(z) \int_{\mathbb{D}} \frac{|f'(w)|^2 (1-|w|^2)^2}{|1-z\overline{w}|^4} \, dA(w).$$

Fix any positive radius R. Then there exists a constant $C > 0$ such that

$$I(f) \geq C \int_{\mathbb{D}} K(1-|z|^2) \, dA(z) \int_{D(z,R)} \frac{|f'(w)|^2 (1-|w|^2)^2}{|1-z\overline{w}|^4} \, dA(w).$$

It is well-known that

$$\frac{(1-|w|^2)^2}{|1-z\overline{w}|^4} \sim \frac{1}{(1-|z|^2)^2} \sim \frac{1}{|D(z,R)|}$$

for $w \in D(z, R)$, see [10, Section 4.3]. It follows that there exists a positive constant C such that

$$I(f) \geq C \int_{\mathbb{D}} K(1-|z|^2) \, dA(z) \frac{1}{|D(z,R)|} \int_{D(z,R)} |f'(w)|^2 \, dA(w).$$

Combining this with Lemma 2.3, we obtain a positive constant C such that

$$I(f) \geq C \int_{\mathbb{D}} |f'(z)|^2 K(1-|z|^2) \, dA(z).$$

This completes the proof. □

The second half of the above proof (the one based on Lemma 2.3) also follows from inequality (5.5) of [3]; see the proof of Lemma 3.2 below. But the proof of (5.5) in [3] transfers the setting to the upper half-plane and makes use of certain additional assumptions on K, while our proof here does not depend on any special property of K. The following lemma, however, makes a key assumption on K.

LEMMA 3.2. *If the function K satisfies condition* (2.1), *then there exists a constant $C > 0$ such that $\int_{\mathbb{D}} |f'(z)|^2 K(1 - |z|^2) \, dA(z) \geq C I(f)$ for all analytic functions f in $\mathbb{D}$, where $I(f)$ is as given in Lemma* 3.1.

PROOF. By Fubini's Theorem, we can rewrite (3.3) as

$$I(f) \sim \int_{\mathbb{D}} |f'(w)|^2 \, dA(w) \int_{\mathbb{D}} \frac{(1 - |w|^2)^2}{|1 - z\overline{w}|^4} K(1 - |z|^2) \, dA(z).$$

The inner integral above is nothing but the Berezin transform of the function $K(1-|z|^2)$ at the point w. The desired estimate now follows from Lemma 2.2. □

We can now prove the main result of this section.

THEOREM 3.3. *Suppose K satisfies condition* (2.1). *Then an analytic function f in $\mathbb{D}$ belongs to Q_K if and only if*

$$(3.4) \qquad \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z) - f(w)|^2}{|1 - z\overline{w}|^4} K(1 - |\varphi_a(z)|^2) \, dA(z) \, dA(w) < \infty.$$

PROOF. Recall that $f \in Q_K$ if and only if

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(1 - |\varphi_a(z)|^2) \, dA(z) < \infty.$$

By a change of variables, we have $f \in Q_K$ if and only if

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |(f \circ \varphi_a)'(z)|^2 K(1 - |z|^2) \, dA(z) < \infty.$$

Replacing f by $f \circ \varphi_a$ in Lemmas 3.1 and 3.2, we conclude that $f \in Q_K$ if and only if

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f \circ \varphi_a(z) - f \circ \varphi_a(w)|^2}{|1 - z\overline{w}|^4} K(1 - |z|^2) \, dA(z) \, dA(w) < \infty.$$

Changing variables and simplifying the result, we find that the double integral above is the same as

$$\int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z) - f(w)|^2}{|1 - z\overline{w}|^4} K(1 - |\varphi_a(z)|^2) \, dA(z) \, dA(w).$$

Therefore, $f \in Q_K$ if and only if condition (3.4) holds. □

4. The mean oscillation in the Bergman metric

In this section we give two closely related characterizations of Q_K spaces, one in terms of the Berezin transform and the other in terms of the mean oscillation in the Bergman metric.

Given a function $f \in L^2(\mathbb{D}, dA)$, it is customary to write

$$MO(f)(z) = \sqrt{B(|f|^2)(z) - |Bf(z)|^2};$$

see [10, Section 7.1]. It is easy to check that

$$
\begin{aligned}
[MO(f)(z)]^2 &= \int_{\mathbb{D}} |f \circ \varphi_z(w) - Bf(z)|^2 \, dA(w) \\
&= \int_{\mathbb{D}} |f(w) - Bf(z)|^2 \frac{(1 - |z|^2)^2}{|1 - z\overline{w}|^4} \, dA(w).
\end{aligned}
$$

If the function f is analytic, then it is easy to see that $Bf = f$, so that

$$
\begin{aligned}
[MO(f)(z)]^2 &= \int_{\mathbb{D}} |f \circ \varphi_z(w) - f(z)|^2 \, dA(w) \\
&= \int_{\mathbb{D}} |f(w) - f(z)|^2 \frac{(1 - |z|^2)^2}{|1 - z\overline{w}|^4} \, dA(w).
\end{aligned}
$$

We can think of $MO(f)(z)$ as the invariant mean oscillation of f in the Bergman metric at the point z, because we always have $MO(f \circ \varphi)(z) = MO(f)(\varphi(z))$, where $\varphi \in \mathrm{Aut}(\mathbb{D})$.

We can now reformulate Theorem 3.3 as follows, which is in the same spirit as condition (i) in Theorem 6.1 of [3].

THEOREM 4.1. *If K satisfies condition* (2.1), *then an analytic function f in $\mathbb{D}$ belongs to Q_K if and only if*

$$(4.1) \qquad \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} [MO(f)(z)]^2 \, K(1 - |\varphi_a(z)|^2) \, d\tau(z) < \infty,$$

where $d\tau(z) = dA(z)/(1 - |z|^2)^2$ is the Möbius invariant measure on the unit disk.

PROOF. Consider the integral from Theorem 3.3

$$I_a(f) = \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z) - f(w)|^2}{|1 - z\overline{w}|^4} K(1 - |\varphi_a(z)|^2) \, dA(z) \, dA(w).$$

We rewrite it as an iterated integral

$$I_a(f) = \int_{\mathbb{D}} K(1 - |\varphi_a(z)|^2) \, dA(z) \int_{\mathbb{D}} \frac{|f(z) - f(w)|^2}{|1 - z\overline{w}|^4} \, dA(w),$$

or

$$I_a(f) = \int_{\mathbb{D}} K(1 - |\varphi_a(z)|^2) \, d\tau(z) \int_{\mathbb{D}} |f(w) - f(z)|^2 \frac{(1 - |z|^2)^2}{|1 - z\overline{w}|^4} \, dA(w).$$

According to the calculations preceding this theorem, we have

$$I_a(f) = \int_{\mathbb{D}} [MO(f)(z)]^2 \, K(1 - |\varphi_a(z)|^2) \, d\tau(z).$$

This proves the desired result. □

Fix a positive radius R and denote by

$$A_R(f)(z) = \frac{1}{|D(z, R)|} \int_{D(z,R)} f(w) \, dA(w)$$

the average of f over the Bergman metric ball $D(z, R)$. We define

$$MO_R(f)(z) = \left[\frac{1}{|D(z, R)|} \int_{D(z,R)} |f(w) - A_R(f)(z)|^2 \, dA(w) \right]^{1/2},$$

and call it the mean oscillation of f in the Bergman metric at the point z. It is easy to verifty that $[MO_R(f)(z)]^2 = A_R(|f|^2)(z) - |A_R(f)(z)|^2$. See [10, Section 7.1] for basic facts about the mean oscillation in the Bergman metric.

The Möbius invariant mean oscillation $MO(f)$ and the localized mean oscillation $MO_R(f)$ in the Bergman metric are useful in the study of Hankel operators on the Bergman space; see [10] and [11].

THEOREM 4.2. *If K satisfies condition* (2.1)*, then an analytic function f in $\mathbb{D}$ belongs to Q_K if and only if*

(4.2) $$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} [MO_R(f)(z)]^2 \, K(1 - |\varphi_a(z)|^2) \, d\tau(z) < \infty,$$

where R is any fixed positive radius.

PROOF. There exists a positive constant C (depending on R only) such that

$$MO_R(f)(z) \le MO(f)(z), \quad z \in \mathbb{D},$$

where f is any function in $L^2(\mathbb{D}, dA)$. See the proof of Theorem 7.1.6 in [10]. Therefore, condition (4.1) implies condition (4.2).

On the other hand, since $D(0, R)$ is a Euclidean disk centered at the origin, we can find a positive constant C (depending on R only) such that

$$|f'(0)|^2 \leq C \int_{D(0,R)} |f(w) - c|^2 \, dA(w)$$

for all analytic f in $\mathbb{D}$ and all complex constants c. Replace f by $f \circ \varphi_z$ and replace c by $A_R(f)(z)$. Then

$$(1 - |z|^2)^2 |f'(z)|^2 \leq C \int_{D(0,R)} |f \circ \varphi_z(w) - A_R(f)(z)|^2 \, dA(w).$$

Through a change of variables on the right-hand side, we arrive at

$$(1 - |z|^2)^2 |f'(z)|^2 \leq C \int_{D(z,R)} |f(w) - A_R(f)(z)|^2 \frac{(1 - |z|^2)^2}{|1 - z\overline{w}|^4} \, dA(w).$$

Since

$$\frac{(1 - |z|^2)^2}{|1 - z\overline{w}|^4} \sim \frac{1}{(1 - |z|^2)^2} \sim \frac{1}{|D(z, R)|}$$

for $w \in D(z, R)$, we can find another positive constant C such that

$$(1 - |z|^2)^2 |f'(z)|^2 \leq C \, [MO_R(f)(z)]^2 , \quad z \in \mathbb{D}.$$

It follows that, for each $a \in \mathbb{D}$, the integral

$$\int_{\mathbb{D}} |f'(z)|^2 K(1 - |\varphi_a(z)|^2) \, dA(z)$$

is less than or equal to C times the integral

$$\int_{\mathbb{D}} [MO_R(f)(z)]^2 \, K(1 - |\varphi_a(z)|^2) \, d\tau(z).$$

This shows that condition (4.2) implies $f \in Q_K$. $\qquad\square$

Recall from [3] that a positive Borel measure μ on $\mathbb{D}$ is called a *K-Carleson measure* if

$$\sup_I \int_{S(I)} K\left(\frac{1 - |z|}{|I|}\right) d\mu(z) < \infty,$$

where the supremum is taken over all sub-arcs $I \subset \partial D$. Here, for a sub-arc I of $\partial \mathbb{D}$, $|I|$ is the length of I and $S(I) = \{r\zeta : \zeta \in I, 1 - |I| < r < 1\}$ is the corresponding Carleson 'square'.

THEOREM 4.3. *Suppose K satisfies the following two conditions:*

(a) *There exists a constant $C > 0$ such that $K(2t) \leq CK(t)$ for all $t > 0$; and*
(b) *The auxiliary function φ_K has the property that $\int_0^1 \varphi_K(s)/s \, ds < \infty$.*

Let μ be a positive Borel measure on $\mathbb{D}$. Then μ is a K-Carleson measure if and only if $\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} K(1 - |\varphi_a(z)|^2) \, d\mu(z) < \infty$.

PROOF. See [3, Corollary 3.2]. □

Since Q_K is defined by the condition

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 K(1 - |\varphi_a(z)|^2) \, dA(z) < \infty,$$

we see that $f \in Q_K$ if and only if the measure $|f'(z)|^2 \, dA(z)$ is a K-Carleson measure. This is one of the conclusions of [3, Theorem 3.1]. The following corollary gives two analogous characterizations.

COROLLARY 4.4. *Suppose K satisfies condition (2.1) and conditions (a) and (b) of Theorem 4.3. Let $R > 0$ be a fixed radius. Then the following conditions are equivalent for an analytic function f in $\mathbb{D}$.*

(a) *The function f belongs to Q_K.*
(b) *The measure $d\mu(z) = [MO(f)(z)]^2 \, d\tau(z)$ is a K-Carleson measure.*
(c) *The measure $d\nu(z) = [MO_R(f)(z)]^2 \, d\tau(z)$ is a K-Carleson measure.*

PROOF. This is a direct consequence of Theorems 4.1–4.3. □

5. The spaces $Q_{K,0}$

The space $Q_{K,0}$ consists of analytic functions f in $\mathbb{D}$ with the property that

$$\lim_{|a| \to 1^-} \int_{\mathbb{D}} |f'(z)|^2 K(1 - |\varphi_a(z)|^2) \, dA(z) = 0.$$

It can be checked that $Q_{K,0}$ is a closed subspace in Q_K.

THEOREM 5.1. *If K satisfies condition (2.1) and $R > 0$ is fixed, then the following conditions are equivalent for all analytic functions f in $\mathbb{D}$.*

(a) $f \in Q_{K,0}$.
(b) $\displaystyle \lim_{|a| \to 1^-} \int_{\mathbb{D}} \int_{\mathbb{D}} \frac{|f(z) - f(w)|^2}{|1 - z\overline{w}|^4} K(1 - |\varphi_a(z)|^2) \, dA(z) \, dA(w) = 0.$

(c) $\displaystyle\lim_{|a|\to 1^-}\int_{\mathbb{D}}[MO(f)(z)]^2 K(1-|\varphi_a(z)|^2)\,d\tau(z)=0.$

(d) $\displaystyle\lim_{|a|\to 1^-}\int_{\mathbb{D}}[MO_R(f)(z)]^2 K(1-|\varphi_a(z)|^2)\,d\tau(z)=0.$

PROOF. All our earlier estimates are pointwise estimates with respect to $a\in\mathbb{D}$. So the little oh version of these results must be valid. □

A positive Borel measure μ on $\mathbb{D}$ is called a *vanishing K-Carleson measure* if

$$\lim_{|I|\to 0}\int_{S(I)} K\left(\frac{1-|z|}{|I|}\right)d\mu(z)=0.$$

Carefully checking the proof of [3, Theorem 3.1], we see that the little oh version of Theorem 4.3 holds as well, from which we obtain the following result.

THEOREM 5.2. *Under the same assumptions in Theorem 4.4, the following conditions are equivalent for analytic functions f in $\mathbb{D}$.*

(a) *The function f belongs to $Q_{K,0}$.*
(b) *The measure $d\mu(z)=|f'(z)|^2\,dA(z)$ is a vanishing K-Carleson measure.*
(c) *The measure $d\mu(z)=[MO(f)(z)]^2\,d\tau(z)$ is a vanishing K-Carleson measure.*
(d) *The measure $d\mu(z)=[MO_R(f)(z)]^2\,d\tau(z)$ is a vanishing K-Carleson measure.*

PROOF. We leave the details to the interested reader. □

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