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MONOTHETIC ALGEBRAIC GROUPS

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Abstract

We call an algebraic group monothetic if it possesses a dense cyclic subgroup. For an arbitrary field k we describe the structure of all, not necessarily affine, monothetic k-groups G and determine in which cases G has a k-rational generator.

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The abstract theory of groups, especially in the infinite case, deals preponderantly with the study of classes of groups which are given by distinguished group theoretical properties. For algebraic groups so far this approach has not been applied systematically. If one wants to follows the pattern given by the abstract theory of groups in the case of algebraic groups, it is indispensable to have detailed knowledge of those algebraic groups that correspond to cyclic groups.

Groups that have a dense cyclic subgroup play a fundamental role in the theory of topological groups. For the class of locally compact groups these groups are called *monothetic* and were introduced by van Dantzig in [14]. The full classification of their structure can be found in [4, Section 25].

In the theory of algebraic groups over a field k we study the analogous class of groups, which we also call *monothetic*. So far, groups having a dense cyclic subgroup with respect to the Zariski topology have not found particular attention. In [10, page 146] the affine monothetic groups over an algebraically closed field of characteristic 0 are described in connection with the Galois groups of differential equations.

In order to treat algebraic groups with group theoretical methods it is necessary to know the monothetic groups in detail. We classify them for k-groups and find

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k-rational generators whenever they exist. Our results may be summarized by the following theorem.

MAIN THEOREM. (I) *A k-group G is monothetic if and only if it is the direct product of a connected monothetic k-group G° and a finite cyclic group.*

(II) *A connected algebraic k-group $G \neq 1$ is monothetic if and only if k is not locally finite and $G = HD$, where H is a monothetic connected affine k-group, D is a connected monothetic algebraic k-group having no non-trivial affine image, and $H \cap D$ is finite.*

(III) *A connected algebraic k-group $G \neq 1$ having no non-trivial affine epimorphic image is monothetic if and only if k is not locally finite.*

(IV) *A connected affine algebraic k-group $H \neq 1$ is monothetic if and only if k is not locally finite and H is the direct product of a torus T and an ϵ-dimensional connected unipotent group V, where $\epsilon = 0$ if $\operatorname{char}(\mathsf{k}) > 0$, whereas $\epsilon \leq 1$ if $\operatorname{char}(\mathsf{k}) = 0$. Both groups T and V are defined over k.*

(V) *A monothetic algebraic k-group G has a k-rational generator if and only if the minimal subgroup D of G with an affine factor group has a k-rational generator.*

In Theorem 16 we show that all closed connected subgroups of a monothetic algebraic group G are monothetic if and only if the maximal connected unipotent subgroup U of G is monothetic. This condition is clearly satisfied for affine groups and for abelian varieties. The same holds over fields of positive characteristic (Corollary 15). However, in characteristic 0, there are counter examples, which are connected monothetic non-affine groups having a vector subgroup of dimension greater than 1 (Remark 12).

Our paper shows that in contrast to the monothetic Lie groups, which are precisely the direct products of a torus and a cyclic discrete group, the structure of algebraic monothetic groups is more subtle.

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A. Prerequisites In this note we consider algebraic k-groups. A general reference for these groups is Borel's book [1]. If not specified, we denote by G the set of elements which are rational over the algebraic closure k^a of k. If we consider the rational points of G over a field F containing the minimal field of definition for G (see [13] and [16, Chapter IV, Corollary 3, page 71]), we denote this set by $G(\mathsf{F})$.

We call an algebraic k-group G *monothetic* if it contains an element x such that $G = \overline{\langle x \rangle}$. In this case G is a commutative group [1, Section 2.1(e)]. We call x a *generator* of G.

Clearly any epimorphic image of a monothetic group is monothetic as well. Conversely we have the following result.

PROPOSITION 1. *Let G be a connected algebraic* k*-group, and let E be a finite subgroup of G such that G/E is monothetic. Then G is monothetic also.*

PROOF. Let $G/E = \overline{\langle uE \rangle}$ and $U = \overline{\langle u \rangle}$. Then $G = UE$. It follows $G = U^\circ$, the connected component of the identity in U. Thus G coincides with U. $\square$

Proposition 1 shows that together with a connected algebraic group G all groups isogenous to G are monothetic.

Let x be a generator of G. As the connected component G° of an algebraic group G has finite index in G, a power x^m of x lies in G°. Put $H = \overline{\langle x^m \rangle}$; then $H \leq G^\circ$. Since the finite union $H \cup Hx \cup \cdots \cup Hx^{m-1}$ is a closed subgroup containing x, it is all of G, and we have $H = G^\circ$. If G is a k-group then $H = G^\circ$ is a k-group, too. Therefore, we arrive at the following result.

PROPOSITION 2. *Let $G = \overline{\langle x \rangle}$ be a monothetic algebraic* k*-group. Then its connected component G° is a monothetic* k*-group also. If G is connected, then for any integer m we have $G = \overline{\langle x^m \rangle}$.*

By this proposition we are justified to restrict our attention mostly to connected monothetic k-groups. We shall discuss the non-connected case in more detail later.

The connected subgroups of a connected monothetic Lie group are always monothetic. In the case of connected algebraic k-groups this is true for affine groups (Corollary 6), as well as for abelian varieties (Lemma 9), but not in general (Remark 12). Of course, subgroups that are not connected do not have to be monothetic.

The set $G(\mathsf{k})$ of k-rational elements of an algebraic k-group G is always a subgroup of $G = G(\mathsf{k}^a)$ ([15, Sections I.7 and II.8–10]). We are interested in the question of whether it is possible to find a k-rational generator of a given monothetic k-group G. It follows from the next proposition that this question is meaningful only if the field k is not locally finite.

PROPOSITION 3. *Let G be a* k*-group.*

(a) *If* k *is a locally finite field, then $G(\mathsf{k})$ is a torsion group.*

(b) *If G has positive dimension and is monothetic, then the field* k *is not locally finite and there is a finite extension* F *of* k *such that $G(\mathsf{k}^a)$ contains an* F*-rational generator.*

PROOF. Observe that the minimal field k_0 of definition for G is finitely generated over the prime field of k, since the representatives of G are described by finitely many polynomials (see [16, page 179]). Consider $a \in G(\mathsf{k}^a)$. Adjoining to k_0 the set of coordinates of a with respect to a finite system of charts for G, one obtains a finite extension $\mathsf{k}_0(a)$ of k_0, over which a is rational and G is defined (compare [16, Chaper VII.3]).

(a) If k is locally finite, then for $g \in G(\mathsf{k})$ the field $\mathsf{k}_0(g)$ is finite. Thus g lies in the finite group $G(\mathsf{k}_0(g))$.

(b) Obviously any generator g of a k-group G of positive dimension has infinite order. Because of (a), the field k is not locally finite. Since k_0 is a subfield of k, the field $\mathsf{F} = \mathsf{k}(g)$ has finite degree over k. □

B. Affine groups For affine groups the structure theorems for algebraic k-groups, found in [1, Chapter III] enable one to determine the structure of monothetic groups and to consider rationality questions.

PROPOSITION 4. *A connected unipotent* k*-group* $G \neq 1$ *is monothetic if and only if* $\mathrm{char}(\mathsf{k}) = 0$ *and* $\dim(G) = 1$. *In this case every element of* G *that is not equal to* 1 *is a generator of* G, *and* G *has* k*-rational generators.*

PROOF. If $\mathrm{char}(\mathsf{k}) > 0$, then G is a torsion group. If $\mathrm{char}(\mathsf{k}) = 0$, then every element of G that is not equal to 1 has infinite order and generates a 1-dimensional subgroup of G. □

THEOREM 5. *A connected affine algebraic* k*-group* $G \neq 1$ *is monothetic if and only if the following conditions hold*:

(a) *the field* k *is not locally finite*;

(b) G *is commutative*; *and*

(c) *the unipotent radical* G_u *is an* ϵ*-dimensional vector group, where* $\epsilon = 0$ *if* $\mathrm{char}(\mathsf{k}) > 0$ *and* $\epsilon \leq 1$ *if* $\mathrm{char}(\mathsf{k}) = 0$.

In this case, the unipotent radical G_u *and the maximal torus* G_s *of* G *are defined over* k *and a generator of* G *can be chosen to be* k*-rational.*

PROOF. Let G be a monothetic k-group. Since G is commutative, it is the direct product of G_u and G_s, where G_u is the set of unipotent elements of G, which is a k-closed subgroup, and G_s is the set of its semisimple elements, which is even defined over k (see [1, Theorems 4.7 and 10.6 (3)]). For a generator x of G one has $x = x_u x_s$ with $x_u \in G_u$ and $x_s \in G_s$. Hence $G = \overline{\langle x_u \rangle} \times \overline{\langle x_s \rangle}$, $G_s = \overline{\langle x_s \rangle}$ and $G_u = \overline{\langle x_u \rangle}$.

It follows from Proposition 4 that (c) holds. Since fields of characteristic 0 are perfect, the unipotent radical of a monothetic affine k-group is defined over k (see [5, Lemma 34.1, page 217]) and we can choose a generator of G_u that is rational over k (see Proposition 4).

The subgroup G_s of the monothetic k-group G is a k-torus. From [1, Proposition 8.8 and Remark] it follows that a k-torus is monothetic if and only if k is not locally finite. In this case, a generator can be chosen to be k-rational.

Conversely, let G be an affine k-group satisfying (a)–(c). Then $G = G_u \times G_s$ and G_u, G_s are k-groups with k-rational generators x_u, x_s. Consider the monothetic group

$X = \overline{\langle x_u x_s \rangle}$. Then the restrictions to X of the projections from G to G_u and from G to G_s, respectively are surjective. Since there are no non-trivial homomorphisms between G_u and G_s it follows that $G = X$ (compare [9, page 40]). □

COROLLARY 6. *Every closed connected subgroup of an affine connected monothetic algebraic* k*-group is monothetic.*

C. Abelian varieties We now turn our attention to monothetic abelian varietes. A simple abelian variety is monothetic if and only if it is not a torsion group. The next statement, which is a variant of Theorem 12 in [2], is needed for the proof of our Theorem 9, where we show that also for abelian varieties the converse of Proposition 3 holds.

A field k is called *Hilbertian* if Hilbert's irreducibility theorem holds over k (compare [3, page 141]). Important examples of Hilbertian fields are finitely generated transcendental extensions of some subfield (compare [3, Theorem 12.10, page 155]) and the algebraic number fields (compare [3, Corollary 12.8, page 154]).

THEOREM 7. *Let* k *be a field that is not locally finite and let* $A \neq 1$ *be an abelian variety, defined over* k*. Then there is a Hilbert field* $\mathsf{E} \subseteq \mathsf{k}$ *such that* A *is defined over* E *and* $A(\mathsf{E}^a)$ *has infinite torsion free rank. The field* E *is finitely generated over its prime field.*

PROOF. Assume that the torsion free rank of $A(\mathsf{k}^a)$ is some finite number $d \geq 0$. Let L be an algebraically closed extension of k of uncountable transcendence degree. Since the torsion subgroup of $A(\mathsf{L})$ is countable (see [8, page 39]), whilst $A(\mathsf{L})$ itself has the same cardinality as L, there are elements $a_1, \ldots, a_{d+1}$ in $A(\mathsf{L})$, that generate a free abelian group of rank $d+1$. If we embed $A(\mathsf{L})$ into a projective space over L, we see that there exists a field F, finitely generated over the prime field of k, such that the elements $a_1, \ldots, a_{d+1}$ are rational in $A(\mathsf{F})$.

Let k_0 be a minimal field of definition for A contained in k. Since A is defined (in any of its finitely many charts) by polynomials with coefficients in k_0, the field k_0 is finitely generated. We assume first that k_0 is locally finite. Then there exists an element t_0 in k, which is transcendental over k_0. We put $\mathsf{E} = \mathsf{k}_0(t_0)$. If k_0 is not locally finite, we put $\mathsf{E} = \mathsf{k}_0$. In both cases the field E is Hilbertian, contained in k, and is a field of definition for the abelian variety A.

By a theorem of Néron [7, Chapter 1, Theorem 7.2, page 41], there exists an injective specialization homomorphism σ from $A(\mathsf{F})$ to $A(\mathsf{E}^a)$. Hence $\{\sigma(a_1), \ldots, \sigma(a_{d+1})\}$ are $\mathbb{Z}$-independent elements in $A(\mathsf{E}^a) \subset A(\mathsf{k}^a)$: a contradiction. □

PROPOSITION 8. *Assume that the field* k *is not locally finite and let* A_1, A_2 *be abelian* k*-varieties. Then for every element* $a_1 \in A_1$ *of infinite order there exists an element*

$a_2 \in A_2$ of infinite order such that $\varphi(a_1) \neq a_2{}^n$ for all algebraic homomorphisms φ from A_1 to A_2 and for any $n \in \mathbb{N}$.

PROOF. By [8, Theorem 3, page 176], the group $\mathsf{Hom}(A_1, A_2)$ of all algebraic homomorphisms from A_1 to A_2 has finite torsion free rank. Thus the abstract group $X = \mathsf{Hom}(A_1, A_2)a_1$ is a commutative group of finite rank. It follows from Theorem 7 that the group X cannot intersect all torsion-free subgroups of rank 1 of A_2 non-trivially. $\square$

Now we treat monothetic abelian varieties in general. Although the discussion of monothetic abelian varieties is in the spirit of Borel's proof for tori (see [1, Proposition 8.8]), it becomes more complicated.

THEOREM 9. *An abelian* k*-variety* $A \neq 1$ *is monothetic if and only if the field* k *is not locally finite.*

PROOF. If k is locally finite, then A is a torsion group by Proposition 3. Conversely, assume that k is not locally finite and that A is a counterexample of minimal dimension. By Theorem 7, the abelian variety A is not simple, hence $A = A_1A_2$, where A_1 is a simple non-trivial abelian variety and A_2 is monothetic, by minimality. Because of Proposition 1 we may assume that $A = A_1 \times A_2$. Choose an arbitrary generator t_2 of A_2 and an element $t_1 \in A_1$ of infinite order such that $t_1{}^n \neq \varphi(t_2)$ for any $n \in \mathbb{N}$ and for all algebraic homomorphisms φ from A_2 to A_1. This is possible thanks to Proposition 8. We put $U = \overline{\langle (t_1, t_2)\rangle}$. Then $U \neq A$ and, changing (t_1, t_2) to a suitable power, by Proposition 2 we can even assume that U is connected. Since A_1 is simple, the group $U \cap A_1$ has some finite order m.

Let $\pi_i : U \to A_i$ be the restriction to U of the canonical projection from A to A_i, $(i = 1, 2)$. Since $t_2 = \pi_2(t_1, t_2) \in A_2$, the homomorphism π_2 is surjective; its kernel is the finite group $U \cap A_1$ of order m. It follows from [8, Chapter IV.18, Remark, page 169] that there is a homomorphism $\eta : A_2 \to U$ such that $\eta\pi_2(x_1, x_2) = (x_1, x_2)^m$ for all $(x_1, x_2) \in U$. But then

$$\pi_1\eta(t_2) = \pi_1\eta\pi_2(t_1, t_2) = \pi_1(t_1, t_2)^m = (\pi_1(t_1, t_2))^m = t_1^m,$$

a contradiction to the choice of t_1. $\square$

In contrast to the affine case (Theorem 5), in general we cannot say anything more about the rationality of a generator of an abelian k-variety A than we said in Proposition 3. For $\mathsf{k} = \mathbb{Q}$ one finds in [6, Chapter I.3, Table 2] examples of elliptic curves, where the group of $\mathbb{Q}$-rational points is finite. Using [6, Theorem 3.3, page 34] one sees that are many cases in which the identity is the only $\mathbb{Q}$-rational point of A. One such example is given by the equation $y^2 = x^3 + 6$.

D. The general case In [11], Rosenlicht described how an arbitrary connected algebraic group is built from an affine group and an abelian variety. We collect these facts in the following theorem.

THEOREM 10 (Rosenlicht [11]). *Let G be a connected algebraic k-group. Then there exists a (unique) maximal connected affine subgroup L_G and a (unique) minimal (connected) normal subgroup D_G such that the factor group G/D_G is affine. The following statements hold*:

(i) $G = L_G D_G$.

(ii) *L_G is a k-closed characteristic subgroup of G, and G/L_G is an abelian variety.*

(iii) *D_G is defined over k and is central and characteristic in G. It has no non-trivial affine epimorphic image, and it has only a finite number of elements of any given finite order.*

(iv) *Any k-closed abelian subvariety A of G, as well as the connected component of $L_G \cap D_G$, is defined over k and is contained in D_G. Moreover, for the subgroup A there exists a connected k-closed algebraic subgroup G_1 of G such that $G = G_1 A$ and $G_1 \cap A$ is finite.*

By the preceding theorem the group D_G is commutative. Hence the connected component of $L_G \cap D_G$ is the direct product of a vector group and a torus ([5, Theorem 15.5, page 100]).

LEMMA 11. *If the field k is not locally finite, then a connected algebraic k-group G having no non-trivial affine epimorphic image is always monothetic. If G is such a group and k has positive characteristic, then L_G is a torus.*

PROOF. Let L be the maximal connected affine subgroup of G. As G/L is an abelian variety, by Theorem 9, we find an element $y \in G/L$ such that $G/L = \overline{\langle y \rangle}$. Let x be a pre-image of y with respect to the canonical projection $G \to G/L$. We put $X = \overline{\langle x \rangle}$. Since XL/L contains y, we see that $XL = G$. Thus $G/X = XL/X \cong L/(L \cap X)$ is affine and consequently we have $G = X$.

Assume finally that $\mathrm{char}(\mathsf{k})$ is positive. Since G has only a finite number of elements of order p (Theorem 10 (iii)), the group G in this case cannot contain any non trivial vector subgroup. □

If a monothetic k-group G having no non-trivial affine epimorphic image is an extension of a vector group H by an abelian variety A, then it follows from Lemma 11 that the field k has characteristic 0. Conversely, if $\mathrm{char}(\mathsf{k}) = 0$, then according to [12, Proposition 11 and Theorem 3] such extensions exist for $\dim(H) \leq \dim(A)$. From our Theorem 5 we see that the vector group H is monothetic if and only if $\dim(H) \leq 1$. Hence, in contrast to Corollary 6, we have the following observation.

REMARK 12. Closed connected subgroups of connected monothetic k-groups G are not necessarily monothetic if the maximal connected affine subgroup of G is not a torus. This can only happen if $\mathrm{char}(\mathsf{k}) = 0$.

THEOREM 13. *A connected algebraic* k*-group* $G \neq 1$ *is monothetic if and only if the field* k *is not locally finite and* G *is a product* DH *with finite* $D \cap H$*, where* H *is a monothetic connected affine* k*-group and* D *is a monothetic connected* k*-group having no non-trivial affine epimorphic image.*

A connected algebraic k*-group* G *has a* k*-rational generator if and only if* D *has a* k*-rational generator*; *the group* H *always has a* k*-rational generator.*

PROOF. Assume first that G is monothetic, and consider the subgroups $L = L_G$, $D = D_G$ of G as in Theorem 10. There is a finite subgroup E of G such that $G/E = C \times D_{G/E}$ with an affine group C ([9, Lemma 3]). Thus by Proposition 1, we may suppose that $L \cap D$ is connected. Furthermore, L is a k-closed subgroup of G by Theorem 10 (ii) and there is a complement H of $L \cap D$ in L; in particular, we have $H \cap D = 1$. The affine algebraic group H is isomorphic to $L/(L \cap D)$.

If $\mathrm{char}(\mathsf{k}) = 0$, then the affine monothetic group $G/D = LD/D$ is the direct product of a torus with an ϵ-dimensional vector group ($\epsilon \leq 1$) by theorem 5. By Theorem 10 (ii) and [5, Lemma 34.1, page 217], the group L is a connected commutative affine algebraic k-subgroup of G.

If $\mathrm{char}(\mathsf{k}) > 0$, then LD/D is a torus (Theorem 5) and D is an extension of a torus by an abelian variety (Lemma 11); in particular, L is the maximal torus of G. By [11, Proposition 4, page 443], it follows that L is defined over k.

Now, in both cases, the group $H \cong L/(L \cap D)$ is defined over k ([11, Theorem 4, page 413]) and is monothetic as an epimorphic image of G. Finally, we observe that G is isogeneous to the direct product of H and D.

Conversely, let $G = HD$ with k-groups $D = \overline{\langle x \rangle}$, $H = \overline{\langle y \rangle}$. Since $H \cap D$ is finite, we may assume that $H \cap D = 1$. Consider the group $U = \overline{\langle xy \rangle}$. If $U \neq G$, then D and H are not contained in U.

Observing that $G = D \times H$ we consider the subgroup $N = (D \cap U) \times (H \cap U)$ and the factor group

$$G_1 = G/N = \big(D \times H\big)/\big((D \cap U) \times (H \cap U)\big) \cong D_1 \times H_1,$$

where $D_1 = D/(D \cap U)$ is the minimal subgroup of G_1 with an affine factor group and $H_1 = H/(H \cap U)$ is the maximal connected affine subgroup of G_1. With $U_1 = U/N$, we have $U_1 \times D_1 = G_1 = U_1 \times H_1$. Since G_1/D_1 is affine, it follows that U_1 is affine. Hence $G_1 = U_1 H_1$ is affine, a contradiction. Thus D or H is contained in U, but in both cases $G = U$ is monothetic.

Since the monothetic group H is defined over k, it follows from Theorem 5 that we can choose a k-rational generator x for H. If y is a k-rational generator of D, then, as just shown, the element xy is a k-rational generator of G. Conversely, if z is a k-rational generator for G, then for any k-rational generator x for H the element $x^{-1}z$ is a k-rational generator of D. □

Corollary 14. *A connected monothetic* k*-group* G *is divisible* (*as an abstract commutative group*).

Proof. We consider monothetic subgroups D, H of G satisfying $G = DH$ as in Theorem 13. It follows from Lemma 11 that D is divisible as an extension of a divisible group by a divisible group. Since H is divisible by Theorem 5, the assertion follows. □

From the preceding theorem, Corollary 6 and Remark 12, we also obtain the following result.

Corollary 15. *Let* k *be a field. Each closed connected subgroup of every monothetic* k*-group is monothetic if and only if* k *has positive characteristic.*

For fields of characteristic 0, our structure theorems, together with Remark 12 yield the following.

Corollary 16. *All closed connected subgroups of a monothetic* k*-group* G *over a field of characteristic* 0 *are monothetic if and only if the maximal connected unipotent subgroup of* G *has dimension at most* 1.

Theorem 17. *An algebraic* k*-group* G *is monothetic if and only if* G *is commutative, the connected component* G° *is monothetic, and* G/G° *is a finite cyclic group.*

Proof. If G is monothetic, then G° is monothetic by Proposition 2 and G/G° is a finite cyclic group.

Conversely, let G° and G/G° be monothetic. Since G° is divisible by Corollary 14, there is a finite cyclic subgroup $F \cong G/G^\circ$ of G such that $G = G^\circ \times F$ is an algebraic group. Let g be a generator of G° and let f be a generator of F. Using Proposition 2 it follows that gf is a generator of G. □

Remark 18. Let $G = G^\circ \times F$ be a non-connected monothetic k-group. One has the decomposition $G^\circ = HD$ as in Theorem 13, since G° is a monothetic k-group by Proposition 2. The group $H \times F$ is a monothetic affine k-group (see [11, Corollary 1, page 430]). Using Theorem 5, one sees that $H \times F$ has a k-rational generator. Thus G has a k-rational generator if and only if D has this property.

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