J. Aust. Math. Soc. **83** (2007), 149–155

RECOGNIZING POWERS IN NILPOTENT GROUPS AND NILPOTENT IMAGES OF FREE GROUPS

GILBERT BAUMSLAG

(Received 1 March 2006; revised 23 May 2006)

Communicated by C. F. Miller

Abstract

An element in a free group is a proper power if and only if it is a proper power in every nilpotent factor group. Moreover there is an algorithm to decide if an element in a finitely generated torsion-free nilpotent group is a proper power.

2000 *Mathematics subject classification*: primary 20F18.

1. Introduction

One of the questions that arose in the development of the software package MAGNUS was whether it is possible to discern that an element in a free group is not a proper power in one of its nilpotent quotients. This question has also arisen in an ongoing attempt to prove that free $\mathbb{Q}$–groups are residually torsion–free nilpotent. One of the objects of this note is to settle the first question by proving the following.

THEOREM 1.1. *An element in a free group is a proper power if and only if it is a proper power in all of its nilpotent images.*

A key idea involved in the proof of Theorem 1.1 goes back to Wilhelm Magnus and is a critical step in his solution to the word problem for groups with a single defining relation (see a detailed discussion of Magnus' method in [4]).

As a companion theorem we also prove the following much easier result.

THEOREM 1.2. *There is an algorithm to decide if an element in a finitely generated torsion-free nilpotent group is a proper power.*

2. The proof of Theorem 1.1

2.1. The groups $H(Y, n, c)$ We shall have need of a family of torsion–free nilpotent groups $H(Y, n, c)$ which depend on three parameters: a set Y and two positive integers n and c. Each of the groups $H(Y, n, c)$ is an extension of a free nilpotent group $N = N(Y, n, c)$ of class c by the infinite cyclic group $T =< t >$ generated by t. N is freely generated by the set

$$\{y_h | y \in Y,\ h = 1, \dots, n\},$$

indexed by Y and $\{1, \dots, n\}$. The action of T on N is defined as follows

$$t^{-1}y_1t = y_1y_2, \dots, t^{-1}y_{n-1}t = y_{n-1}y_n, \quad t^{-1}y_nt = y_n \tag{2.1}$$

for $y \in Y$. In order to ensure that the action τ of T on N given by (2.1) defines an automorphism of N we need first to observe that in a nilpotent group any set of elements that generates the group modulo its derived group, generates the group itself. Consequently τ is an epimorphism. Next we observe that an epimorphism of a finitely generated nilpotent group is an automorphism. Notwithstanding the fact that N need not be finitely generated, the very nature of N allows one to deduce that τ is monic. Thus the definition (2.1) makes sense (for some additional explanation, see the proof of Lemma 2.2, if desired). Denoting the c^{th} term of the lower central series of a group G by $\gamma_c(G)$, we observe that, modulo $\gamma_2(N)$, $H(Y, n, c)$ is nilpotent of class n. Hence, by a theorem of P. Hall [2], $H(Y, n, c)$ is also nilpotent. It is clearly torsion–free. So we have proved the following.

LEMMA 2.1. *The groups $H(Y, n, c)$ are torsion–free nilpotent for every choice of Y, n and c.*

The following lemma is a consequence of the fact that N is free nilpotent of class c (here we denote the conjugate $u^{-1}vu$ of v by u by v^u).

LEMMA 2.2. *Let N be as above and let*

$$z(y, h) = y_1^{t^h} \qquad (h = 0, \dots, n-1).$$

Then

$$z(y, 0), \dots, z(y, n-1) \quad (y \in Y)$$

freely generate N.

PROOF. In order to prove Lemma 2.2, observe first that in a free nilpotent group any set of elements which are independent modulo the derived group, freely generate a free nilpotent group (see, e.g., [5]). Notice that $H(Y,n,c)/\gamma_2(H(Y,n,c)) \cong H(Y,n,1)$. Thus it suffices for the proof of Lemma 2.2 to prove that that in $H(Y,n,1)$ the elements $z(y,h) = y_1^{t^h}$ $(h = 0,\ldots,n-1,\ y \in Y)$ freely generate N. Now $H(Y,n,1)$ is an extension of the free abelian group N on $y_1,\ldots,y_n$ $(y \in Y)$ by the infinite cyclic group T on t, where t acts on N as above:

$$t^{-1}y_1t = y_1y_2,\ldots,t^{-1}y_{n-1}t = y_{n-1}y_n, \quad t^{-1}y_nt = y_n$$

for every $y \in Y$. Observe that for each integer $0 \le k \le n-1$ we have

$$z(y,k) = y_1^{\binom{k}{0}}y_2^{\binom{k}{1}}\cdots y_{k+1}^{\binom{k}{k}}.$$

It follows that for $1 \le k \le n$ we have

$$gp(y_1,\ldots,y_k) = gp(z(y,0),\ldots,z(y,k-1))$$

and thence that

$$z(y,0),\ldots,z(y,n-1) \quad (y \in Y)$$

generate N. So, by the remark at the outset of the proof, $z(y,0),\ldots,z(y,n-1)$ freely generate N. □

Finally we shall have need of the following simple observation.

LEMMA 2.3. *Let F be a free group in a variety $\mathscr{V}$ of groups, freely generated by a set X. Let f be an element of F and suppose that f can be written as a word in the finitely many elements $x_1,\ldots,x_q$ of X. If f is not a proper power in the subgroup S of F generated by the elements $x_1,\ldots,x_q$, then f is not a proper power in F.*

PROOF. Suppose if possible that $f = f_1^m$, for some $f_1 \in F$, where $m > 1$. Let ρ be the retraction of F onto S defined by mapping each of the x_i to itself for $i = 1,\ldots,q$ and the remaining elements of X to the identity. Then ρ maps f to itself and hence

$$f = f\rho = (f_1^m)\rho = (f_1\rho)^m.$$

Since $f_1\rho \in S$, this contradiction proves the lemma. □

2.2. The main step The main step in the proof of Theorem 1.1 is the following lemma, which will be used in the inductive step in the proof. This will be made clear in what follows.

LEMMA 2.4. *Let F be a free group freely generated by a set X, let $s \in X$ and let $X' = X - \{s\}$. Furthermore, let K be the normal closure in F of X' and suppose that $f' \in K$. If f' is not a proper power modulo some term of the lower central series of K, then f' is not a proper power modulo some term of the lower central series of F.*

PROOF. Observe that K is freely generated by the elements

$$x_i = s^{-i} x s^i \quad (x \in X',\ i \in \mathbb{Z}).$$

We can assume, replacing f' by one of its conjugates if necessary, that f' can be expressed as a word in the generators

$$x_i = s^{-i} x s^i \quad (x \in X',\ i \in \{0, \ldots, n-1\})$$

for a suitably large choice of the integer n. By hypothesis, there exists an integer c such that $f'\gamma_{c+1}(K)$ is not a proper power in $K/\gamma_{c+1}(K)$.

We now choose the set Y in Lemma 2.3 in such a way that there is a one-to-one correspondence ϕ between X' and Y. Next we define a homomorphism θ from F into $H(Y, n, c)$ by sending s to t and $x \in X'$ to $(x\phi)_1$. In the event that ϕ maps $x \in X'$ to $y \in Y$, we will denote y_1 also by $y(x, 1)$. Thus $(x\phi)_1 = y(x, 1)$. Consequently x_i maps onto $(y(x, 1)^{t^i})$ for $i = 0, \ldots, n-1$. Now as y ranges over Y, by Lemma 2.2, the conjugates of the elements y_1 by the powers t^h $(h = 0, \ldots, n-1)$ of t generate N. Consequently θ is onto. Hence θ induces a homomorphism $\theta_\star$ of $F/\gamma_{c+1}(K)$ onto $H(Y, n, c)$. Observe that $\theta_\star$ maps the elements $x_i\gamma_{c+1}(K)$ onto $(y(x, 1)^{t^i})$ where here $i = 0, \ldots, n-1$ and $x \in X'$. Now, by Lemma 2.2, the elements $(y(x, 1)^{t^i})$ where again $i = 0, \ldots, n-1$ and $x \in X'$, freely generate the free nilpotent group N. Thus $\theta_\star$ when restricted to $gp(x_0\gamma_{c+1}(K), \ldots, x_{n-1}\gamma_{c+1}(K))$ with x ranging over X', is an isomorphism. Hence $(f'\gamma_{c+1}(K))\theta_\star$ is not a proper power in $gp((x_0\gamma_{c+1}(K))\theta_\star, \ldots, (x_{n-1}\gamma_{c+1}(K))\theta_\star)$. Since the elements

$$(x_0\gamma_{c+1}(K))\theta_\star, \ldots, (x_{n-1}\gamma_{c+1}(K))\theta_\star$$

are part of a free basis for N, it follows from Lemma 2.3 that $(f'\gamma_{c+1}(K))\theta_\star$ is not a proper power in N. But $H(Y, n, c)/N$ is infinite cyclic. Hence $(f'\gamma_{c+1}(K))\theta_\star$ is not a proper power in $H(Y, n, c)$. Now $H(Y, n, c)$ is nilpotent of class, say j. It follows that $f'\gamma_{j+1}(F)$ is not a proper power in $F/\gamma_{j+1}(F)$ since its image under the homomorphism from $F/\gamma_{j+1}(F)$ onto $H(Y, n, c)$ induced by θ is not a proper power in $H(Y, n, c)$. This completes the proof of Lemma 2.4 □

2.3. The proof of Theorem 1.1 Suppose again that F is a free group, freely generated by a set X and that $f \in F$ is not a proper power. The proof that f is not a proper power modulo some term of the lower central series of F will be by induction on the length ℓ of f. If $\ell = 1$, $f\gamma_2(F)$ is not a proper power in $F/\gamma_2(F)$. So we can focus on the case where $\ell > 1$. Inductively then we assume that an element of length at most $\ell - 1$ in a free group is a proper power if and only if it is a proper power in every nilpotent image of that free group. We will make this assumption in the lemma that follows.

LEMMA 2.5. *Let F be a free group, freely generated by a set X and let s be an element of X. Furthermore, let $f \in F$ be of length ℓ, suppose that f is not a proper power and suppose that s occurs in f with exponent sum zero. Then there exists an integer j such that $f\gamma_{j+1}(F)$ is not a proper power in $F/\gamma_{j+1}(F)$.*

PROOF. In order to prove Lemma 2.5, put $X' = X - \{s\}$ and let K be the normal closure in F of X'. Let, as before,

$$x_i = s^{-i}xs^i \quad (x \in X', i \in \mathbb{Z}).$$

Then K is freely generated by the x_i and $f \in K$. So we can express f as a word f' in terms of the x_i. Moreover, because s occurs with exponent sum 0 in f, f' has length at most $\ell - 2$. So, inductively, there exists a nilpotent quotient of K in which the image of f' is not a proper power. It follows then from Lemma 2.4 that there is a nilpotent quotient $F/\gamma_{j+1}(F)$ of F such that $f\gamma_{j+1}(F) = f'\gamma_{j+1}(F)$ is not a proper power in $F/\gamma_{j+1}(F)$, as required. □

In order to complete the proof of Theorem 1.1, we are left with the case of an element f of length $\ell > 1$ in the free group F freely generated by the set X which is not a proper power and such that none of the elements of X occurs in f with exponent sum 0. It follows that f must involve at least two elements of X. We can, by appealing to Lemma 2.3, also assume that X is finite. Suppose then that f involves the elements a and b of X and inductively that in a free group any element of length less than ℓ is a proper power only if it is a proper power in every nilpotent factor group. We express the involvement of a and b in f by using functional notation, that is, by writing $f = f(a, b, \ldots)$. Suppose that a occurs with exponent sum α and b with exponent sum β in f. We freely adjoin a β^{th} root r to a, so $r^\beta = a$. The resultant group E is free on $X - \{a\} \cup \{r\}$. Moreover, f now takes the form

$$f = f(r^\beta, b, \ldots)$$

and r occurs with exponent sum $\alpha\beta$ in $f = f(r^\beta, b, \ldots)$. Next put $u = br^\alpha$. Then we find that E is free on r, u and all of the elements of X exclusive of a and b. Observe next that

$$f = f(r^\beta, ur^{-\alpha}, \ldots).$$

It follows that r now occurs with exponent sum 0 in this new form for f. Let L be the normal closure of $X' = \{u\} \cup (X - \{a, b\})$. Then $f \in L$. In addition, L is free on the conjugates $x_i = r^{-i}xr^i$, where x ranges over X' and i over the integers. If we now express f as a word f' in terms of these free generators of L, we find that the length of f' is at most $\ell - 1$. On appealing now to Lemma 2.4 and making use of the induction hypothesis, we find that f is not a proper power modulo some term of the lower central series of E. Consequently f is not a proper power modulo some term of the lower central series of F. This completes the proof of Theorem 1.1.

3. The proof of Theorem 1.2

Let G be a finitely generated, torsion–free nilpotent group given by a consistent polycyclic presentation (see [1]). We take for granted here the notions and results in that paper [1].

The proof that there is an algorithm to decide if an element g in G is a proper power will be by induction on the class c of G. If $c = 1$, that is, if G is abelian, then we can decompose G into a direct product of finitely many infinite cyclic groups. This allows us to express g as a product of powers of the generators of these infinite cyclic groups. Then g is a proper power if and only if these powers have a common divisor $d > 1$.

Now suppose that $c > 1$ and that C is the centre of G. By [1] we can compute a consistent polycyclic presentation for C and one for G/C. By a theorem of Kantorovich (see, for example, [3]), G/C is again a torsion–free nilpotent group. So inductively we can determine if gC is a proper power in G/C. If gC is not a proper power in G/C then g is not a proper power in G. Suppose that gC is a proper power in G/C. Now, according to Kantorovich, extraction of roots in torsion–free nilpotent groups is unique in so far as they exist (see again [3]). It follows that we can express gC uniquely in the form $gC = (hC)^n$, where n is chosen maximal. Thus if gC is a proper power, then g can be written in the form $g = h^e h'$, where $h' \in C$. Let $A = gp(C, h)$. Then A is an isolated subgroup of G (see [3] for this notion of isolated). Now $g \in A$ and g is a proper power in G if and only if it is a proper power in A. A is a finitely generated torsion–free abelian group. Since we can effectively find a consistent polycyclic presentation for A, it follows that we can decide if g is a proper power in A. This completes the proof of Theorem 1.2.

It should be noted that since the torsion subgroup of a finitely generated nilpotent group is finite, the restriction in Theorem 1.2 that G be torsion–free can easily be omitted. Moreover one can, also easily, fashion a proof that there is an algorithm to decide if an element in a finitely generated nilpotent group is a proper power by using two facts. First that in a nilpotent group of class $c > 1$ an element together with the derived group generates a nilpotent subgroup of class at most $c - 1$. The second fact is that in a torsion–free nilpotent group the isolator of a nilpotent subgroup has the same class as the subgroup itself.

Acknowledgements

Support from the National Science Foundation is gratefully acknowledged.

References

[1] G Baumslag, F. B. Cannonito, D. Robinson and D. Segal, 'The algorithmic theory of polycyclic–by–finite groups', *J. Algebra* **142** (1994), 118–149.

[2] P. Hall, 'Some sufficient condtions for a group to be nilpotent', *Illinois J. Math.* **2** (1958), 787–801.

[3] A. G. Kurosh, *The theory of groups, vol. 2* (Chelsea, New York, 1960).

[4] W. Magnus, A. Karrass and D. Solitar, *Combinatorial group theory* (Wiley, New York, 1966).

[5] S. Moran, 'Errata and addenda to "A subgroup theorem for free nilpotent groups"', *Trans. Amer. Math. Soc.* **112** (1964), 79–83.

Department of Mathematics and Computer Science
City College of New York
New York, N.Y. 10031
USA
e-mail: gilbert@sci.ccny.cuny.edu