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PERMUTABLE FUNCTIONS CONCERNING DIFFERENTIAL EQUATIONS

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Abstract

Let f and g be two permutable transcendental entire functions. Assume that f is a solution of a linear differential equation with polynomial coefficients. We prove that, under some restrictions on the coefficients and the growth of f and g, there exist two non-constant rational functions R_1 and R_2 such that $R_1(f) = R_2(g)$. As a corollary, we show that f and g have the same Julia set: $J(f) = J(g)$. As an application, we study a function f which is a combination of exponential functions with polynomial coefficients. This research addresses an open question due to Baker.

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1. Introduction and Main Results

Let f be a meromorphic function. We denote by $T(r, f)$ the Nevanlinna characteristic of f. The order and the lower order of f are defined by

$$\lambda = \lambda(f) = \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r} = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}$$

and

$$\rho = \rho(f) = \liminf_{r\to\infty} \frac{\log\log M(r, f)}{\log r} = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r},$$

respectively, where $M(r, f) = \max\{|f(z)| : |z| = r\}$ is the maximum modulus (see for example [8] for an introduction to Nevanlinna Theory).

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Let f and g denote two meromorphic functions. If

$$f(g) = g(f), \tag{1.1}$$

then we call f and g *permutable*. Many mathematicians have studied the analytic and dynamical properties of f and g. The following general results are known.

- For a given f, there exist infinitely many transcendental entire functions g such that $f(g) = g(f)$, for example, $g = f^n$ will do, where f^n denotes the n-th iterate of f: $f^n = f^{n-1}(f)$. There should be no confusion with ordinary powers, which will be explicitly written as $(f(z))^n$ if necessary.
- For a given f, there are only countably many entire functions g such that (1.1) holds (see [2]).
- Let $f(z) = ae^{bz} + c$ ($ab \neq 0$, $a, b, c \in \mathbb{C}$). If $f(g) = g(f)$ then $g = f^n$ for some $n \geq 0$ (see [1]).

In this paper we shall study relations between permutable entire functions and differential equations. In fact, we shall consider functions which are solutions of some linear differential equations of the form

$$p_n(z)f^{(n)}(z) + p_{n-1}(z)f^{(n-1)}(z) + \cdots + p_1(z)f'(z) + p_0(z)f(z) + p(z) = 0, \tag{1.2}$$

where n is a positive integer, and p_i ($0 \leq i \leq n$) and p are polynomials, with $p_n \not\equiv 0$.

THEOREM 1.1. *Let f and g be two permutable transcendental entire functions with $\rho(f) > 0$ and $\lambda(g) < \infty$. If*

(i) *$f(z)$ satisfies (1.2) with $p_0(z) \not\equiv 0$ and $p(z)/p_0(z) \not\equiv$ constant;*
(ii) *$f(z)$ cannot be a solution of any linearly differential equation of order $\leq n-1$ with polynomial coefficients,*

then there exist two nonconstant rational functions $R_1(z)$ and $R_2(z)$ such that $R_1(f) \equiv R_2(g)$.

As an application, we consider the following function $f(z)$:

$$f(z) = p(z) + p_1(z)e^{q_1(z)} + p_2(z)e^{q_2(z)} + \cdots + p_n(z)e^{q_n(z)}, \tag{1.3}$$

where $p(z)$ is a polynomial, $p_i(z)$ ($i = 1, \ldots, n$) are non-zero polynomials and $q_i(z)$ ($i = 1, \ldots, n$) are polynomials with $q_i(z) - q_j(z) \not\equiv$ constant for $1 \leq i \neq j \leq n$.

THEOREM 1.2. *Let f and g be two permutable transcendental entire functions with $\lambda(g) < \infty$, where f satisfies (1.3). Assume that $p(z)$ is not a constant. Then there exist two rational functions $P_1(z)$ and $P_2(z)$ such that $P_1(f) = P_2(g)$.*

REMARK. In [17], we studied a special case where $n \leq 2$.

Next, we list some well-known permutable transcendental entire functions of exponential type (see Ng [14] for other examples).

EXAMPLE 1. Let $f(z) = z + \gamma e^z$ and $g(z) = z + \gamma e^z + 2k\pi i$, where $\gamma (\neq 0) \in \mathbb{C}$ and $k \in \mathbb{Z}$. Then $f \circ g = g \circ f$.

EXAMPLE 2. Let $g_1(z) = z + \gamma \sin z + 2k\pi$ and $g_2(z) = -z - \gamma \sin z + 2k\pi$ and $f(z) = z + \gamma \sin z$, where $\gamma (\neq 0) \in \mathbb{C}$ and $k \in \mathbb{Z}$. Then $f \circ g_1 = g_1 \circ f$ and $f \circ g_2 = g_2 \circ f$.

EXAMPLE 3. Let

$$f(z) = ia\left[\exp\left(\frac{(4k+3)\pi}{8a^2} iz^2\right) + \exp\left(-\frac{(4k+3)\pi}{8a^2} iz^2\right)\right],$$

$$g(z) = a\left[\exp\left(\frac{(4k+3)\pi}{8a^2} iz^2\right) - \exp\left(-\frac{(4k+3)\pi}{8a^2} iz^2\right)\right], \quad q(z) = \frac{(4k+3)\pi}{8a^2} iz^2,$$

where $a \in \mathbb{C}$, $a \neq 0$ and $k \in \mathbb{N}$. It is easy to check that $q(g) = -q(f) - (2k+1.5)\pi i$ and $f(g) = g(f)$.

The motivation for this research comes from the following open question in complex dynamics.

Let f be a nonlinear meromorphic function. We define

$$F = F(f) = \{z \in \overline{\mathbb{C}} : \text{ the sequence } \{f^n\} \text{ is well defined and normal at } z\}$$

and

$$J = J(f) = \overline{\mathbb{C}} - F(f),$$

where $\overline{\mathbb{C}} = \overline{\mathbb{C}} \cup \{\infty\}$, and the concept *normal* is in the sense of Montel. $F(f)$ and $J(f)$ are called the Fatou and Julia sets of f, respectively. When there is no confusion, we briefly write F and J instead of $F(f)$ and $J(f)$. Clearly $F(f)$ is open and it is well-known that $J(f)$ is a nonempty perfect set which either coincides with $\mathbb{C}$ or is nowhere dense in $\mathbb{C}$. For the basic results in the dynamical system theory of transcendental functions, we refer the reader to the books [9] and [13].

OPEN QUESTION 1 (Baker [1]). For two given permutable transcendental entire functions f and g, does it follow that $F(f) = F(g)$?

This is a difficult question to answer. So far, affirmative answers to special cases of functions of f and g have been obtained (see [1, 16, 18]). When f and g are permutable rational functions, Fatou [4, 5, 6] and Julia [10] proved that they have the same Julia set.

COROLLARY 1.3. *Let f and g satisfy the assumptions of Theorem 1.1 or Theorem 1.2. Then $J(f) = J(g)$.*

2. Some Lemmas

LEMMA 2.1 ([7]). *Let $G_0, G_1, \ldots, G_m$ and f be non-constant entire functions and let $h_0, h_1, \ldots, h_m$ ($m \geq 1$) be nonzero meromorphic functions. Suppose that K is a positive number and $\{r_i\}$is an unbounded monotone increasing sequence of positive numbers such that, for each $j \geq 1$,*

$$T(r_j, h_i) \leq KT(r_j, f) \quad (i = 0, \ldots, m)$$

and

$$T(r_j, f') \leq (1 + o(1))T(r_j, f).$$

If

$$h_0 G_0(f) + h_1 G_1(f) + \cdots + h_m G_m(f) \equiv 0$$

then there exist polynomials $\{p_j\}$ ($j = 0, 1, \ldots, m$), not all identically zero, such that

$$p_0(z)G_0(z) + p_1(z)G_1(z) + \cdots + p_m(z)G_m(z) \equiv 0.$$

LEMMA 2.2 ([3]). *Let $f_j(z)$ ($j = 1, 2, \ldots, n$) and $g_j(z)$ ($j = 1, 2, \ldots, n$, $n \geq 2$) be two systems of entire functions satisfying the following conditions:*

(1) $\sum_{j=1}^{n} f_j(z)e^{g_j(z)} \equiv 0$;
(2) *for* $1 \leq j, k \leq n$, $j \neq k$, $g_j(z) - g_k(z)$ *is non-constant;*
(3) *for* $1 \leq h, k \leq n$, $h \neq k$, $1 \leq j \leq n$, $T(r, f_j) = o\{T(r, e^{g_h - g_k})\}$.

Then $f_j(z) \equiv 0$ ($j = 1, 2, \ldots, n$).

To state the following result, we denote by $W(f_1, f_2, \ldots, f_n)$ the Wronskian of the functions $f_1, \ldots, f_n$:

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

LEMMA 2.3 ([15], Problem 60, Chapter VII). *Let $f_j(z)$ ($j = 1, 2, \ldots, n$) be transcendental entire functions. If $W(f_1, \ldots, f_n) \equiv 0$ but $W(f_1, \ldots, f_{n-1}) \not\equiv 0$, then there exist constants $c_1, c_2, \ldots, c_{n-1}$ such that*

$$f_n(z) = c_1 f_1(z) + c_2 f_2(z) + \cdots + c_{n-1} f_{n-1}(z).$$

This implies the following lemma.

LEMMA 2.4. *If $f_j(z)$ $(j = 1, 2, \ldots, n)$ are linearly independent transcendental entire functions then $W(f_1, \ldots, f_n) \not\equiv 0$.*

LEMMA 2.5 ([19]). *Let f and g be two permutable entire functions satisfying*

$$0 < \rho(f) < \lambda(f) < \infty, \quad \lambda(g) < \infty.$$

Then for any given positive integer n there exist a positive constant K and a sequence $\{r_j\}$ tending to ∞ such that

$$T\left(r_j, g^{(i)}\right) \leq KT(r_j, f)$$

for all $j \geq 1$ and $0 \leq i \leq n$.

LEMMA 2.6 ([12]). *If f and g are two permutable transcendental entire functions and there exists a nonconstant polynomial $\Phi(x, y)$ in both x and y such that $\Phi(f, g) \equiv 0$, then $J(f) = J(g)$.*

LEMMA 2.7. *Let $n \geq 1$, f and g be two permutable transcendental entire functions with $\rho(f) > 0$ and $\lambda(g) < \infty$, and let $p_i(z)$ $(0 \leq i \leq n)$ and $p(z)$ be polynomials. If*

(1) *$p_n(z) \not\equiv 0$ and $p_0(z) \not\equiv 0$,*
(2) *$f(z)$ satisfies the following differential equation:*

$$p_n(z)f^{(n)}(z) + p_{n-1}(z)f^{(n-1)}(z) + \cdots + p_1(z)f'(z) + p_0(z)f(z) + p(z) = 0, \tag{2.1}$$

(3) *$f(z)$ cannot be a solution of any linearly differential equation with polynomial coefficients of order $\leq n - 1$,*

then there exist four polynomials $Q(z)$, $Q_0(z)$, $Q_{n-1}(z)$ and $Q_n(z)$, with $Q_0(z) \not\equiv 0$ and $Q_n(z) \not\equiv 0$ such that

$$Q_{n-1}(f)p_n(g)\left(\frac{f'}{g'}\right)^n - Q_n(f)\left[p_n(g)\frac{a_n f'^{n-2}(f''g' - f'g'')}{(g')^{n+1}} + p_{n-1}(g)\left(\frac{f'}{g'}\right)^{n-1}\right] = 0,$$

$$Q_0(f)p_n(g)\left(\frac{f'}{g'}\right)^n - Q_n(f)p_0(g) = 0 \tag{2.2}$$

and

$$Q(f)p_n(g)\left(\frac{f'}{g'}\right)^n - Q_n(f)p(g) = 0, \tag{2.3}$$

where $a_1 = a_2 = 1$ and $a_n = n(n-1)/2$ for $n \geq 3$.

PROOF. From (2.1) we see that $\lambda(f) < \infty$ (see [11]). By (1.1) we have

$$
\begin{aligned}
&f'(g)g' = g'(f)f',\\
&f''(g)g'^2 + f'(g)g'' = g''(f)f'^2 + g'(f)f'',\\
&f'''(g)g'^3 + 3f''(g)g'g'' + f'(g)g''' = g'''(f)f'^3 + 3g''(f)f'f'' + g'(f)f''',\\
&f^{(4)}(g)g'^4 + 6f'''(g)g'^2g'' + f''(g)A_{4,2}(g', g'', g''') + f'(g)g^{(4)}\\
&\quad = g^{(4)}(f)f'^4 + 6g'''(f)f'^2f'' + g''(f)B_{4,2}(f', f'', f''') + g'(f)f^{(4)},\\
&f^{(5)}(g)g'^5 + 10f^{(4)}(g)g'^3g'' + f'''(g)A_{5,3}(g', \ldots, g^{(4)}) + f''(g)A_{5,2}(g', \ldots, g^{(4)})\\
&\quad + f'(g)g^{(5)}\\
&\quad = g^{(5)}(f)f'^5 + 10g^{(4)}(f)f'^3f'' + g'''(f)B_{5,3}(f', \ldots, f^{(4)})\\
&\qquad + g''(f)B_{5,2}(f', \ldots, f^{(4)}) + g'(f)f^{(5)},\\
&\qquad\cdots\\
&f^{(n)}(g)(g')^n + f^{(n-1)}(g)a_n(g')^{n-2}g'' + f^{(n-2)}(g)A_{n,n-2}(g', \ldots, g^{(n-1)}) + \cdots\\
&\quad + f''(g)A_{n,2}(g', \ldots, g^{(n-1)}) + f'(g)g^{(n)}\\
&\quad = g^{(n)}(f)(f')^n + g^{(n-1)}(f)a_n(f')^{n-2}f'' + g^{(n-2)}(f)B_{n,n-2}(f', \ldots, f^{(n-1)}) + \cdots\\
&\qquad + g''(f)B_{n,2}(f', \ldots, f^{(n-1)}) + g'(f)f^{(n)},
\end{aligned}
$$

where $A_{i,j}(g', \ldots, g^{(i-2)})$ $(i \geq 3, 2 \leq j \leq i-2)$ are polynomials of $g', g'' \ldots, g^{(i-2)}$ and $B_{i,j}(f', \ldots, f^{(i-2)})$ $(i \geq 3, 2 \leq j \leq i-2)$ are polynomials of $f', f'', \ldots, f^{(i-2)}$. Solving the above system yields

$$
(2.4)\quad
\begin{cases}
f'(g) = \dfrac{f'}{g'}g'(f),\\[2mm]
f''(g) = \left(\dfrac{f'}{g'}\right)^2 g''(f) + \left[\dfrac{f''}{g'^2} - \dfrac{f'g''}{g'^3}\right]g'(f),\\[2mm]
f'''(g) = \left(\dfrac{f'}{g'}\right)^3 g'''(f) + \left[\dfrac{3f'(f''g' - f'g'')}{g'^4}\right]g''(f) + C_{3,1}(f', g')g'(f),\\[2mm]
f^{(4)}(g) = \left(\dfrac{f'}{g'}\right)^4 g^{(4)}(f) + \left[\dfrac{6(f')^2(f''g' - f'g'')}{g'^5}\right]g'''(f)\\
\qquad + C_{4,2}(f', g')g''(f) + C_{4,1}(f', g')g'(f),\\
\qquad \ldots,\\
f^{(n)}(g) = \left(\dfrac{f'}{g'}\right)^n g^{(n)}(f) + \left[\dfrac{a_n(f')^{n-2}(f''g' - f'g'')}{(g')^{n+1}}\right]g^{(n-1)}(f)\\
\qquad + C_{n,n-2}(f', g')g^{n-2}(f) + \cdots + C_{n,1}(f', g')g'(f),
\end{cases}
$$

where $C_{i,j}(f', g')$ $(i \geq 3, 1 \leq j \leq i-2)$ are rational functions of $g', g'', \ldots, g^{(i)}$ and $f', f'', \ldots, f^{(i)}$.

Replacing z by $g(z)$ in Equation (2.1) yields

(2.5)

$$p_n(g)f^{(n)}(g) + p_{n-1}(g)f^{(n-1)}(g) + \cdots + p_1(g)f'(g) + p_0(g)f(g) + p(g) = 0.$$

Substituting (1.1) and (2.4) into (2.5) we deduce that

(2.6)

$$p_n(g)\left(\frac{f'}{g'}\right)^n g^{(n)}(f) + \left[p_n(g)\frac{a_n(f')^{n-2}(f''g'-f'g'')}{(g')^{n+1}} + p_{n-1}(g)\left(\frac{f'}{g'}\right)^{n-1}\right]g^{(n-1)}(f)$$
$$+ D_{n,n-2}(f',g')g^{(n-2)}(f) + \cdots + D_{n,1}(f',g')g'(f) + p_0(g)g(f) + p(g) = 0,$$

where $D_{n,i}(f',g')$ $(n \geq 3,\ 1 \leq i \leq n-2)$ are rational functions of $g, g', g'', \ldots, g^{(n)}$ and $f', f'', \ldots, f^{(n)}$.

Claim 2.8. *Let $\{r_j\}_{j=1}^{\infty}$ tending to ∞ be the sequence of positive numbers in Lemma 2.5. Then there exists a positive number K such that, for sufficiently large j,*

$$T\left(r_j, p_n(g)\left(\frac{f'}{g'}\right)^n\right) \leq KT(r_j, f),$$
$$T\left(r_j, p_n(g)\frac{a_n(f')^{n-2}(f''g'-f'g'')}{(g')^{n+1}} + p_{n-1}(g)\left(\frac{f'}{g'}\right)^{n-1}\right) \leq KT(r_j, f) \quad \text{and}$$
$$T\left(r_j, D_{n,i}(f',g')\right) \leq KT(r_j, f)$$

for all $n \geq 3,\ 1 \leq i \leq n-2$.

Proof of Claim 2.8. We shall prove a more general result.
Let $P(f, f', \ldots, f^{(n)}, g, g', \ldots, g^{(n)})$ be a linear combination of

$$V(z) = b(z)f(z)^{s_0}f'(z)^{s_1}\cdots f^{(n)}(z)^{s_n}g(z)^{t_0}g'(z)^{t_1}\cdots g^{(n)}(z)^{t_n},$$

where s_i, t_i $(0 \leq i \leq n)$ are integers and $b(z)$ is a rational function. We shall prove that there exists a positive constant K such that, for all sufficiently large j,

$$T(r_j, P) \leq KT(r_j, f).$$

In fact, by Nevanlinna's Logarithmic Derivative Lemma, (see [8, Page 105]) we have $T(r, f') \leq T(r, f) + O(\log r)$. Then, for $0 \leq i \leq n$,

$$T(r_j, f^{(i)}) \leq T(r_j, f) + O(\log r).$$

Since $\rho(f) > 0$, $\liminf_{r\to\infty}(\log T(r, f)/\log r) = \rho(f) > 0$. Thus, for sufficiently large r,

$$\log r \leq \frac{\rho(f)}{2}\log T(r, f) = o\{T(r, f)\}.$$

Combining this with the above inequality implies that

$$T\left(r_j, f^{(i)}\right) \leq 2T\left(r_j, f\right). \tag{2.7}$$

Now, by Lemma 2.5, there exists a positive constant K_1 such that

$$T\left(r_j, g^{(i)}\right) \leq K_1 T\left(r_j, f\right) \quad \text{for } j \geq 1 \text{ and } 0 \leq i \leq n. \tag{2.8}$$

By Nevanlinna's First Fundamental Theorem, $T(r, 1/f) \leq 2T(r, f)$ for sufficiently large r. Note that $T(r, b) = o(T(r, f))$. Thus, from (2.7) and (2.8),

$$T(r_j, V) \leq T\left(r_j, b\right) + \sum_{i=0}^{n} 2|s_i| T\left(r_j, f^{(i)}\right) + \sum_{i=0}^{n} 2|t_i| T\left(r_j, g^{(i)}\right) \leq K_2 T\left(r_j, f\right)$$

for some positive constant K_2 and for sufficiently large j. Therefore, there exists a positive constant K such that

$$T(r_j, P) \leq KT(r_j, f)$$

for sufficiently large j. Claim 2.8 follows. □

Now by (2.6), Claim 2.8 and Lemma 2.1, there exist $n+2$ polynomials $Q_n(z)$, $Q_{n-1}(z), \ldots, Q_1(z)$, $Q_0(z)$ and $Q(z)$, not all identically zero, such that

$$Q_n(z)g^{(n)}(z) + Q_{n-1}(z)g^{(n-1)}(z) + \cdots + Q_0(z)g(z) + Q(z) = 0.$$

Substituting z by $f(z)$ in this equation, we get

$$Q_n(f)g^{(n)}(f) + Q_{n-1}(f)g^{(n-1)}(f) + \cdots + Q_0(f)g(f) + Q(f) = 0.$$

Eliminating the term $g^{(n)}(f)$ from this and (2.6), we have

$$H_{n-1}g^{(n-1)}(f) + H_{n-2}g^{(n-2)}(f) + \cdots + H_1 g'(f) + H_0 g(f) + H = 0, \tag{2.9}$$

where

$$\begin{aligned} H_{n-1} &= Q_{n-1}(f)p_n(g)\left(\frac{f'}{g'}\right)^n \\ &\quad - Q_n(f)\left[p_n(g)\frac{a_n(f')^{n-2}(f''g' - f'g'')}{(g')^{n+1}} + p_{n-1}(g)\left(\frac{f'}{g'}\right)^{n-1}\right], \\ &\ldots, \\ H_0 &= Q_0(f)p_n(g)\left(\frac{f'}{g'}\right)^n - Q_n(f)p_0(g), \\ H &= Q(f)p_n(g)\left(\frac{f'}{g'}\right)^n - Q_n(f)p(g). \end{aligned}$$

CLAIM 2.9. *$H_i \equiv 0$ for $0 \le i \le n-1$ and $H \equiv 0$.*

PROOF OF CLAIM 2.9. Without loss of generality, we suppose on the contrary that $H_{n-1} \not\equiv 0$.

Then, from (1.1), (2.4) and (2.9) we deduce that

$$H_{n-1}\left(\frac{g'}{f'}\right)^{n-1} f^{(n-1)}(g) + E_{n-1,1}(f', g') f^{(n-2)}(g) + \cdots + E_{n-1,n-2}(f', g') f'(g)$$
$$+ H_0 f(g) + H = 0,$$

where $E_{n-1,i}(f', g')$ $(n \ge 3, 1 \le i \le n-2)$ are rational functions of $g, g', g'', \ldots, g^{(n)}$ and $f', f'', \ldots, f^{(n)}$. By Claim 2.8 and Lemma 2.1, there exist $n+1$ polynomials $t_{n-1}(z), t_{n-2}(z), \ldots, t_1(z), t_0(z)$ and $t(z)$, not all identically zero, such that

$$t_{n-1}(z) f^{(n-1)}(z) + t_{n-2}(z) f^{(n-2)}(z) + \cdots + t_0(z) f(z) + t(z) = 0.$$

This contradicts condition 3 of the lemma. Claim 2.9 follows. □

Thus, we have

$$Q_{n-1}(f) p_n(g)\left(\frac{f'}{g'}\right)^n - Q_n(f)\left[p_n(g)\frac{a_n f'^{n-2}(f''g' - f'g'')}{g'^{n+1}} + p_{n-1}(g)\left(\frac{f'}{g'}\right)^{n-1}\right] = 0,$$

$$Q_0(f) p_n(g)\left(\frac{f'}{g'}\right)^n - Q_n(f) p_0(g) = 0 \quad \text{and} \tag{2.10}$$

$$Q(f) p_n(g)\left(\frac{f'}{g'}\right)^n - Q_n(f) p(g) = 0.$$

CLAIM 2.10. *$Q_n \not\equiv 0$ and $Q_0 \not\equiv 0$.*

PROOF OF CLAIM 2.10. In fact, if $Q_n \equiv 0$ then, by the same arguments as used in the proof of Claim 2.8, we can deduce that $f(z)$ must be a transcendental entire solution of some differential equation with order at most $n-1$ and with polynomial coefficients. This is a contradiction. Hence $Q_n \not\equiv 0$. It follows from (2.10) that $Q_0 \not\equiv 0$. Claim 2.10 follows. □

This completes the proof of Lemma 2.7. □

3. Proof of Theorem 1.1

By (2.2) and (2.3), we have

$$\frac{Q(f)}{Q_0(f)} = \frac{p(g)}{p_0(g)}. \tag{3.1}$$

Let

$$R_1(z) = \frac{Q(z)}{Q_0(z)}, \qquad R_2(z) = \frac{p(z)}{p_0(z)}.$$

By assumption, $p(z)/p_0(z)$ is not a constant. Therefore $R_2(z)$ is not constant. Also $R_1(z)$ is not constant by (3.1).

4. Proof of Theorem 1.2

Recall that

$$f(z) = p(z) + p_1(z)e^{q_1(z)} + p_2(z)e^{q_2(z)} + \cdots + p_n(z)e^{q_n(z)}.$$

Without loss of generality, we assume that $\deg q_1 \le \deg q_2 \le \cdots \le \deg q_n$. Then $\rho(f) = \lambda(f) = \max\{\deg q_1, \ldots, \deg q_n\} = \deg q_n$.

For $1 \le i \le n$ and $0 \le j \le n$, set

$$u_{i,j} = \frac{(p_i e^{q_i})^{(j)}}{e^{q_i}}. \tag{4.1}$$

It is easy to see that all $u_{i,j}$ are non-zero polynomials. Note from (4.1) that

$$u'_{i,j} + q'_i u_{i,j} = u_{i,j+1} \tag{4.2}$$

for all $j \ge 0$ and $1 \le i \le n$. From

$$f = p + p_1 e^{q_1} + p_2 e^{q_2} + \cdots + p_n e^{q_n} \tag{4.3}$$

we get

$$f^{(j)} = p^{(j)} + u_{1,j}e^{q_1} + u_{2,j}e^{q_2} + \cdots + u_{n,j}e^{q_n}, \tag{4.4}$$

for all $j \ge 0$.

Set

$$A = \begin{vmatrix} p_1 & p_2 & \cdots & p_n \\ u_{1,1} & u_{2,1} & \cdots & u_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{1,n-1} & u_{2,n-1} & \cdots & u_{n,n-1} \end{vmatrix}, \quad R_0 = \begin{vmatrix} u_{1,n} & u_{2,n} & \cdots & u_{n,n} \\ u_{1,1} & u_{2,1} & \cdots & u_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{1,n-1} & u_{2,n-1} & \cdots & u_{n,n-1} \end{vmatrix},$$

$$B_1 = \begin{vmatrix} f - p & p_2 & \cdots & p_n \\ f' - p' & u_{2,1} & \cdots & u_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ (f-p)^{(n-1)} & u_{2,n-1} & \cdots & u_{n,n-1} \end{vmatrix}, \quad \ldots$$

$$B_n = \begin{vmatrix} p_1 & p_2 & \cdots & f-p \\ u_{1,1} & u_{2,1} & \cdots & f'-p' \\ \vdots & \vdots & \ddots & \vdots \\ u_{1,n} & u_{2,n} & \cdots & (f-p)^{(n-1)} \end{vmatrix}.$$

It is easy to check that A and R_0 are two polynomials with $A \not\equiv 0$ and $R_0 \not\equiv 0$. In fact, if $A \equiv 0$, that is

$$\begin{vmatrix} p_1 e^{q_1} & p_2 e^{q_2} & \cdots & p_n e^{q_n} \\ u_{1,1} e^{q_1} & u_{2,1} e^{q_2} & \cdots & u_{n,1} e^{q_n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{1,n-1} e^{q_1} & u_{2,n-1} e^{q_2} & \cdots & u_{n,n-1} e^{q_n} \end{vmatrix} \equiv 0,$$

then

$$\begin{vmatrix} p_1 e^{q_1} & p_2 e^{q_2} & \cdots & p_n e^{q_n} \\ (p_1 e^{q_1})' & (p_2 e^{q_2})' & \cdots & (p_n e^{q_n})' \\ \vdots & \vdots & \ddots & \vdots \\ (p_1 e^{q_1})^{(n-1)} & (p_2 e^{q_2})^{(n-1)} & \cdots & (p_n e^{q_n})^{(n-1)} \end{vmatrix} \equiv 0.$$

Now, by Lemma 2.4, the functions $p_1 e^{q_1}$, $p_2 e^{q_2}, \ldots, p_n e^{q_n}$ are linearly dependent. This obviously contradicts Lemma 2.2. Similarly we can show that $R_0 \not\equiv 0$.

Now from (4.3) and (4.4), we deduce that

$$e^{q_s} = B_s / A \quad (s = 1, \ldots, n).$$

Substituting these into (4.4) for $j = n$, we have

$$A f^{(n)} = p^{(n)} + u_{1,n} B_1 + u_{2,n} B_2 + \cdots + u_{n.n} B_n.$$

Note that each B_i $(1 \le i \le n)$ is a linear combination of $f, f', \ldots, f^{(n-1)}$. We deduce that

$$A f^{(n)} + R_{n-1} f^{(n-1)} + \cdots + R_1 f' + R_0 f + R = 0,$$

where $R_{n-1}, \ldots, R_0$ and R are polynomials.

Further, f cannot be a solution of a differential equation

$$t_{n-1} f^{(n-1)} + t_{n-2} f^{(n-2)} + \cdots + t_1 f' + t_0 f + t = 0, \tag{4.5}$$

where $t_{n-1}, \ldots, t_1, t_0$ and t are polynomials, not all of them zero. For suppose to the contrary that $f(z)$ is a solution of (4.5). Then this, (4.3) and (4.4) (with $j = n$) imply that

$$\begin{aligned} &(u_{1,n-1} t_{n-1} + u_{1,n-2} t_{n-2} + \cdots + u_{1,1} t_1 + t_0 p_1) e^{q_1} + \cdots \\ &\quad + (u_{n,n-1} t_{n-1} + u_{n,n-2} t_{n-2} + \cdots + u_{n,1} t_1 + t_0 p_n) e^{q_n} \\ &\quad\quad + (t_{n-1} p^{(n-1)} + t_{n-2} p^{(n-2)} + \cdots + t_1 p + t) \equiv 0. \end{aligned}$$

Combining this with Lemma 2.2, we get

$$\begin{aligned} u_{1,n-1}t_{n-1} + u_{1,n-2}t_{n-2} + \cdots + u_{1,1}t_1 + t_0 p_1 &= 0 \\ &\cdots \\ u_{n,n-1}t_{n-1} + u_{n,n-2}t_{n-2} + \cdots + u_{n,1}t_1 + t_0 p_n &= 0. \end{aligned}$$

These obviously contradict $A \not\equiv 0$.

Thus the conditions of Lemma 2.7 are satisfied, so there exist polynomials $Q(z)$, $Q_0(z)$, $Q_{n-1}(z)$ and $Q_n(z)$, with $Q_0(z) \not\equiv 0$ and $Q_n(z) \not\equiv 0$, such that

$$Q_{n-1}(f)A(g)\left(\frac{f'}{g'}\right)^n - Q_n(f)\left[A(g)\frac{a_n f'^{n-2}(f''g' - f'g'')}{g'^{n+1}} + R_{n-1}(g)\left(\frac{f'}{g'}\right)^{n-1}\right] = 0,$$

$$Q_0(f)A(g)\left(\frac{f'}{g'}\right)^n - Q_n(f)R_0(g) = 0 \quad \text{and}$$

$$Q(f)A(g)\left(\frac{f'}{g'}\right)^n - Q_n(f)R(g) = 0.$$

We remark that

$$\begin{aligned} R_{n-1} &= -u_{1,n}(-1)^{n+1}\begin{vmatrix} p_2 & p_3 & \cdots & p_n \\ u_{2,1} & u_{3,1} & \cdots & u_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{2,n-2} & u_{3,n-2} & \cdots & u_{n,n-2} \end{vmatrix} \\ &\quad - u_{2,n}(-1)^{n+2}\begin{vmatrix} p_1 & p_3 & \cdots & p_n \\ u_{1,1} & u_{3,1} & \cdots & u_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{1,n-2} & u_{3,n-2} & \cdots & u_{n,n-2} \end{vmatrix} - \cdots \\ &\quad - u_{n,n}(-1)^{n+n}\begin{vmatrix} p_1 & p_2 & \cdots & p_{n-1} \\ u_{1,1} & u_{2,1} & \cdots & u_{n-1,1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{1,n-2} & u_{2,n-2} & \cdots & u_{n-1,n-2} \end{vmatrix} \\ &= -\begin{vmatrix} p_1 & p_2 & \cdots & p_n \\ u_{1,1} & u_{2,1} & \cdots & u_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{1,n-2} & u_{2,n-2} & \cdots & u_{n,n-2} \\ u_{1,n} & u_{2,n} & \cdots & u_{n,n} \end{vmatrix}. \end{aligned}$$

Similarly, for any i with $1 \le i \le n-1$, one has

$$R_{n-i} = -\begin{vmatrix} p_1 & p_2 & \cdots & p_n \\ \vdots & \vdots & \ddots & \vdots \\ u_{1,n-i-1} & u_{2,n-i-1} & \cdots & u_{n,n-i-1} \\ u_{1,n-i+1} & u_{2,n-i+1} & \cdots & u_{n,n-i+1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{1,n} & u_{2,n} & \cdots & u_{n,n} \end{vmatrix}.$$

Also, it is easy to verify that

(4.6) $$R = pR_0 + p'R_1 + \cdots + p^{(n-1)}R_{n-1} - p^{(n)}A,$$

with

(4.7) $$\deg R_0 \ge \max\{\deg R_i\ (1 \le i \le n-1), \deg A\}.$$

Here we only prove that $\deg R_0 \ge \deg A$. In fact, we have

(4.8) $$\deg R_0 = \deg A + \sum_{i=1}^{n}(\deg q_i - 1).$$

Set

$$Z = \begin{vmatrix} u_{1,1} & u_{2,1} & \cdots & u_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{1,n-1} & u_{2,n-1} & \cdots & u_{n,n-1} \\ u_{1,n} & u_{2,n} & \cdots & u_{n,n} \end{vmatrix}.$$

To establish (4.8), we need only prove that $\deg Z = \deg A + \sum_{i=1}^{n}(\deg q_i - 1)$. Rewrite Z by (4.2) as $Z = Z_{11} + Z_{12}$ where

$$Z_{11} = \begin{vmatrix} u_{1,1} & u_{2,1} & \cdots & u_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{1,n-1} & u_{2,n-1} & \cdots & u_{n,n-1} \\ u'_{1,n-1} & u'_{2,n-1} & \cdots & u'_{n,n-1} \end{vmatrix},$$

$$Z_{12} = \begin{vmatrix} u_{1,1} & u_{2,1} & \cdots & u_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ u_{1,n-1} & u_{2,n-1} & \cdots & u_{n,n-1} \\ q'_1 u_{1,n-1} & q'_2 u_{2,n-1} & \cdots & q'_n u_{n,n-1} \end{vmatrix}.$$

We easily deduce that $\deg Z = \deg Z_{12}$. Now we decompose Z_{12} as $Z_{12} = Z_{121} + Z_{122}$, where

$$Z_{121} = \begin{vmatrix} u_{1,1} & u_{2,1} & \cdots & u_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ u'_{1,n-2} & u'_{2,n-2} & \cdots & u'_{n,n-2} \\ q'_1 u_{1,n-1} & q'_2 u_{2,n-1} & \cdots & q'_n u_{n,n-1} \end{vmatrix},$$

$$Z_{122} = \begin{vmatrix} u_{1,1} & u_{2,1} & \cdots & u_{n,1} \\ \vdots & \vdots & \ddots & \vdots \\ q'_1 u_{1,n-2} & q'_2 u_{2,n-2} & \cdots & q'_2 u_{n,n-2} \\ q'_1 u_{1,n-1} & q'_2 u_{2,n-1} & \cdots & q'_n u_{n,n-1} \end{vmatrix}.$$

We easily deduce that $\deg Z_{12} = \deg Z_{122}$. Thus $\deg Z = \deg Z_{122}$. This procedure can be repeated. Finally we can assert that $\deg Z = \deg Z_{12\cdots 2}$, where

$$Z_{12\cdots 2} = \begin{vmatrix} q'_1 p_1 & q'_2 p_2 & \cdots & q'_n p_n \\ \vdots & \vdots & \ddots & \vdots \\ q'_1 u_{1,n-2} & q'_2 u_{2,n-2} & \cdots & q'_2 u_{n,n-2} \\ q'_1 u_{1,n-1} & q'_2 u_{2,n-1} & \cdots & q'_n u_{n,n-1} \end{vmatrix} = q'_1 q'_2 \cdots q'_n A,$$

so establishing (4.8)

From (4.6) it follows that $p \equiv$ constant implies $R/R_0 \equiv$ constant. We now prove the converse. Let us suppose that

$$\frac{R}{R_0} = c.$$

By (4.6),

$$(c - p)R_0 = p' R_1 + \cdots + p^{(n-1)} R_{n-1} - p^{(n)} A.$$

If $p \not\equiv$ constant, this contradicts (4.7).

Finally, the theorem follows from Lemma 2.7 and Theorem 1.1.

5. Proof of Corollary 1.3

By (2.2) and (2.3), we have

$$\frac{Q(f)}{Q_0(f)} = \frac{p(g)}{p_0(g)}. \tag{5.1}$$

Let

$$R_1(z) = \frac{Q(z)}{Q_0(z)}, \quad R_2(z) = \frac{p(z)}{p_0(z)}.$$

By assumption, $p(z)/p_0(z)$ is not a constant, therefore, $R_2(z)$ is not constant. Also $R_1(z)$ is not constant by (5.1).

We rewrite (5.1) in the form

$$\frac{Q_0(x)p(y) - Q(x)p_0(y)}{Q_0(x)p_0(y)} = 0 \tag{5.2}$$

and consider two subcases.

If $Q_0(x)p(y) - Q(x)p_0(y) \equiv$ constant, then by (5.2)

$$Q_0(x)p(y) - Q(x)p_0(y) \equiv 0.$$

Hence

$$\frac{Q(z)}{Q_0(z)} = \frac{p(z)}{p_0(z)} = S(z)$$

for some rational function $S(z)$. It follows from (5.1) that

$$S(f) = S(g).$$

Therefore $f = \pm g + c$ for some constant c. By Baker [1], we obtain $J(f) = J(g)$.

If $Q_0(x)p(y) - Q(x)p_0(y) \not\equiv$ constant, then by (3.1) we get a nonconstant polynomial $Q_0(x)p(y) - Q(x)p_0(y)$ such that

$$Q_0(f)p(g) - Q(f)p_0(g) \equiv 0.$$

The conclusion follows from Lemma 2.6.

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