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THE MONGE-AMPÈRE EQUATION AND WARPED PRODUCTS OF HIGHER RANK

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Abstract

We show that a warped product $M_f = \mathbb{R}^n \times_f \mathbb{R}$ has higher rank and nonpositive curvature if and only if f is a convex solution of the Monge-Ampère equation. In this case we show that M contains a Euclidean factor.

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1. Introduction

A Riemannian manifold has higher rank if along each of its geodesics there are at least two linearly independent parallel Jacobi fields. For nonpositively curved manifolds this seems to be a rather restrictive condition. An irreducible Hadamard manifold of higher rank is symmetric if it additionally admits a cocompact discontinuous group Γ of isometries. In fact it suffices if Γ satisfies the duality condition (see [1, 2, 3, 6]). It is an open question whether this additional condition may be entirely omitted ('Rank Rigidity Conjecture').

In this note we consider warped products $M = \mathbb{R}^n \times_f \mathbb{R}$. It turns out that nonpositive sectional curvature on M translates to convexity of f and that higher rank of M requires f to satisfy the Monge-Ampère equation. We show in Proposition 3.2 that a positive convex solution $f \in C^\infty(\mathbb{R}^n)$ of the Monge-Ampère equation has the form $f(x) = q(Ax)$, where $q \in C^\infty(\mathbb{R}^k)$ is a positive convex function with $k < n$ and $A\colon \mathbb{R}^n \to \mathbb{R}^k$ is an orthogonal projection, that is $AA^* = \mathrm{id}_{\mathbb{R}^k}$. Hence a Riemannian manifold $M = \mathbb{R}^n \times_f \mathbb{R}$ of nonpositive curvature and higher rank splits as a Riemannian product $(\mathbb{R}^k \times_q \mathbb{R}) \times \mathbb{R}^{n-k}$ (Theorem 2.1).

2. Warped Products of Higher Rank

Let M be a Riemannian manifold with curvature tensor R. The Jacobi operator J_v of a tangent vector $v \in T_pM$ at $p \in M$ is the symmetric endomorphism $x \mapsto R(x, v)v$ of T_pM. M has higher infinitesimal rank if the kernel of the Jacobi operator of each $v \in TM$ is at least 2-dimensional. By the Jacobi equation, manifolds with higher rank have this property. On the other hand, the semisymmetric spaces of conullity 2 considered in [5], [8] and [4, Theorem 2] provide examples of manifolds of rank 1 which have higher infinitesimal rank.

Given a smooth real positive function f on $\mathbb{R}^n$ we let M denote $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ with the warped product metric given at a point $(x, t) \in M$ by

$$g = dx^2 \oplus f(x)^2 dt^2,$$

where dx^2, dt^2 denote the standard metrics on $\mathbb{R}^n$, $\mathbb{R}$ respectively. We show that rank rigidity holds for these warped products.

THEOREM 2.1. *The warped product $M = \mathbb{R}^n \times_f \mathbb{R}$, $n > 0$, has nonpositive curvature and higher infinitesimal rank if and only if $f(x) = q(Ax)$, where $q \in C^\infty(\mathbb{R}^k)$ is a positive convex function and $A\colon \mathbb{R}^n \to \mathbb{R}^k$ is an orthogonal projection with $k < n$. Thus M is isometric to the product $(\mathbb{R}^k \times_q \mathbb{R}) \times \mathbb{R}^{n-k}$.*

PROOF. This follows from Lemma 2.2 and Proposition 3.2. □

LEMMA 2.2. *$M = \mathbb{R}^n \times_f \mathbb{R}$ has nonpositive curvature and higher infinitesimal rank if and only if f is a positive convex solution of the Monge-Ampère equation* (3.1).

PROOF. We first express the curvature tensor of M in terms of f. To that end, let X, Y and T be vector fields on $M = \mathbb{R}^n \times_f \mathbb{R} \cong \mathbb{R}^{n+1}$ which are parallel with respect to the Euclidean metric on $\mathbb{R}^{n+1}$ and such that X, Y are tangent to the first factor and $T = \partial/\partial t$ is tangent to the second factor. We have $g(T, T) = f^2$. For the covariant derivative we compute

$$\nabla_X Y = 0, \quad \nabla_X T = \nabla_T X = \frac{Xf}{f} T \quad \text{and} \quad \nabla_T T = -f \operatorname{grad} f .$$

The sectional curvature is given by

$$\begin{aligned} K(X+T, Y) &= \langle R(X+T, Y)Y, T+X \rangle = -\langle \nabla_Y \nabla_T Y, X+T \rangle \\ &= -\left\langle \nabla_Y \frac{Yf}{f} T, X+T \right\rangle = -(YYf)f = -f\, d^2 f(Y) . \end{aligned} \tag{2.1}$$

For $n = 1$, M has higher infinitesimal rank if and only if f is constant. Let now $n \geq 2$. By (2.1), M has nonpositive curvature if and only if the Hessian $d^2 f$ is everywhere nonnegative. For any X the Jacobi operator of X has at least n-dimensional kernel. The Jacobi operator of $X + T$ has at least 2-dimensional kernel if and only if $\langle R(X+T, Y)Y, T+X\rangle = -f\, d^2 f(Y) = 0$ for some $Y \neq 0$. But this is equivalent to $\det d^2 f = 0$. □

3. Global Convex Solutions of the Monge-Ampère Equation

Henceforth $f\colon \mathbb{R}^n \to \mathbb{R}$ always denotes a smooth nonnegative convex solution of the Monge-Ampère equation. Thus the Hessian $d^2 f$ of f satisfies

$$\det d_x^2 f = 0 \quad \text{and} \quad d_x^2 f \geq 0 \tag{3.1}$$

for all $x \in \mathbb{R}^n$. On a strictly convex domain $\Omega \subset \mathbb{R}^n$ we have from [7] that

$$f(x) = \sup_L\{L(x)\}, \quad x \in \Omega\,, \tag{3.2}$$

where L runs through all affine linear functions on $\mathbb{R}^n$ with $L \leq f$ on the boundary $\partial\Omega$.

Lemma 3.1. *For any $q \in \mathbb{R}^n$ with $f(q) = 0$ and any $\rho > 0$ we have that q is contained in the convex hull of*

$$f^{-1}(0) \cap \left(q + \rho S^{n-1}\right) = \{z \in \mathbb{R}^n : \ f(z) = 0,\ |z - q| = \rho\}\ .$$

In particular there are $z_0, \ldots, z_n \in f^{-1}(0) \cap \left(q + \rho S^{n-1}\right)$ and $\lambda_0, \ldots, \lambda_n \geq 0$ with $\sum_{i=0}^n \lambda_i = 1$ such that $\sum_{i=0}^n \lambda_i z_i = q$.

Proof. We assume the contrary and suppose $q = 0$ and $\rho = 1$, since (3.1) is invariant under translations and homotheties. The convex hull of $f^{-1}(0) \cap S^{n-1}$ is an intersection of halfspaces of $\mathbb{R}^n$. Hence after a suitable choice of coordinates we may further suppose that it is contained in $\{x_1 \leq -2\delta\}$ for some $\delta > 0$. We apply (3.2) to the unit ball $\Omega = D^n$. Since $S^{n-1} \cap \{x_1 \geq -\delta\}$ is compact, there is $\epsilon > 0$ such that $f|_{S^{n-1}\cap\{x_1 \geq -\delta\}} \geq \epsilon$. Let α be the affine linear function on $\mathbb{R}^n$ given by

$$\alpha(x_1, \ldots, x_n) = (\delta + x_1)\frac{\epsilon}{\delta + 1}\,.$$

Clearly $\alpha|_{\partial\Omega} \leq f|_{\partial\Omega}$, hence $f(0) \geq \alpha(0) = \delta\epsilon/(\delta + 1) > 0$, contradicting the assumption that $f(0) = 0$. □

Proposition 3.2. *Let $f \in C^\infty(\mathbb{R}^n)$ be a smooth positive function with $d_x^2 f \geq 0$ and $\det d_x^2 f = 0$ for all $x \in \mathbb{R}^n$. Then there is a $k < n$, an orthogonal projection $A\colon \mathbb{R}^n \to \mathbb{R}^k$ and a $q \in C^\infty(\mathbb{R}^k)$ such that $f(x) = q(Ax)$ for all x.*

PROOF. We may assume $f \geq 0$ and $f(0) = 0$, replacing $f(x)$, if necessary, by $f(x) - f(0) - d_0 f x$. For $\rho > 0$ let $z_i(\rho)$, $i = 0, \ldots, n$, be as in Lemma 3.1, so that $f(z_i(\rho)) = 0$, $|z_i| = \rho$ and $\sum_{i=0}^n \lambda_i z_i = 0$ for some $\lambda_i \geq 0$ with $\sum_{i=0}^n \lambda_i = 1$. Choosing a suitable sequence $\rho_k \to \infty$, we get convergent sequences $z_i(\rho_k)/|z_i(\rho_k)| \to z_i(\infty)$. Then $U = \operatorname{span}\{z_0(\infty), \ldots, z_n(\infty)\}$ is a subspace of $\mathbb{R}^n$ of dimension $n - k > 0$. We have

$$f(\lambda z_i(\infty)) = \lim_{k\to\infty} f\left(\frac{\lambda z_i(\rho_k)}{|z_i(\rho_k)|}\right) \leq \lim_{k\to\infty}\left(\left(1 - \frac{\lambda}{\rho_k}\right) f(0) + \frac{\lambda}{\rho_k} f(z_i(\rho_k))\right) = 0$$

for each $\lambda \in \mathbb{R}^+$, since f is convex and $f(0) = 0$. For $\lambda \to \infty$ the convex hulls of $\{\lambda z_0(\infty), \ldots, \lambda z_n(\infty)\}$ exhaust U. Hence $f|_U = 0$.

By convexity

$$f((1-\mu)p + u) \leq (1-\mu) f(p) + \mu f(u/\mu) = (1-\mu) f(p) \tag{3.3}$$

for any $p \in \mathbb{R}^n$, $0 < \mu \leq 1$ and $u \in U$. In the limit $\mu \to 0$ the left hand side of (3.3) tends to $f(p+u)$, hence $f(p+u) \leq f(p)$. Since p and $p+u$ are arbitrary elements of the affine space $p + U$ we conclude that f is constant on the affine spaces $p + U$. We choose an isometry of the orthogonal complement $U^\perp$ of U with $\mathbb{R}^k$. Let A be the orthogonal projection $\mathbb{R}^n \to U^\perp \cong \mathbb{R}^k$ and q be the restriction of f to $U^\perp \cong \mathbb{R}^k$. □

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