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NUMERICAL RANGE OF THE DERIVATION OF AN INDUCED OPERATOR

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Abstract

Let V be an n-dimensional inner product space over $\mathbb{C}$, let H be a subgroup of the symmetric group on $\{1, \ldots, m\}$, and let $\chi : H \to \mathbb{C}$ be an irreducible character. Denote by $V_\chi^m(H)$ the symmetry class of tensors over V associated with H and χ. Let $K(T) \in \mathrm{End}(V_\chi^m(H))$ be the operator induced by $T \in \mathrm{End}(V)$, and let $D_K(T)$ be the derivation operator of T. The decomposable numerical range $W^*(D_K(T))$ of $D_K(T)$ is a subset of the classical numerical range $W(D_K(T))$ of $D_K(T)$. It is shown that there is a closed star-shaped subset $\mathcal{S}$ of complex numbers such that

$$\mathcal{S} \subseteq W^*(D_K(T)) \subseteq W(D_K(T)) = \mathrm{conv}\,\mathcal{S},$$

where $\mathrm{conv}\,\mathcal{S}$ denotes the convex hull of $\mathcal{S}$. In many cases, the set $\mathcal{S}$ is convex, and thus the set inclusions are actually equalities. Some consequences of the results and related topics are discussed.

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1. Introduction

Let V be an n-dimensional inner product space over $\mathbb{C}$. Let S_m be the symmetric group of degree m on the set $\{1, \ldots, m\}$. Each $\sigma \in S_m$ gives rise to a linear operator $P(\sigma)$ on $\otimes^m V$:

$$P(\sigma)(v_1 \otimes v_2 \otimes \cdots \otimes v_m) := v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(m)}, \quad v_1, \ldots, v_m \in V.$$

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Let H be a subgroup of S_m and let $\chi : H \to \mathbb{C}$ be an irreducible character of H. The *symmetrizer*

$$S_\chi := \frac{\chi(e)}{|H|} \sum_{\sigma \in H} \chi(\sigma) P(\sigma) \in \mathrm{End}(\otimes^m V)$$

is an orthoprojector with respect to the inner product on $\otimes^m V$ induced by the inner product on V via

$$(u_1 \otimes \cdots \otimes u_m, v_1 \otimes \cdots \otimes v_m) := \prod_{i=1}^{m} (u_i, v_i).$$

The range of S_χ

$$V_\chi^m(H) := S_\chi(\otimes^m V)$$

is called the *symmetry class of tensors* over V associated with H and χ. The elements in $V_\chi^m(H)$ of the form $S_\chi(v_1 \otimes \cdots \otimes v_m)$ are called *decomposable symmetrized tensors* and are denoted by $v_1 * \cdots * v_m$.

Let $T \in \mathrm{End}(V)$. There is a unique induced operator $K(T)$ acting on $V_\chi^m(H)$ satisfying

$$K(T) v_1 * \cdots * v_m = T v_1 * \cdots * T v_m.$$

Furthermore, one can define the derivation operator $D_K(T)$ of T by

$$D_K(T) = \frac{\mathrm{d}}{\mathrm{d}t} K(I + tT)\Big|_{t=0},$$

which acts on $V_\chi^m(H)$ in the following way:

$$D_K(T) v_1 * \cdots * v_m = \sum_{j=1}^{m} v_1 * \cdots * v_{j-1} * T v_j * v_{j+1} * \cdots * v_m.$$

Clearly $T \mapsto D_K(T)$ is linear. See [16, 17] for general background on $K(T)$ and $D_K(T)$.

Define the *numerical range* and the *decomposable numerical range* of a linear operator L acting on $V_\chi^m(H)$ by

$$W(L) = \{(Lx, x) : x \in V_\chi^m(H),\ (x, x) = 1\},$$

and

$$W^*(L) = \{(Lx, x) : x \in V_\chi^m(H) \text{ is decomposable},\ (x, x) = 1\},$$

respectively. The numerical range and the decomposable numerical range are useful concepts for studying linear operators, and they also have applications to other areas such as quantum physics (see [2, 3, 13, 14, 15, 19]). Clearly

$$W^*(L) \subseteq W(L).$$

However, since a unit vector in $V_\chi^m(H)$ need not be decomposable [16, 17, 20, 25], one cannot expect that the above set inclusion is equality in general. Nonetheless, in this paper we show that there is a closed star-shaped subset $\mathscr{S}$ of $\mathbb{C}$ such that

$$\mathscr{S} \subseteq W^*(D_K(T)) \subseteq W(D_K(T)) = \operatorname{conv} \mathscr{S}.$$

We give several examples for which $\mathscr{S}$ is actually convex so that all of the above sets are equal. For instance, this is the case when $H = S_m$ (see Theorem 4.2). In particular $\mathscr{S} = \operatorname{conv} \mathscr{S}$ when χ is the alternating character of S_m with $m \leq n$. In this case $V_\chi^m(H)$ is the mth exterior space $\wedge^m V$, which has the special feature that each unit decomposable vector $u_1 * \cdots * u_m$ is equal to some $v_1 * \cdots * v_m$ with $v_1, \ldots, v_m$ orthonormal vectors in V. Consequently, the decomposable numerical range $W^*(D_K(T))$ of $D_K(T)$ is equal to the mth *higher numerical range* of T [19]

$$W_m(T) = \left\{ \sum_{j=1}^{m} (Tv_j, v_j) : \{v_1, \ldots, v_m\} \text{ is an orthonormal set in } V \right\},$$

which is convex by a result of Berger [19].

We present some preliminaries in Section 2, and prove the set inclusion result in Section 3. In Section 4 we discuss the situation where H is the dihedral group, the alternating group, or the full symmetric group (or, more generally, a Young subgroup). Some consequences are deduced in Section 5 and related results are discussed in Section 6.

2. Preliminaries

In this section, we present some preliminary results for induced operators; see [16, 17, 20, 25] for general background.

Let $I(H)$ be the set of irreducible characters of $H \leq S_m$. If $\chi, \xi \in I(H)$ and $\chi \neq \xi$, then $S_\chi S_\xi = 0$. Moreover $\sum_{\chi \in I(H)} S_\chi$ is the identity operator on $\otimes^m V$. So we have the orthogonal sum

$$\otimes^m V = \dot{\sum}_{\chi \in I(H)} V_\chi^m(H).$$

Let $\Gamma_{m,n}$ be the set of sequences $\alpha = (\alpha(1), \ldots, \alpha(m))$ with $1 \leq \alpha(j) \leq n$ for $j = 1, \ldots, m$. Two sequences α and β in $\Gamma_{m,n}$ are said to be equivalent modulo H, denoted by $\alpha \sim \beta$, if there exists $\sigma \in H$ such that $\beta = \alpha\sigma$, where $\alpha\sigma := (\alpha(\sigma(1)), \ldots, \alpha(\sigma(m)))$. This equivalence relation partitions $\Gamma_{m,n}$ into equivalence classes. Let Δ be a system of representatives for the equivalence classes such

that each sequence in Δ is first in its equivalence class relative to the lexicographic order. Define $\bar{\Delta}$ as the subset of Δ consisting of those sequences $\alpha \in \Delta$ such that

$$\sum_{\sigma \in H_\alpha} \chi(\sigma) \neq 0,$$

where $H_\alpha := \{\sigma \in H : \alpha\sigma = \alpha\}$ is the stabilizer subgroup of α.

Let $\mathscr{B} = \{e_1, \dots, e_n\}$ be a basis for V. Then $\{e_\alpha^\otimes := e_{\alpha(1)} \otimes \cdots \otimes e_{\alpha(m)} : \alpha \in \Gamma_{m,n}\}$ is a basis for $\otimes^m V$. Let

$$e_\alpha^* := S_\chi e_\alpha^\otimes = \frac{\chi(e)}{|H|} \sum_{\sigma \in H} \chi(\sigma) e_{\alpha\sigma^{-1}(1)} \otimes \cdots \otimes e_{\alpha\sigma^{-1}(m)},$$

for each $\alpha \in \Gamma_{m,n}$. Then $\{e_\alpha^* : \alpha \in \Gamma_{m,n}\}$ is a spanning set for the space $V_\chi^m(H)$, but it may not be linearly independent. Indeed some of these vectors may even be zero. It is known that $e_\alpha^* \neq 0$ if and only if the restriction of χ to H_α contains the principal character as an irreducible constituent [20, page 163]. Let

$$\Omega := \{\alpha \in \Gamma_{m,n} : (\chi, 1)_{H_\alpha} \neq 0\}.$$

Note that $\bar{\Delta} = \Delta \cap \Omega$ and

$$\Omega = \bigcup_{\alpha \in \bar{\Delta}} \{\alpha\sigma : \sigma \in H\}; \tag{2.1}$$

see [20, Equation (6.15), page 164]. The set $\{e_\alpha^* : \alpha \in \Omega\}$ consists of the nonzero elements of $\{e_\alpha^* : \alpha \in \Gamma_{m,n}\}$. Moreover

$$V_\chi^m(H) = \bigoplus_{\alpha \in \bar{\Delta}} O_\alpha, \tag{2.2}$$

where $O_\alpha := \langle e_{\alpha\sigma}^* : \sigma \in H \rangle$ (called an *orbital subspace*). Freese's theorem [20, page 165] asserts that

$$s_\alpha := \dim O_\alpha = \frac{\chi(e)}{|H_\alpha|} \sum_{\sigma \in H_\alpha} \chi(\sigma) = \chi(e)(\chi, 1)_{H_\alpha}. \tag{2.3}$$

We now construct a basis for $V_\chi^m(H)$. For each $\alpha \in \bar{\Delta}$, we find a basis for the orbital subspace O_α: choose a lexicographically ordered set $\{\alpha_1, \dots, \alpha_{s_\alpha}\}$ from $\{\alpha\sigma : \sigma \in H\}$ such that $\{e_{\alpha_1}^*, \dots, e_{\alpha_{s_\alpha}}^*\}$ is a basis for O_α. Execute this procedure for each $\gamma \in \bar{\Delta}$. If $\{\alpha, \beta, \dots\}$ is the lexicographically ordered set $\bar{\Delta}$, take

$$\hat{\Delta} = \{\alpha_1, \dots, \alpha_{s_\alpha}, \beta_1, \dots, \beta_{s_\beta}, \dots\}$$

to be ordered as indicated. Then $\{e_\alpha^* : \alpha \in \hat{\Delta}\}$ is a basis for $V_\chi^m(H)$ (but the elements of $\hat{\Delta}$ need not be lexicographically ordered). Clearly $\bar{\Delta} \subseteq \hat{\Delta} \subseteq \Omega$. Although $\hat{\Delta}$ is not unique, it does not depend on the basis $\mathscr{B}$ since Δ and $\bar{\Delta}$ do not depend on $\mathscr{B}$. Thus if $\mathscr{B}' = \{f_1, \dots, f_n\}$ is another basis for V, then $\{f_\alpha^* : \alpha \in \hat{\Delta}\}$ is still a basis for $V_\chi^m(H)$.

The inner product of V induces an inner product on $V_\chi^m(H)$:

$$(u^*, v^*) = \frac{\chi(e)}{|H|} \sum_{\sigma \in H} \chi(\sigma) \prod_{t=1}^{m} (u_t, v_{\sigma(t)}). \tag{2.4}$$

Let $\mathscr{B} = \{e_1, \dots, e_n\}$ be an orthonormal basis for V. Then

$$(e_\alpha^*, e_\beta^*) = \begin{cases} 0 & \text{if } \alpha \not\sim \beta, \\ \frac{\chi(e)}{|H|} \sum_{\sigma \in H_\alpha} \chi(\sigma) & \text{if } \alpha = \beta, \end{cases}$$

and thus

$$\|e_\alpha^*\|^2 = \frac{\chi(e)}{|H|} \sum_{\sigma \in H_\alpha} \chi(\sigma).$$

Hence (2.2) becomes $V_\chi^m(H) = \dot{\sum}_{\alpha \in \bar{\Delta}} \langle e_{\alpha\sigma}^* : \sigma \in H \rangle$, an orthogonal sum. However, those e_α^*'s of $\{e_\alpha^* : \alpha \in \hat{\Delta}\}$ belonging to the same orbital subspace need not be orthogonal.

It is known [25, page 103], and also follows from (2.3), that $\bar{\Delta} = \hat{\Delta}$ if and only if χ is linear. In such cases, $\{e_\alpha^* : \alpha \in \bar{\Delta}\}$ is an orthogonal basis for $V_\chi^m(H)$.

We give several common examples of symmetry classes of tensors and induced operators.

EXAMPLE 1. Assume $1 \le m \le n$, $H = S_m$, and χ is the alternating character, that is, $\chi(\sigma) = \operatorname{sgn}(\sigma)$. Then $V_\chi^m(H)$ is the mth exterior space $\wedge^m V$, $\bar{\Delta} = \hat{\Delta} = Q_{m,n}$, the set of strictly increasing sequences in $\Gamma_{m,n}$, $\Delta = G_{m,n}$, the set of nondecreasing sequences in $\Gamma_{m,n}$, and $K(T)$ is the mth compound of $T \in \operatorname{End}(V)$, usually denoted by $C_m(T)$.

EXAMPLE 2. Assume $H = S_m$ and $\chi \equiv 1$, the principal character. Then $V_\chi^m(H)$ is the mth completely symmetric space over $V = \mathbb{C}^n$, $\bar{\Delta} = \hat{\Delta} = \Delta = G_{m,n}$, and $K(T)$ is the mth induced power of $T \in \operatorname{End}(V)$, usually denoted by $P_m(T)$.

EXAMPLE 3 (see [13]). Assume $H = \{e\}$, where e is the identity in S_m (and $\chi \equiv 1$, which is the only irreducible character). Then $V_\chi^m(H) = \otimes^m V$, $\bar{\Delta} = \hat{\Delta} = \Delta = \Gamma_{m,n}$, and $K(T) = \otimes^m T$ is the mth tensor power of $T \in \operatorname{End}(V)$.

We now provide an example with nonlinear irreducible character.

EXAMPLE 4 (see [13]). Assume $H = S_3$ and $\chi = \chi_3$ (notation as in [12, page 157]), the only nonlinear irreducible character. We have $\chi(e) = 2$, $\chi((12)) = 0$, $\chi((123)) = -1$. Assume $n := \dim V = 2$. Then

$$\bar{\Delta} = \{(1,1,2),(1,2,2)\}, \quad \hat{\Delta} = \{(1,1,2),(1,2,1),(1,2,2),(2,1,2)\};$$

see [20, page 164]. Let $\mathscr{B} = \{e_1, e_2\}$ be a basis for V. Then

$$\mathscr{B}^* = \{e^*_{(1,1,2)}, e^*_{(1,2,1)}, e^*_{(1,2,2)}, e^*_{(2,1,2)}\}$$

is a basis for $V^m_\chi(H)$, and (see [25, pages 98–101])

$$e^*_{(2,1,1)} = -e^*_{(1,1,2)} - e^*_{(1,2,1)}, \quad e^*_{(2,2,1)} = -e^*_{(1,2,2)} - e^*_{(2,1,2)}.$$

Let $T \in \mathrm{End}(V)$ be defined by

$$[T]_{\mathscr{B}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

By direct computation,

$$[K(T)]_{\mathscr{B}^*} = \begin{pmatrix} a^2d - abc & 0 & abd - b^2c & 0 \\ 0 & a^2d - abc & abd - b^2c & b^2c - abd \\ acd - bc^2 & 0 & ad^2 - bcd & 0 \\ acd - bc^2 & bc^2 - acd & 0 & ad^2 - bcd \end{pmatrix}$$

and

$$[D_K(T)]_{\mathscr{B}^*} = \begin{pmatrix} 2a + d & 0 & b & 0 \\ 0 & 2a + d & b & -b \\ c & 0 & a + 2d & 0 \\ c & -c & 0 & a + 2d \end{pmatrix}.$$

Observe that $\mathscr{B}^*$ is not an orthogonal basis even if $\mathscr{B}$ is an orthonormal basis, since

$$(e^*_{(1,1,2)}, e^*_{(1,2,1)}) = (e^*_{(1,2,2)}, e^*_{(2,1,2)}) = -1/3.$$

Let $\mathbf{m}_j(\alpha)$ denote the number of occurrences of $j = 1, \ldots, n$ in the sequence $\alpha \in \Gamma_{m,n}$. The vector $\mathbf{m}(\alpha) = (\mathbf{m}_1(\alpha), \ldots, \mathbf{m}_n(\alpha))$ is called the *multiplicity vector* of α.

Recall that $\Omega := \{\alpha \in \Gamma_{m,n} : (\chi, 1)_{H_\alpha} \neq 0\}$. Let $\mathbf{m}(\Omega) := \{\mathbf{m}(\alpha) : \alpha \in \Omega\} \subseteq \mathbb{N}^n$ be the collection of multiplicity vectors of all $\alpha \in \Omega$. A vector $k \in \mathbb{N}^m$ is said to be *admissible* if $k \in \mathbf{m}(\Omega)$. We claim that $\mathbf{m}(\Omega)$ is invariant under the usual action of S_n on $\mathbb{N}^n$ given by $k\tau = (k_{\tau(1)}, \ldots, k_{\tau(n)})$ for $k \in \mathbb{N}^n$, $\tau \in S_n$. Let $\alpha \in \Omega$ and $\tau \in S_n$. With the definition $\tau\alpha = (\tau(\alpha(1)), \ldots, \tau(\alpha(m))$, we clearly have $H_{\tau\alpha} = H_\alpha$ so that $\tau\Omega = \Omega$. One easily checks that $\mathbf{m}(\alpha)\tau = \mathbf{m}(\tau^{-1}\alpha)$. Thus, $\mathbf{m}(\Omega)\tau = \mathbf{m}(\tau^{-1}\Omega) = \mathbf{m}(\Omega)$, as desired.

3. Set inclusion and convexity

Let $T \in \mathrm{End}(V)$ have eigenvalues $\lambda_1(T), \ldots, \lambda_n(T)$ and let $\mathscr{B} = \{v_1, \ldots, v_n\}$ be an orthonormal basis of V. One calls $\mathscr{B}$ a *Schur basis* for T if the matrix representation A of T with respect to $\mathscr{B}$ is upper triangular and $\mathrm{diag}\, A = (\lambda_1(T), \ldots, \lambda_n(T))$. According to the Schur Triangularization Theorem, a Schur basis always exists no matter how the $\lambda_i(T)$ are preordered.

Assume that $\mathscr{B}$ is a Schur basis for T. The matrix of $D_K(T)$ relative to the basis $\{v_\alpha^* : \alpha \in \hat{\Delta}\}$ of $V_\chi^m(H)$ is upper triangular. Moreover, the diagonal entries of this matrix are $\lambda_\alpha = \sum_{j=1}^n m_j(\alpha)\lambda_j(T)$, $\alpha \in \hat{\Delta}$. So, in particular, these are the eigenvalues of $D_K(T)$. We point out that it might not be possible to choose $\hat{\Delta}$ for which $\{v_\alpha^* : \alpha \in \hat{\Delta}\}$ is orthogonal [8, 9, 26] and thus this need not be a Schur basis for $D_K(T)$.

Denote by $\mathrm{Spec}(L)$ the spectrum of a linear operator L. The following is an extension of the well-known result $\mathrm{Spec}\, L \subset W(L)$; see [7].

PROPOSITION 3.1. *Let* $T \in \mathrm{End}(V)$. *Then* $\mathrm{Spec}(D_K(T)) \subset W^*(D_K(T))$.

PROOF. Let $x_1, \ldots, x_n \in V$ and let $\alpha \in \Omega$. We have

$$
\begin{aligned}
(3.1)\qquad (D_K(T)x_\alpha^*, x_\alpha^*) &= (D_K(T)x_{\alpha(1)} * \cdots * x_{\alpha(m)}, x_{\alpha(1)} * \cdots * x_{\alpha(m)}) \\
&= \left(\sum_{i=1}^m x_{\alpha(1)} * \cdots T x_{\alpha(i)} * \cdots x_{\alpha(m)}, x_{\alpha(1)} * \cdots * x_{\alpha(m)} \right) \\
&= \sum_{i=1}^m \frac{\chi(e)}{|H|} \sum_{\sigma \in H} \chi(\sigma)(T x_{\alpha(i)}, x_{\alpha\sigma(i)}) \prod_{j \neq i} (x_{\alpha(j)}, x_{\alpha\sigma(j)}).
\end{aligned}
$$

Now suppose $\{x_1, \ldots, x_n\}$ is an orthonormal basis for V. Let $\sigma \in H$ and suppose that $\prod_{j \neq i}(x_{\alpha(j)}, x_{\alpha\sigma(j)}) \neq 0$. Then $\alpha(j) = \alpha\sigma(j)$ for all $j \neq i$. However since $\mathbf{m}_t(\alpha) = \mathbf{m}_t(\alpha\sigma)$ for $t = 1, \ldots, n$, we also have $\alpha(i) = \alpha\sigma(i)$. So $\sigma \in H_\alpha$. Thus

$$
\begin{aligned}
(3.2)\qquad (D_K(T)x_\alpha^*, x_\alpha^*) &= \sum_{i=1}^m \frac{\chi(e)}{|H|} \sum_{\sigma \in H_\alpha} \chi(\sigma)(T x_{\alpha(i)}, x_{\alpha(i)}) \\
&= \frac{\chi(e)}{|H|} \sum_{\sigma \in H_\alpha} \chi(\sigma) \sum_{i=1}^m (T x_{\alpha(i)}, x_{\alpha(i)}) \\
&= \|x_\alpha^*\|^2 \sum_{t=1}^n m_t(\alpha)(T x_t, x_t).
\end{aligned}
$$

Since $\|x_\alpha^*\|^2 = (\chi(e)/|H|) \sum_{\sigma \in H_\alpha} \chi(\sigma) = (\chi, 1)_{H_\alpha} \neq 0$, the desired result follows if $\{x_1, \ldots, x_n\}$ is a Schur basis. □

Since $\operatorname{Re} W(L) = W(\operatorname{Re}(L))$ where Re denotes the real part of a complex number as well as the Hermitian part of an operator, the right vertical support line of $W(L)$ equals $\{\mu \in \mathbb{C} : \operatorname{Re}\mu = \max \operatorname{Spec}(\operatorname{Re} L)\}$.

Given $c = (c_1, \ldots, c_n) \in \mathbb{C}^n$, the *c-numerical range* of $T \in \operatorname{End}(V)$ is the following subset of $\mathbb{C}$:

$$W_c(T) := \left\{ \sum_{j=1}^{n} c_j (Tx_j, x_j) : \{x_1, \ldots, x_n\} \text{ is an orthonormal basis for } V \right\}.$$

It is known that $W_c(T)$ is convex if $c \in \mathbb{R}^n$, but it can fail to be convex if $c \in \mathbb{C}^n$; see [27] (see [21] for another proof and [23] for a generalization in the context of compact Lie groups).

We say that $c = (c_1, \ldots, c_n) \in \mathbb{R}^n$ is *majorized* by $d = (d_1, \ldots, d_n) \in \mathbb{R}^n$, written $c \prec d$, if for each $1 \le k \le n$ the sum of the k largest entries of c is not larger than that of d, and $\sum_{j=1}^n c_j = \sum_{j=1}^n d_j$. Majorization induces a partial order on the set of orbits (under the entry permutation action of S_n) of those vectors in $\mathbb{R}^n$ having the same entry sum. Two vectors $x, y \in \mathbb{R}^n$ are said to be *comparable* if either $x \prec y$ or $y \prec x$. Clearly not all pairs of vectors are comparable; for example, $x = (4, 1, 1)$ and $y = (3, 3, 0)$ are not comparable.

It is known that if c is majorized by d, then $W_c(T) \subseteq W_d(T)$; see [1, 6].

THEOREM 3.2. *The set*

$$\mathscr{S} := \bigcup_{\alpha\in\Omega} W_{\mathbf{m}(\alpha)}(T) = \bigcup_{\alpha\in\hat{\Delta}} W_{\mathbf{m}(\alpha)}(T) = \bigcup_{\alpha\in\bar{\Delta}} W_{\mathbf{m}(\alpha)}(T)$$

is star-shaped with $(m/n)\operatorname{tr} T$ *as a star-center. Moreover,*

$$\mathscr{S} \subseteq W^*(D_K(T)) \subseteq W(D_K(T)) = \operatorname{conv} \mathscr{S}.$$

All these sets are equal to $W_{\mathbf{m}(\delta)}(T)$ *if there is* $\delta \in \Omega$ *such that* $\mathbf{m}(\delta)$ *majorizes* $\mathbf{m}(\alpha)$ *for all* $\alpha \in \Omega$ *(respectively,* $\alpha \in \bar{\Delta}$*).*

PROOF. Recall (2.1) $\Omega = \bigcup_{\alpha\in\bar{\Delta}} \{\alpha\sigma : \sigma \in H\}$. Clearly $\mathbf{m}(\alpha) = \mathbf{m}(\alpha\sigma)$ and thus $W_{\mathbf{m}(\alpha)}(T) = W_{\mathbf{m}(\alpha\sigma)}(T)$ for any $\alpha \in \Gamma_{m,n}$, $\sigma \in S_m$. So $\mathscr{S} := \bigcup_{\alpha\in\Omega} W_{\mathbf{m}(\alpha)}(T) = \bigcup_{\alpha\in\bar{\Delta}} W_{\mathbf{m}(\alpha)}(T)$. Since $\Omega \supseteq \hat{\Delta} \supseteq \bar{\Delta}$, we have $\mathscr{S} = \bigcup_{\alpha\in\hat{\Delta}} W_{\mathbf{m}(\alpha)}(T)$ as well.

For any $\alpha \in \Omega$, we have $(m/n)\operatorname{tr} T \in W_{\mathbf{m}(\alpha)}(T)$. This is because there is an orthonormal basis $\{u_1, \ldots, u_n\}$ of V such that $(Tu_i, u_i) = (1/n)\operatorname{tr} T$ for all $i = 1, \ldots, n$ (see [5]), and $m = \sum_{t=1}^n \mathbf{m}_t(\alpha)$. Thus the set $\mathscr{S}$ is star-shaped with $(m/n)\operatorname{tr} T$ as a star-center by the convexity of each $W_{\mathbf{m}(\alpha)}(T)$ for $\alpha \in \Omega$ (also see [24]).

Each element of $W_{\mathbf{m}(\alpha)}(T)$ is of the form $\sum_{t=1}^n m_t(\alpha)(Tx_t, x_t)$ for some orthonormal basis $\{x_1, \ldots, x_n\}$. By (3.2),

$$W_{\mathbf{m}(\alpha)}(T) \subseteq W^*(D_K(T)).$$

Hence

$$\mathscr{S} \subseteq W^*(D_K(T)) \subseteq W(D_K(T)).$$

Next, we are going to prove that $W(D_K(T)) = \operatorname{conv}\mathscr{S}$. Since $W(D_K(T))$ is convex [7, page 110] and contains $\mathscr{S}$, we get $W(D_K(T)) \supseteq \operatorname{conv}\mathscr{S}$. Therefore, it remains to prove the other inclusion, and for this it suffices to prove that every extreme point of $W(D_K(T))$ belongs to $W_{\mathbf{m}(\alpha)}(T)$ for some $\alpha \in \hat{\Delta}$.

First, we consider an exposed boundary point $\mu \in W(D_K(T))$. By definition, there exists a support line of $W(D_K(T))$ intersecting $W(D_K(T))$ at the point μ alone. Then there is $r \in [0, 2\pi)$ such that $e^{ir}\mu$ is the only point of $W(e^{ir}D_K(T)) = e^{ir}W(D_K(T))$ on the right vertical support line of $W(e^{ir}D_K(T))$. Since $T \mapsto D_K(T)$ is linear, $D_K(\operatorname{Re}(e^{ir}T)) = \operatorname{Re}(e^{ir}D_K(T))$. By the discussion before the theorem,

$$\begin{aligned}\operatorname{Re} e^{ir}\mu &= \max \operatorname{Spec}(\operatorname{Re}(e^{ir}D_K(T))) = \max \operatorname{Spec}(D_K(\operatorname{Re}(e^{ir}T)))\\ &= \max_{\alpha\in\hat{\Delta}} \sum_{j=1}^{n} m_j(\alpha)\lambda_j(\operatorname{Re}(e^{ir}(T))).\end{aligned}$$

Let $\{v_1, \dots, v_n\}$ be an orthonormal basis of eigenvectors corresponding to the eigenvalues $\lambda_1(\operatorname{Re}(e^{ir}T)) \geq \cdots \geq \lambda_n(\operatorname{Re}(e^{ir}T))$ of the Hermitian operator $\operatorname{Re}(e^{ir}T)$. Then

$$\lambda_j(\operatorname{Re}(e^{ir}T)) = (\operatorname{Re}(e^{ir}T)v_j, v_j) = \operatorname{Re}(e^{ir}Tv_j, v_j), \quad j = 1, \dots, n.$$

So there is $\alpha \in \hat{\Delta}$ such that

$$\operatorname{Re} e^{ir}\mu = \operatorname{Re} \sum_{j=1}^{n} m_j(\alpha)(e^{ir}Tv_j, v_j).$$

Hence, the point

$$\sum_{j=1}^{n} m_j(\alpha)(e^{ir}Tv_j, v_j) \in W_{\mathbf{m}(\alpha)}(e^{ir}T) \subseteq W(D_K(e^{ir}T)) = W(e^{ir}D_K(T))$$

lies on the right vertical support line of $W(e^{ir}D_K(T))$. Since $e^{ir}\mu$ is the only point of $W(e^{ir}D_K(T))$ lying on the right vertical support line, $e^{ir}\mu = \sum_{j=1}^{n} m_j(\alpha)(e^{ir}Tv_j, v_j)$ and hence $\mu = \sum_{j=1}^{n} m_j(\alpha)(Tv_j, v_j) \in W_{\mathbf{m}(\alpha)}(T)$.

Straszewicz's Theorem [22, page 167] asserts that the set of exposed points of any closed convex set C is a dense subset of the set of extreme points of C. The convex hull of two circular disks in $\mathbb{C}$ of the same radius, but of different centers, is a simple example for seeing how an extreme point is the limit of a sequence of exposed points. Now $W_{\mathbf{m}(\alpha)}(T) \subset \mathbb{C}$, $\alpha \in \hat{\Delta}$, are finitely many closed sets. So $\mathscr{S} = \bigcup_{\alpha\in\hat{\Delta}} W_{\mathbf{m}(\alpha)}(T)$ is closed and contains all the exposed points of the closed convex set $W(D_K(T))$. We conclude that every extreme point of $W(D_K(T))$ belongs to $\mathscr{S}$.

Finally, suppose there is $\delta \in \Omega$ such that $\mathbf{m}(\delta)$ majorizes $\mathbf{m}(\alpha)$ for all $\alpha \in \Omega$ (respectively, $\alpha \in \bar{\Delta}$). Then $W_{\mathbf{m}(\delta)}(T) \supseteq W_{\mathbf{m}(\alpha)}(T)$ for each $\alpha \in \Omega$ (respectively, $\alpha \in \bar{\Delta}$), whence $\mathscr{S} = W_{\mathbf{m}(\delta)}(T)$. Since $W_{\mathbf{m}(\delta)}(T)$ is convex, we have conv $\mathscr{S} = \mathscr{S}$ and so the inclusions in the statement of the theorem are all equalities. The last assertion of the theorem follows. □

4. Largest multiplicity vector and convexity

A vector $x \in \mathbf{m}(\Omega)$ is said to be a *largest* vector of $\mathbf{m}(\Omega)$ if $y \prec x$ for all $y \in \mathbf{m}(\Omega)$. A largest vector of $\mathbf{m}(\Omega)$, if it exists, is unique up to permutation of its entries.

In this section, we give examples for which the set equalities

$$\mathscr{S} = W^*(D_K(T)) = W(D_K(T)) = \text{conv}\,\mathscr{S} \tag{4.1}$$

hold. We establish the indicated equalities in each case by showing that either Ω is empty, so that these sets are all empty as well, or $\mathbf{m}(\Omega)$ has a largest vector k, so that these sets all equal $W_k(T)$ by Theorem 3.2. We also give examples to illustrate that some of these equalities can fail to hold (see Theorems 4.5 and 4.6). We point out that in all of the examples in Section 2 there is a largest vector of $\mathbf{m}(\Omega)$, so (4.1) holds in those cases.

Our first example was already observed in [3, Theorem 8.1].

PROPOSITION 4.1. *Suppose χ is the principal character of the subgroup H of S_m. Then $\mathbf{m}(\Omega)$ has a largest vector, namely $(m, 0, \ldots, 0)$, and so (4.1) holds. Moreover the sets in (4.1) all equal $mW(T)$.*

PROOF. As observed in [3, Theorem 8.1], $(m, 0, \ldots, 0) = \mathbf{m}(\delta)$, where $\delta = (1, 1, \ldots, 1) \in \Omega$, so $(m, 0, \ldots, 0)$ is in $\mathbf{m}(\Omega)$, and it is clearly a largest vector of $\mathbf{m}(\Omega)$. Therefore, Theorem 3.2 states that the sets in (4.1) all equal $W_{\mathbf{m}(\delta)}(T) = \{m(Tx, x) : x \in V, (x, x) = 1\}$. □

THEOREM 4.2. *If $H = S_m$, then (4.1) holds.*

PROOF. Assume $H = S_m$ and let π_χ be the partition of m corresponding to χ [20]. If π_χ has greater than n parts, then Ω is empty [20, Corollary 6.38, page 169]. Assume π_χ has at most n parts, so that, in particular, it is a multiplicity vector. It is known that a multiplicity vector $k \in \mathbb{N}^n$ is admissible if and only if k is majorized by π_χ [13], [20, page 169]. It follows that π_χ is a largest vector of $\mathbf{m}(\Omega)$. □

For a partition $\mu = (\mu_1, \ldots, \mu_r)$ of m, the corresponding *Young subgroup* of S_m is the internal direct product $S_\mu = S_{M_1} \times \cdots \times S_{M_r}$, where S_{M_i} is the subgroup of S_m consisting of those permutations that fix every integer *not* contained in the set

$$M_i := \left\{ 1 \le k \le m \,:\, \sum_{j=1}^{i-1} \mu_j < k \le \sum_{j=1}^{i} \mu_j \right\}$$

(an empty sum being interpreted as zero).

THEOREM 4.3. *Let $\mu = (\mu_1, \ldots, \mu_r)$ be a partition of m. If $H = S_\mu$ (Young subgroup), then (4.1) holds.*

PROOF. Assume $H = S_\mu = \prod_{i=1}^r S_{M_i}$ (notation as above). Then $\chi = \prod_{i=1}^r \chi_i$ with χ_i an irreducible character of S_{M_i} $(1 \le i \le r)$.

Denoting by $\Gamma_{M_i,n}$ the set of functions from M_i to $\{1, \ldots, n\}$, we have a bijection $\Gamma_{m,n} \to \prod_{i=1}^r \Gamma_{M_i,n}$ given by $\alpha \mapsto (\alpha_1, \ldots, \alpha_r)$, where α_i denotes the restriction of α to M_i. For $\alpha \in \Gamma_{m,n}$ we have $H_\alpha = \prod_{i=1}^r (S_{M_i})_{\alpha_i}$ so that

$$(\chi, 1)_{H_\alpha} = \prod_{i=1}^{r} (\chi_i, 1)_{(S_{M_i})_{\alpha_i}}.$$

Therefore, $\alpha \in \Omega$ if and only if $\alpha_i \in \Omega_i := \{\beta \in \Gamma_{M_i,n} : (\chi_i, 1)_{(S_{M_i})_\beta} \ne 0\}$ for each i. Clearly $\mathbf{m}(\alpha) = \sum_i \mathbf{m}(\alpha_i)$ $(\alpha \in \Gamma_{m,n})$, so it follows from the observations above that $\mathbf{m}(\Omega) = \sum_i \mathbf{m}(\Omega_i)$. Identifying S_{M_i} with S_{μ_i} in the natural way, we have that χ_i equals χ_{π_i} for some partition π_i of μ_i. If some π_i has greater than μ_i parts, then Ω_i is empty, whence Ω is empty and (4.1) holds. Assume each π_i has at most μ_i parts. Then the proof of Theorem 4.2 shows that π_i is a largest vector of $\mathbf{m}(\Omega_i)$ and thus $\pi := \sum_i \pi_i$ is a largest vector of $\mathbf{m}(\Omega)$. □

PROPOSITION 4.4. *If $m \le 5$, then (4.1) holds.*

PROOF. Assume $m \le 5$. Then majorization is a total order on the set of partitions of m. Since each vector in $\mathbf{m}(\Omega)$ lies in the orbit (under the entry permutation action of S_n) of some partition of m, it follows that $\mathbf{m}(\Omega)$ either is empty or has a largest vector. □

In each of the examples so far, $\mathbf{m}(\Omega)$ has had a largest vector (or it has been empty). We next give an example for which $\mathbf{m}(\Omega)$ is nonempty and has no largest vector.

Assume $H = D_m$ $(m \ge 3)$, the dihedral group, which is generated by

$$r = \begin{pmatrix} 1 & 2 & \cdots & m-1 & m \\ 2 & 3 & \cdots & m & 1 \end{pmatrix} \quad \text{and} \quad s = \begin{pmatrix} 1 & 2 & 3 & \cdots & m-1 & m \\ 1 & m & m-1 & \cdots & 3 & 2 \end{pmatrix}.$$

If m is even, there are 4 irreducible characters of degree 1, given by the following table:

	ψ_1	ψ_2	ψ_3	ψ_4
r^k	1	1	$(-1)^k$	$(-1)^k$
sr^k	1	-1	$(-1)^k$	$(-1)^{k+1}$

If m is odd, then ψ_1 and ψ_2 are the only irreducible characters of degree 1. The other irreducible characters of D_m are of degree 2 and are induced by certain irreducible characters of the cyclic subgroup $C_m = \langle r \rangle$.

THEOREM 4.5. *Assume $H = D_m$ and let the notation be as above.*

(1) *Assume $\chi = \psi_2$, $m > 5$, and $n \geq 3$. Then $\mathbf{m}(\Omega)$ contains $(m-2, 1, 1, 0, \ldots, 0)$ and $(m-3, 3, 0, \ldots, 0)$ and each vector in $\mathbf{m}(\Omega)$ is majorized by one of these vectors. In particular, there is no largest admissible vector. In fact, there exists $T \in End(V)$ for which $\mathscr{S}$ is not convex and hence for which (4.1) fails to hold.*

(2) *If $\chi \neq \psi_2$ or $m \leq 5$ or $n < 3$, then (4.1) holds.*

PROOF. We first prove (1). We claim that $(m-1, 1, 0, \ldots, 0)$ is not admissible. Otherwise there is a $\gamma \in \Omega$ such that $\mathbf{m}(\gamma) = (m-1, 1, 0, \ldots, 0)$. We may assume $\gamma = (2, 1, \ldots, 1)$ since $r \in D_m$. However now

$$(\chi, 1)_{H_\gamma} = [\chi(1) + \chi(s)]/2 = (1-1)/2 = 0,$$

contradicting that $\gamma \in \Omega$.

We next claim that $(m-2, 2, 0, \ldots, 0)$ is not admissible. Otherwise there is a $\beta \in \Omega$ such that $\mathbf{m}(\beta) = (m-2, 2, 0, \ldots, 0)$. We may assume that $\beta = (2, 1, \ldots, 1, 2, 1, \ldots, 1)$, where the second 2 is in the ith position for some $i \neq 1$. Then $H_\beta = \{e, sr^{m-i+1}\}$ and $(\chi, 1)_{H_\beta} = (1-1)/2 = 0$, a contradiction.

Let $\delta = (3, 2, 1, \ldots, 1)$. Then $H_\delta = \{e\}$ and thus $(\chi, 1)_{H_\delta} = 1$. So $(m-2, 1, 1, 0, \ldots, 0) = \mathbf{m}(\delta)$ is admissible.

Let $\xi = (2, 2, 1, 2, 1, \ldots, 1)$. Any element of H_ξ must fix 3, since this is the only position i for which $\xi_i = 1$ and $\xi_{i-1} = 2 = \xi_{i+1}$ (subscripts modulo m), so that $H_\xi \subseteq \{e, sr^{m-4}\}$. However $sr^{m-4}(1) = 5$, so $H_\xi = \{e\}$. Thus $(\chi, 1)_{H_\xi} = 1$ and $(m-3, 3, 0, \ldots, 0) = \mathbf{m}(\xi)$ is admissible.

Using these observations, we see that all possible admissible vectors are majorized by either $\mathbf{m}(\delta)$ or $\mathbf{m}(\xi)$ and, since these two vectors are admissible and not comparable, we conclude that there is no largest admissible vector. Moreover, since $W_c(T) \subseteq W_d(T)$ whenever $c \prec d$, we find that

$$\mathscr{S} := \bigcup_{\alpha \in \Omega} W_{\mathbf{m}(\alpha)}(T) = W_{\mathbf{m}(\delta)}(T) \cup W_{\mathbf{m}(\xi)}(T).$$

Suppose $T = \operatorname{diag}(2+i, 1) \oplus 0_{n-2}$. It is a normal operator acting on $V = \mathbb{C}^n$. Let $t := (t_1, \dots, t_n) = (2+i, 1, 0, \dots, 0)$. Then (for example, see [18])

$$W_c(T) = \operatorname{conv}\{c_\sigma^T t : \sigma \in S_n\},$$

where $c_\sigma = (c_{\sigma(1)}, \dots, c_{\sigma(n)})$ for real $c \in \mathbb{R}^n$. So

$$W_{\mathbf{m}(\delta)}(T) = \operatorname{conv}\left\{ (m-2)t_u + t_v + t_w \;\middle|\; \begin{array}{l}\{u, v, w\} \text{ is a three-element} \\ \text{subset of } \{1, \dots, n\}\end{array} \right\},$$

and

$$W_{\mathbf{m}(\xi)}(T) = \operatorname{conv}\left\{ (m-3)t_u + 3t_v \;\middle|\; \begin{array}{l}\{u, v\} \text{ is a two-element} \\ \text{subset of } \{1, \dots, n\}\end{array} \right\}.$$

In particular, $z_1 = (2m-3) + (m-2)i$ is a vertex of $W_{\mathbf{m}(\delta)}(T)$ and $z_2 = (2m-3) + (m-3)i$ is a vertex of $W_{\mathbf{m}(\xi)}(T)$. However z_1 is the only vertex of $W_{\mathbf{m}(\delta)}(T)$ with real part $2m-3$ and clearly $2m-3$ is the largest possible among the real parts of vertices of $W_{\mathbf{m}(\delta)}(T)$. A similar statement holds for z_2 and $W_{\mathbf{m}(\xi)}(T)$. Therefore $(z_1 + z_2)/2 = (2m-3) + (m-5/2)i$ is a point on the segment joining z_1 and z_2 that does not belong to $W_{\mathbf{m}(\delta)}(T) \cup W_{\mathbf{m}(\xi)}(T)$. We conclude that $\mathscr{S}$ is not convex.

Now we prove (2). If $m \le 5$, then the claim follows from Proposition 4.4. Assume $\chi \ne \psi_2$ and $m > 5$. We consider cases:

(a) (χ is the principal character). This case is handled by Proposition 4.1.

In the remaining cases, Ω is empty if $n = 1$, in which case (4.1) holds. So we assume $n > 1$.

(b) (χ is of degree 2). Since χ is induced from a character of the normal subgroup C_m of H, it vanishes on the complement of C_m. Let $\gamma = (2, 1, \dots, 1)$. Then clearly $H_\gamma = \{e, s\}$ and hence $(\chi, 1)_{H_\gamma} = 1$. Thus $(m-1, 1, 0, \dots, 0) = \mathbf{m}(\gamma)$ is admissible and it is clearly a largest such vector. Our claim now follows from Theorem 3.2 as usual.

(c) ($\chi = \psi_3$). Let $\gamma = (2, 1, \dots, 1)$ as before. Then $(\chi, 1)_{H_\gamma} = [\chi(1) + \chi(s)]/2 = (1+1)/2 = 1$. Thus, $(m-1, 1, 0, \dots, 0) = \mathbf{m}(\gamma)$ is admissible and it is a largest such vector.

(d) ($\chi = \psi_4$). First, $(m-1, 1, 0, \dots, 0)$ is not admissible by an argument similar to that in the proof of (1). Let $\beta = (2, 2, 1, \dots, 1)$. Then $H_\beta = \{e, sr^{m-1}\}$ and $(\chi, 1)_{H_\beta} = (1 + (-1)^m)/2 = 1$, using that m must be even in this case. Thus, $(m-2, 2, 0, \dots, 0) = \mathbf{m}(\beta)$ is a largest admissible vector.

Finally, assume $n < 3$, $\chi = \psi_2$, and $m > 5$. Since $n < 3$, $(m-2, 1, 1, 0, \dots, 0)$ cannot be admissible. Therefore, in view of the proof of (1), we have that $(m-3, 3, 0, \dots, 0) = \mathbf{m}(\xi)$ is a largest admissible vector, where $\xi = (2, 2, 1, 2, 1, \dots, 1)$. This completes the proof. □

For the T in the proof of part (1) of the theorem, we know only that one of the first two equalities of (4.1) fails to hold, in view of Theorem 3.2. That is, either $\mathscr{S} \neq W^*(D_K(T))$ or $W^*(D_K(T)) \neq W(D_K(T))$. It would be interesting to determine whether the line segment joining z_1 and z_2 in the above proof belongs to $W^*(D_K(T))$, since an answer in the negative would imply that $W^*(D_K(T)) \neq W(D_K(T))$.

Assume $H = A_m$ ($m \geq 2$), the alternating group. Suppose the irreducible character χ of H is invariant under the conjugation action of S_m, that is, ${}^\sigma\chi = \chi$ for all $\sigma \in S_m$, where ${}^\sigma\chi(\tau) = \chi(\tau^\sigma) = \chi(\sigma^{-1}\tau\sigma)$ ($\tau \in H$). Then $\chi^{S_m} = \chi_\pi + \chi_{\pi'}$ for some partition π of m (written $\pi \vdash m$) with $\pi' \neq \pi$, where π' denotes the conjugate partition of π (see [10, (6.20), (6.17)] and [20, Theorem 4.47]).

Now suppose χ is not invariant. Then $\chi^{S_m} = \chi_\pi$ for some $\pi \vdash m$ with $\pi' = \pi$. Moreover, $({}^\sigma\chi)^{S_m} = \chi_\pi$ for every $\sigma \in S_m$ (see [10, (6.11)] and [20, Corollary 4.48]).

In summary, there exists $\pi \vdash m$ such that for every $\sigma \in S_m$,

$$({}^\sigma\chi)^{S_m} = 2^{-\iota}(\chi_\pi + \chi_{\pi'}),$$

where

$$\iota = \begin{cases} 0, & \pi' \neq \pi, \\ 1, & \pi' = \pi. \end{cases}$$

Note that ι is 0 or 1 according to whether χ is or is not invariant. By Frobenius Reciprocity, π may be taken to be any partition of m for which χ is a constituent of the restriction of χ_π to H.

THEOREM 4.6. *Assume H is the alternating group A_m with $n := \dim V \geq m \geq 2$, and let π and π' be described as above.*

(1) *Then $\mathscr{S} = W_\pi(T) \cup W_{\pi'}(T)$.*

(2) *If π and π' are comparable, then (4.1) holds. In particular, if χ is not invariant, then (4.1) holds.*

PROOF. We begin by establishing a formula. Let $x \vdash m$ and assume $x \neq [1^m]$. Then S_x (Young subgroup) contains an odd permutation, whence $HS_x = S_m$. Therefore, Mackey's Subgroup Theorem [10, page 74] states $((1_{S_x})^{S_m})_H = (1_{S_x \cap H})^H$. Using this observation together with Frobenius Reciprocity and linearity of the inner product, we obtain for each $\sigma \in S_m$

$$\begin{aligned} 2^{-\iota}\left[\left(\chi_\pi, (1_{S_x})^{S_m}\right)_{S_m} + \left(\chi_{\pi'}, (1_{S_x})^{S_m}\right)_{S_m}\right] &= \left(({}^\sigma\chi)^{S_m}, (1_{S_x})^{S_m}\right)_{S_m} = \left({}^\sigma\chi, (1_{S_x})^{S_m}\right)_H \\ &= \left({}^\sigma\chi, (1_{S_x \cap H})^H\right)_H = ({}^\sigma\chi, 1)_{S_x \cap H}. \end{aligned}$$

Next, we claim that $\mathbf{m}(\Omega) = \{x \in \mathbb{N}^n : x \prec \pi \text{ or } x \prec \pi'\}$. Put

$$M = \{x\tau : x \vdash m, \tau \in S_n, \text{ and either } x \prec \pi \text{ or } x \prec \pi'\}.$$

Since $n \geq m$ and $x \vdash m$ has at most m parts, one can view x as an element of $\mathbb{N}^n$ so that the expression $x\tau = (x_{\tau(1)}, \ldots, x_{\tau(n)})$ $(\tau \in S_n)$ is defined. According to our definition of majorization, we have $M = \{x \in \mathbb{N}^n : x \prec \pi \text{ or } x \prec \pi'\}$. So it suffices to show that $\mathbf{m}(\Omega) = M$.

Let $x_1 \in \mathbf{m}(\Omega)$. Then $x_1 = \mathbf{m}(\alpha_1)$ for some $\alpha_1 \in \Omega$. Now $x := x_1\tau^{-1} \vdash m$ for some $\tau \in S_n$. We have $x = \mathbf{m}(\alpha_1)\tau^{-1} = \mathbf{m}(\tau\alpha_1) = \mathbf{m}(\alpha)$, where $\alpha = \tau\alpha_1 \in \Omega$. Since the stabilizer in S_m of α is conjugate to S_x, it follows that $H_\alpha = (S_x)^\sigma \cap H = (S_x \cap H)^\sigma$ for some $\sigma \in S_m$. Then

$$({}^\sigma\chi, 1)_{S_x \cap H} = (\chi, 1)_{(S_x \cap H)^\sigma} = (\chi, 1)_{H_\alpha} \neq 0.$$

If $x \neq [1^m]$, then the computation above implies that either $(\chi_\pi, (1_{S_x})^{S_m})_{S_m}$ or $(\chi_{\pi'}, (1_{S_x})^{S_m})_{S_m}$ is nonzero (because each of them is a nonnegative integer) so that $x \prec \pi$ or $x \prec \pi'$; see [11]. Since $[1^m] \prec \pi$, it follows that in all cases $x \prec \pi$ or $x \prec \pi'$. Thus $x_1 \in M$.

Now let $x_1 \in M$. Then $x_1 = x\tau$ for some $x \vdash m$ and $\tau \in S_n$ with $x \prec \pi$ or $x \prec \pi'$. Set $\alpha = (1^{x_1}, 2^{x_2}, \ldots, t^{x_t})$, where t is the length of x. Note that α is in $\Gamma_{m,n}$ by the assumption $n \geq m$ and since $t \leq m$. Then $H_\alpha = S_x \cap H$. Assume $x \neq [1^m]$. Since $x \prec \pi$ or $x \prec \pi'$, we see (by [11]) that the first member in the computation above (with $\sigma = 1$) is nonzero. Thus $(\chi, 1)_{H_\alpha} \neq 0$, implying $\alpha \in \Omega$. If $x = [1^m]$, then $\alpha = (1, 2, \ldots, m)$, so $H_\alpha = \{e\}$, whence $\alpha \in \Omega$. Thus, in all cases $x = \mathbf{m}(\alpha) \in \mathbf{m}(\Omega)$. Finally, $x_1 = \mathbf{m}(\alpha)\tau = \mathbf{m}(\tau^{-1}\alpha) \in \mathbf{m}(\Omega)$ as desired.

Now we prove (1). Since $W_c(T) \subseteq W_d(T)$ whenever $c \prec d$, we have

$$\mathscr{S} := \bigcup_{\alpha \in \Omega} W_{\mathbf{m}(\alpha)}(T) = W_\pi(T) \cup W_{\pi'}(T).$$

Finally, we prove (2). If $\pi \succ \pi'$, then π is a largest vector in $\mathbf{m}(\Omega)$ and (4.1) follows from Theorem 3.2. The same holds with π and π' exchanged. Finally, if χ is not invariant, then $\pi' = \pi$, so π and π' are comparable and the proof is complete. □

If $\pi = [5, 2^3] \vdash 11$, then $\pi' = [4^2, 1^3]$ so that π and π' are not comparable and there is no largest admissible vector (see proof of theorem). It is easy to check that 11 is the least m for which there exists a partition of m not comparable to its conjugate.

5. Some consequences

Using Theorem 3.2, we extend some results on the classical numerical range to the decomposable numerical range. The irreducible character χ is said to be of *determinant type* [13] if $K(S) = (\det S)^{m/n} I$ for all $S \in \mathrm{End}(V)$, or equivalently, each $\alpha \in \bar{\Delta}$ satisfies $\mathbf{m}_1(\alpha) = \cdots = \mathbf{m}_n(\alpha)$ (in which case each $\mathbf{m}_j(\alpha) = m/n$). In

this case, $D_K(T) = (m/n)(\operatorname{tr} T)I$, and $W^*(D_K(T)) = W(D_K(T)) = \{(m/n)(\operatorname{tr} T)\}$, which does not convey much information about T. Otherwise, we have the following.

THEOREM 5.1. *Suppose χ is not of determinant type. Then $D_K(T)$ is scalar (respectively, Hermitian) if and only if T is scalar (respectively, Hermitian). Consequently,*

(a) *$W^*(D_K(T))$ or $W(D_K(T))$ equals $\{\mu\}$ if and only if $T = (\mu/m)I$;*
(b) *$W^*(D_K(T))$ or $W(D_K(T))$ is a subset of $\mathbb{R}$ if and only if $T = T^*$.*

PROOF. If T is scalar or Hermitian, then clearly $D_K(T)$ has the corresponding property. Suppose $D_K(T) = \mu I$. We claim that both $\operatorname{Re} T$ and $\operatorname{Im} T$ are scalar operators. If $\operatorname{Re} T$ were not a scalar operator, then the spectrum of $\operatorname{Re} D_K(T) = D_K(\operatorname{Re} T)$ would not be a singleton set. It is because it contains $\sum_{j=1}^n m_j(\alpha)\lambda_j(\operatorname{Re} T)$, $\alpha \in \hat{\Delta}$. Since χ is not of determinant type, we may assume that $\mathbf{m}_1(\alpha) > \mathbf{m}_2(\alpha)$ for some $\alpha \in \hat{\Delta}$ and $\lambda_1(\operatorname{Re} T) > \lambda_2(\operatorname{Re} T)$. Thus

$$\begin{aligned}&\mathbf{m}_1(\alpha)\lambda_1(\operatorname{Re} T) + \mathbf{m}_2(\alpha)\lambda_2(\operatorname{Re} T) + \sum_{j=3}^n m_j(\alpha)\lambda_j(\operatorname{Re} T)\\ &\quad > \mathbf{m}_2(\alpha)\lambda_1(\operatorname{Re} T) + \mathbf{m}_1(\alpha)\lambda_2(\operatorname{Re} T) + \sum_{j=3}^n m_j(\alpha)\lambda_j(\operatorname{Re} T),\end{aligned}$$

since $(\mathbf{m}_1(\alpha) - \mathbf{m}_2(\alpha))(\lambda_1(\operatorname{Re} T) - \lambda_2(\operatorname{Re} T)) > 0$. Notice that $(\mathbf{m}(\alpha))(12)$ is still a multiplicity vector, where (12) denotes the transposition in S_n. So $D_K(T)$ would have more than one eigenvalue. Thus $D_K(T)$ would be non-scalar, which is a contradiction. Applying the same arguments, we see that $\operatorname{Im} T$ is a scalar, and the result follows.

Suppose $D_K(T)$ is Hermitian. Then $\operatorname{Im} D_K(T)$ is the zero operator. By the previous argument, we see that $\operatorname{Im} T$ is a scalar and thus the zero operator.

Now, (a) and (b) follow from the fact that $\operatorname{Spec} \operatorname{Re} e^{ir} D_K(T) \subseteq W^*(\operatorname{Re} e^{ir} D_K(T))$ for any $r \in [0, 2\pi)$. □

THEOREM 5.2. *Suppose χ is not of determinant type, and $\alpha, \beta \in \mathbb{C}$. The following conditions are equivalent.*

(a) *$W^*(D_K(T)) = W(D_K(T)) \subseteq \{\alpha t + \beta : t \in \mathbb{R}\}$.*
(b) *$D_K(T) = \alpha\tilde{H} + \beta I$ for some Hermitian operator $\tilde{H}$.*
(c) *$T = \alpha H + (\beta/m)I$ for some Hermitian operator H.*

PROOF. The equivalence of (a) and (b) is well known. That (c) implies (b) is clear. Suppose $D_K(T) = \alpha\tilde{H} + \beta I$ for some Hermitian operator $\tilde{H}$. Consider two cases. (1) $\alpha \neq 0$. $D_K[(T - (\beta/m)I)/\alpha] = (D_K(T) - \beta I)/\alpha = \tilde{H}$ is Hermitian, since $D_K(I) = mI$. Thus, $[T - (\beta/m)I]/\alpha$ is Hermitian by Theorem 5.1 (b). Condition (c)

follows. (2) $\alpha = 0$. Then $D_K(T) = \beta I$ and $W(D_K(T)) = \{\beta\}$. By Theorem 5.1 (a), $T = \beta I/m$. □

6. Related topics

Suppose $m \le n$. Researchers also consider

$$W_\perp^*(L) = \left\{ (Lv^*, v^*) \;\middle|\; \begin{array}{l} v^* = v_1 * \cdots * v_m, \{v_1, \ldots, v_m\} \text{ is} \\ \text{an orthonormal set in } V \end{array} \right\}.$$

Using (3.2) with $\alpha = (1, 2, \ldots, m)$, so that $\mathbf{m}_t(\alpha) = 1$ for all $t = 1, \ldots, m$, we have the following.

THEOREM 6.1. *Suppose $m \le n$. Then*

$$W_\perp^*(D_K(T)) = \left\{ \frac{\chi(e)^2}{|H|} \sum_{j=1}^{m} (Tv_j, v_j) : \{v_1, \ldots, v_m\} \text{ is an orthonormal set in } V \right\}.$$

Thus, $W_\perp^*(D_K(T))$ is, up to a scalar, the mth numerical range of T as defined by Halmos [7].

Another generalization is

$$\widetilde{W}^*(L) := \{(L(u_1 * \cdots * u_m), u_1 * \cdots * u_m) : u_1, \ldots, u_m \text{ are unit vectors in } V\}.$$

Clearly $W_\perp^*(L) \subseteq \widetilde{W}^*(L)$.

THEOREM 6.2. *Suppose $1 < m < n$, $H = S_m$, and χ is the alternating character. Then $\widetilde{W}^*(D_K(T))$ equals the convex hull of $(1/n!)W_m(T) \cup \{0\}$.*

PROOF. Assume $1 < m < n$ and put $L = D_K(T)$. Let $\mu \in (1/n!)W_m(T)$. By Theorem 6.1, $\mu \in W_\perp^*(L)$, so $\mu = (Lx^*, x^*)$ where $x^* = x_1 * \cdots * x_m$ with $x_1, \ldots, x_m$ orthonormal. Let $0 \le t \le 1$. Set $u_1 = \sqrt{t}\, x_1 + \sqrt{1-t}\, x_2$ and $u_j = x_j$, $j = 2, \ldots, m$. Then $u^* = \sqrt{t}x^*$ and thus $t\mu = (Lu^*, u^*) \in \widetilde{W}^*(L)$. Since $W_m(T)$ is convex [7], the convex hull of $(1/n!)W_m(T) \cup \{0\}$ is contained in $\widetilde{W}^*(L)$.

Conversely, let $\mu \in \widetilde{W}^*(L)$. Then $\mu = (Lx^*, x^*)$ with $x_1, \ldots, x_m$ unit vectors. If $x_1, \ldots, x_m$ are linearly dependent, then $x^* = 0$ and $\mu = 0$. Assume $x_1, \ldots, x_m$, are linearly independent and let $\{u_1, \ldots, u_m\}$ be an orthonormal basis of the linear span of $x_1, \ldots, x_m$. Then, for $i = 1, \ldots, m$, we can write $x_i = \sum_{j=1}^m a_{ij}u_j$ with $a_{ij} \in \mathbb{C}$, and $x^* = (\det A)\, u^*$, where $A = (a_{ij})$. So $\mu = |\det A|^2 (Lu^*, u^*)$. However $|\det A|^2 = \det(AA^*)$, which is less than or equal to 1 by the Hadamard inequality [4], so μ is contained in the convex hull of $W_\perp^*(L) \cup \{0\}$. Since $W_\perp^*(L) = (1/n!)W_m(T)$ by Theorem 6.1, we have the desired result. □

REMARK. As pointed out by the referee, it would be interesting to find a non-convex example of $W^*(D_K(T))$.

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