Zeros and Nodal Lines of Modular Forms

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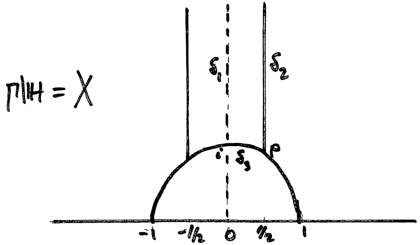
Zeros of Modular Forms

Classical modular forms

$$\Gamma = \mathrm{SL}_2(\mathbb{Z}) \quad \text{acting on } \mathbb{H}.$$

$$z \overset{\gamma}{\mapsto} \frac{az+b}{cz+d}, \quad \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma.$$



(i) $f(z)$ holomorphic in z.

(ii) $f(\gamma z) = (cz+d)^k f(z)$, weight k even, $k \geq 4$. Finite dimensional space.

A winding number argument or Riemann–Roch gives that $f \not\equiv 0$ has essentially $k/12$ zeros; $\nu_p(f)$ order of vanishing

$$\nu_\infty(f) + \frac{\nu_i(f)}{2} + \frac{\nu_\rho(f)}{3} + \sum_{p \in X} \nu_p(f) = \frac{k}{12}.$$

There are no real restrictions on the location of the zeros.

Arithmetically, we look at Hecke eigenforms:

- Hecke operators T_n act on the space of forms of weight k.
- T_n is defined via arithmetic correspondences $\langle 1, z \rangle = \Lambda_z$ lattice in $\mathbb{C}$ corresponding to z, $z \mapsto \tau$, where Λ_τ is index n in Λ_z.
- T_n's commute and can be diagonalized.
- $T_n T_m = \sum_{d|(n,m)} T_{\frac{nm}{d^2}}$.

f a Hecke eigenform. Where are its zeros?

Simplest such forms are Eisenstein series;

$$E_k(z) = \sum_{(c,d)=1} (cz + d)^{-k}, \qquad k \geq 4.$$

Theorem (Rankin–Swinnerton-Dyer)

All the zeros of E_k are on δ_3.

The rest of the forms are cusp forms, $f(i\infty) = 0$, or

$$f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e(nz), \qquad e(z) = e^{2\pi i z}.$$

Key facts about $\lambda_f(n)$:

(i) $\lambda_f(n)$ obey the same multiplicative laws as T_n.

(ii) $|\lambda_f(n)| \ll \sum_{d|n} 1$ (Ramanujan Conjecture — now Deligne Theorem).

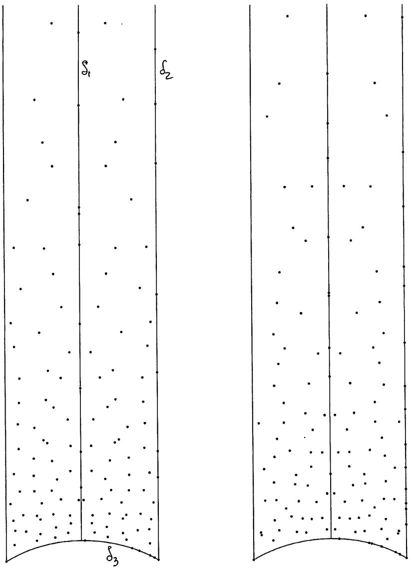
(iii) "Sato–Tate" law [Barnet-Lamb, Geraghty, Harris, Taylor (2010)]; for f fixed as $p \to \infty$, $\lambda_f(p)$ follows a statistical law.

Theorem (QUE Conjecture, now Holowinsky–Soundararajan Theorem (2010))

$c_f > 0$ normalizing constant, $\mu_f = c_f |f(z)|^2 y^k \frac{dx\,dy}{y^2}$ a probability measure on X, then $\mu_f \to \frac{3}{\pi} \frac{dx\,dy}{y^2}$ as $k \to \infty$.

QUE $\Longrightarrow$ zeros of f are equidistributed in $\mathcal{F}$ as $k \to \infty$; $\mathcal{Z}(f)$ the zero set,

$$\frac{|\mathcal{Z}(f) \cap \Omega|}{|\mathcal{Z}(f)|} \to \frac{\mathrm{Area}(\Omega)}{\mathrm{Area}(X)} \quad \text{as } k \to \infty.$$



Zeros for y < 4.5 of two cuspforms of weight 2000 as computed by F. Stromberg. All the zeros found for y > 10 were real.

Let $\delta = \delta_1 \cup \delta_2 \cup \delta_3$.

Symmetry: $\lambda_f(n)$ is real (T_n's are self-adjoint; in fact, $\lambda_f(n)$ lie in a totally real number field).

$\Longrightarrow f$ is real on δ_1 and δ_2 "and δ_3".

$\Longrightarrow \mathcal{Z}(f)$ is symmetric about reflections in each of $\delta_1, \delta_2, \delta_3$.

We call the zeros of f in δ real zeros.

How many real zeros should we expect?

- Treating $\lambda_f(n)$ as random for $n \leq k$ (this is generally believed for n less than square-root of the conductor $= k$ here) a probabilistic computation yields
  expect $N_{\mathrm{real}}(f) \sim c_1 \sqrt{k} \log k$.
  (agree with numerics).

Theorem (A. Ghosh–S. 2011)

$$N_{\mathrm{real}}(f) \gg k^{\frac{1}{4}-\frac{1}{80}}$$

[In recent work, K. Matomaki has removed the $1/80$.]

The proof actually produces this number of zeros on $\delta_1 \cup \delta_2$ but does not tell on which. It is much harder to produce zeros on each separately.

Theorem (Ghosh–S.)

$$|\mathcal{Z}(f) \cap \delta_j| \gg \log k \quad \textit{for each of } j = 1 \textit{ and } 2.$$

Window of opportunity: $$f(z) = \sum_{k=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e^{-2\pi ny} e(nx)$$

as $k \to \infty$, steepest descent with $\xi^k e^{-\xi}$, max at $\xi = k$, very localized. Leads to:

Lemma

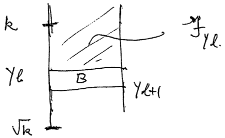
There is $\delta > 0$ such that for $1 \leq \ell \leq \sqrt{k}$ and $y_\ell = \frac{k-1}{4\pi\ell}$,

$$f(x + iy_\ell) = I(y_\ell)\left[\lambda_f(\ell)e(\ell x) + O(k^{-\delta})\right],$$

$I(y_\ell) \in (0, \infty)$.

$\Longrightarrow$ if $\lambda_f(\ell)$ is not too small, the phase of f on $0 \leq x \leq 1$, $y = y_\ell$ is ℓ,

$$\Longrightarrow \quad |\mathcal{Z}(f) \cap \mathcal{F}_{y_\ell}| = \ell.$$



"local QUE on these small sets"

$\Longrightarrow$ if $\lambda_f(\ell)$ and $\lambda_f(\ell+1)$ are both not small, then there is <u>one</u> zero of f in B and symmetry $\Longrightarrow$ the zero must be on one of δ_1 or δ_2.

Suggests (and is probably true) that for $y \geq \sqrt{k}$ all of the zeros of f are real!

So the problem becomes one of finding ℓ, ℓ' opposite parity and close to each other with $\lambda_f(\ell)$ and $\lambda_f(\ell')$ not small.

- Use Hecke relations and sieving arguments to find many primes or squares of primes in short intervals ...

Producing zeros on δ_1 is more difficult. Need to find $\ell < k^{1/2-\delta}$ with $\lambda_f(\ell) \leq -1/10$ say. Luckily techniques from sharp subconvexity for $L(s,f)$ [Peng 2003, Jutila Motohashi] and optimizations in smooth number arguments with differential delay euations [Kaisa Matmaki 2011] allow us to squeak in by the tiniest of margins.

Joint work with A. Ghosh and A. Reznikov

Zeros of Maass Forms

$\phi(z)$ a Maass form, $\phi : \mathbb{H} \to \mathbb{C}$ (real-valued)

(i) $\Delta\phi + \lambda\phi = 0$, $\lambda = \frac{1}{4} + t^2$.

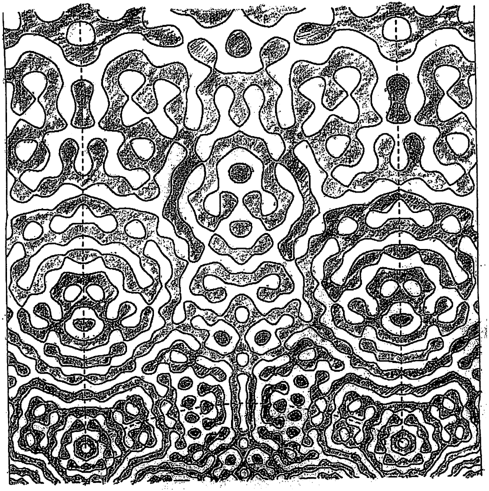
(ii) $\phi(\gamma z) = \phi(z)$, $\gamma \in \Gamma$.

Assume

- ϕ is cuspidal.
- ϕ is Hecke eigenform; $T_n\phi = \lambda_\phi(n)\phi$.

Zeros of ϕ are curves in X, 'nodal lines' denoted $\nu(\phi)$. The connected components of $X \setminus \nu(\phi)$ are the nodal domains. Their number is the number of zeros.

Below a picture of the zero set $\phi = 0$ of such a "Maass form" for $\mathrm{SL}_2(\mathbb{Z})$, $\lambda = \frac{1}{4} + t^2$, $t = 125.34\ldots$ (Hejhal–Rackner).
Is the zero set behaving randomly? How many components does it have?



58 nodal domains in A

A remarkable (if true!) conjecture of Bogomolny and Schmit (physicists) (2002) asserts that for such eigenstates of quantizations of classically chaotic systems,

$$N(\phi) = \# \text{ of nodal domains} \sim \frac{3\sqrt{3}-5}{\pi} n \quad \text{as } n \to \infty.$$

Here ϕ is the n-th eigenfunction. The constant (universal) comes from an exactly solvable critical bond percolation model!

In our example, $N_\phi(A) = 58$.

$$\frac{N_\phi(A) \cdot 4\pi}{\mathrm{Area}(A) \cdot R^2} = 0.0678 \ldots .$$

The prediction is $\frac{3\sqrt{3}-5}{\pi} = 0.0624 \ldots$.

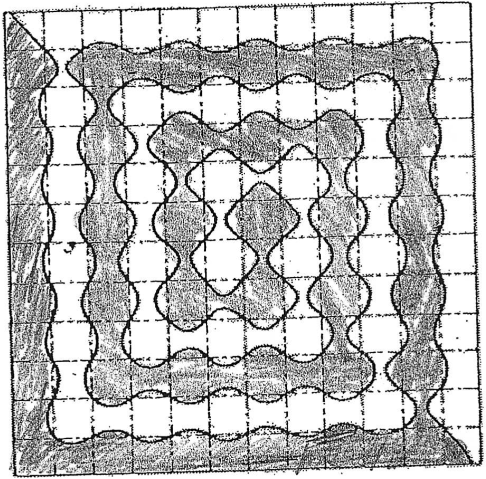
What do we know about such nodal domains in general?

(i) Courant nodal domain theorem: $N(\phi) \leq n$ if $\lambda = \lambda_n$.

(ii) Toth–Zelditch (2008) If $X = \Omega \subset \mathbb{R}^2$ (compact) domain with real analytic boundary, and ϕ satisfies a Neumann eigenfunction, then $\#\{\nu(\phi) \cap \partial\Omega\} \ll t$, $t^2 = \lambda$.

(iii) Donnelly–Fefferman (real analytic manifold): $t \ll \mathrm{length}(\nu(\phi)) \ll t$.

The trouble is giving a lower bound for $N(\phi)$. In general it need not grow!



Nodal domain of an eigenfunction on the square, $N(\phi) = 2$.
From Courant–Hilbert; Vol I. Thesis A. Stern, Gottingen, 1925.

Symmetry for X:

The isometry $z \mapsto -\bar{z}$ of $\mathbb{H}$ induces an isometry $\sigma : X \to X$.

$$\delta = \mathrm{Fix}(\sigma) = \{z : \sigma(z) = z\} = \delta_1 \cup \delta_2 \cup \delta_3.$$

ϕ is either even or odd with respect to σ. We stick to the even ones.

- If Ω is a nodal domain for ϕ, then so is $\sigma(\Omega)$. $\implies$ either $\sigma(\Omega) = \Omega$, we call Ω inert (or real), or $\sigma(\Omega) \cap \Omega = \emptyset$, we call Ω split.
- Ω is inert iff Ω meets δ nontrivially.

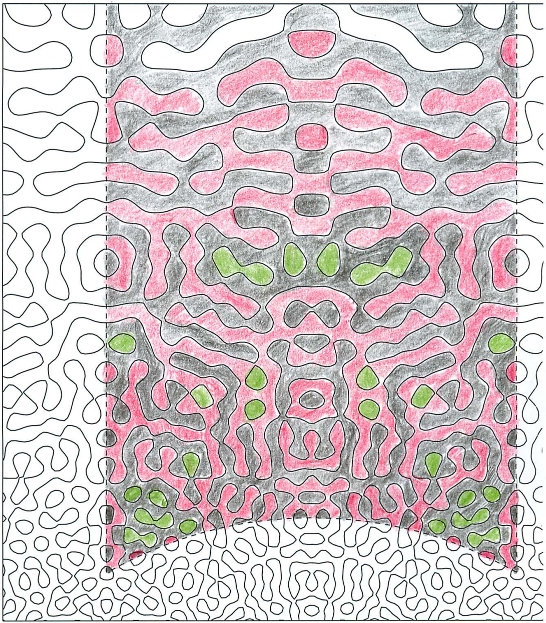
$$N_i(\phi) = \#\text{ inert nodal domains}$$
$$N_s(\phi) = \#\text{ split nodal domains (which is even)}$$
$$N(\phi) = N_i(\phi) + N_s(\phi).$$

Topological Proposition

Let n_ϕ be the number of sign changes of ϕ going around δ and let m_ϕ be the number of zeros of ϕ as one traverses δ. Then

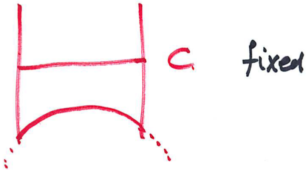
$$\frac{n_\phi}{2} + 1 \leq N_i\phi \leq m_\phi.$$

We are led to study the number of intersections of ν_ϕ with a given curve (namely δ).



Hejhal–Rackner nodal lines for $\lambda = 1/4 + R^2$, $R = 125.313840$

Restriction and intersection with closed horocycles C:



Theorem

(i) *Sharp L^2-restriction,* $\|\phi\|_2 = 1$, $\lambda = \frac{1}{4} + t^2$. *For* $\varepsilon > 0$,

$$t^{-\varepsilon} \ll_\varepsilon \| \phi|_C \|_2^2 \ll_\varepsilon t^\varepsilon.$$

(ii) $t^{1/12} \ll |\nu(\phi) \cap C| \ll t$.

The lower bound (ii) should probably also be t. The upper bound ("Bezout") (ii) can be proven without arithmetic assumption (J. Jung).

For δ, we need to assume a quantitative QUE (which follows from subconvexity for certain L-functions).

Theorem

Let $\beta \subset \delta$ be a compact segment. Assume subconvexity for automorphic L-functions. Then

$$1 \ll \| \phi|_\beta \|_2^2 \ll t^\varepsilon \qquad \longleftarrow \textit{(unconditional).}$$

Theorem

Let $\beta \subset \delta$ be a compact segment and assume the Lindelöf Hypothesis (which is a consequence of the Riemann Hypothesis) for automorphic L-functions. Then

$$t^{1/12} \ll |\nu(\phi) \cap \beta| \ll t \qquad \longleftarrow \textit{(unconditional).}$$

Corollary

Assume Lindelöf. Then

$$t^{1/12} \ll N_i^{(\beta)}(\phi) \ll t.$$

Here $N_i^{(\beta)}(\phi)$ is the number of inert nodal domains that meet β. Again, the upper bound is unconditional.

- In particular, the number of nodal domains goes to infinity with t!
- We don't know how to produce split domains, which presumably are most of them.
- For $y \gg t$, we have a quite complete understanding of the nodal domains. All are inert.

The proofs use the full force of analytic tools and results from automorphic forms — L-functions, Ramanujan bounds, ...

Also critical are asymptotics of classical Bessel functions, transition ranges, Airy function.

The basic method to produce sign changes of ϕ on C or on β is to show that

$$\int_\beta \phi \quad \text{and} \quad \int_\beta |\phi|$$

are unequal. This is achieved using the restriction (L^2) lower bounds and L^∞ upper bounds for ϕ (Iwaniec–S. 1995).

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