Developing mathematical thinking

through puzzles and games

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University of South Australia

2015 AustMS Conference

www.tinyurl.com/austms-15-aa

@nomad_penguin

So I’m going to talk about a new course at the University of South Australia, now halfway through its second offering.

By ‘course’ I mean one of the four subjects that students take in each semester.

As the title suggests, the aim is to use puzzles and games to develop mathematical thinking and problem solving skills in students. In particular, the course is intended for education students aiming to primary and middle school teachers. To fulfil their maths specialisation, these students need four mathematics content courses, and this is now one that they can elect to take. These students come in with mixed mathematical ability; some did not do maths in Year 12.

< Just a side note, these slides (including additional slides with weblinks) can be downloaded from this tinyurl that I’ll show again at the end >

So I’m going to whet your appetite right now with a tasty little puzzle.

What is the minimum number of steps required
to break the bar into its smallest pieces?

Puzzle-based learning, Michalewicz and Michalewicz

A rectangular chocolate bar consists of mxn small rectangles and you wish to break it into its smallest pieces. At each step, you can only pick up one piece and break it along any of its vertical or horizontal lines. How should you break the chocolate bar using the minimum number of steps (breaks)?

If you do not know the answer, which textbook would you search to discover the solution? Textbooks on optimization? Simulation? Strategies? Games? Other textbooks? Or it might be that someone wrote a book on chocolates where in Chapter 7 there is a full discussion on efficient breaking strategies of a chocolate bar? Very unlikely.

What I like about this example is that it is not immediately obvious (to some) how to go about solving the problem, or even what approach might be required, and so for me, it makes it an ideal puzzle for my course.

So by now, you might already have some questions about how you design and teach a course around puzzles and games. For instance you might want to know:

Your possible questions

- Why did I **create** this subject? What are the **aims**?
- What's in the **syllabus**?
- What does a **typical class** look like?
- How do you **assess** students?
- Where do the **puzzles and games** come from?
- …

Talk outline

- Where do the **puzzles and games** come from?
- Why did I **create** this subject? What are the **aims**?
- What's in the **syllabus**?
- What does a **typical class** look like?
- How do you **assess** students?
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- …

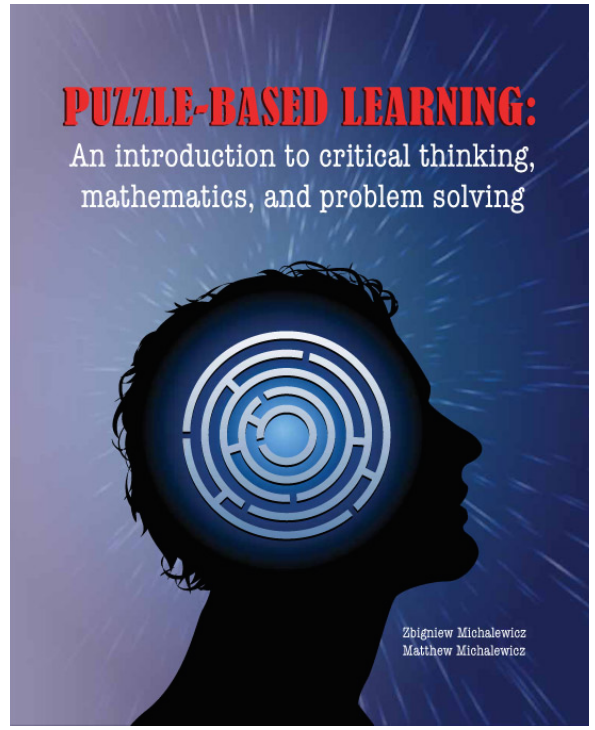
So my aim for this talk is a bit of a 'show and tell' and to also scout for connections with other people who teach like this.

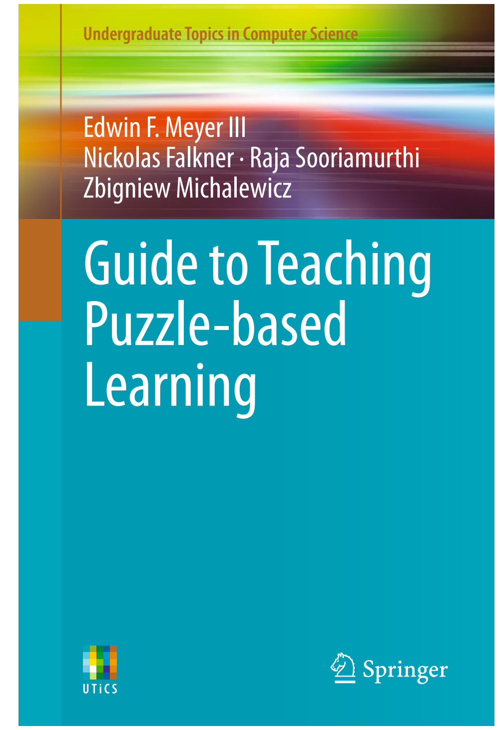
Why puzzles?

Puzzles and games are fun and can be very thought-provoking. They often engage people that don't traditionally 'like maths'— think of old examples like Rubik's Cube and Tetris, or new examples like Cheryl's birthday problem which went viral earlier this year.

People generally don't think that logical thinking or strategising is mathematics and so are more likely to 'give it a go'.

So where do these puzzles come from?

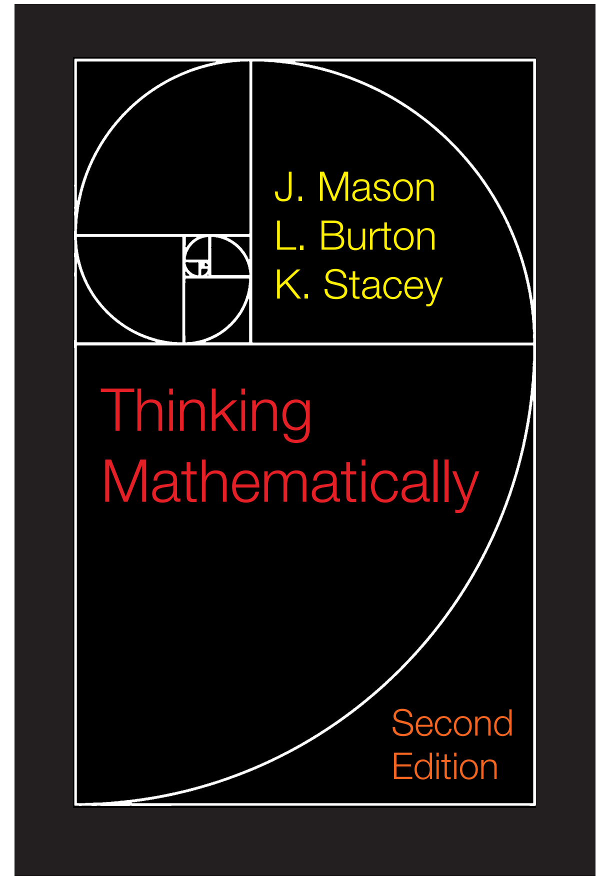




One source of puzzles—including the chocolate bar puzzle—is the puzzle-based learning approach of Michalewicz, at Uni Adelaide. Of course, the educational use of puzzles is not new, think of 20th century champions: Gyorgy Polya and Martin Gardner—another source of puzzles.

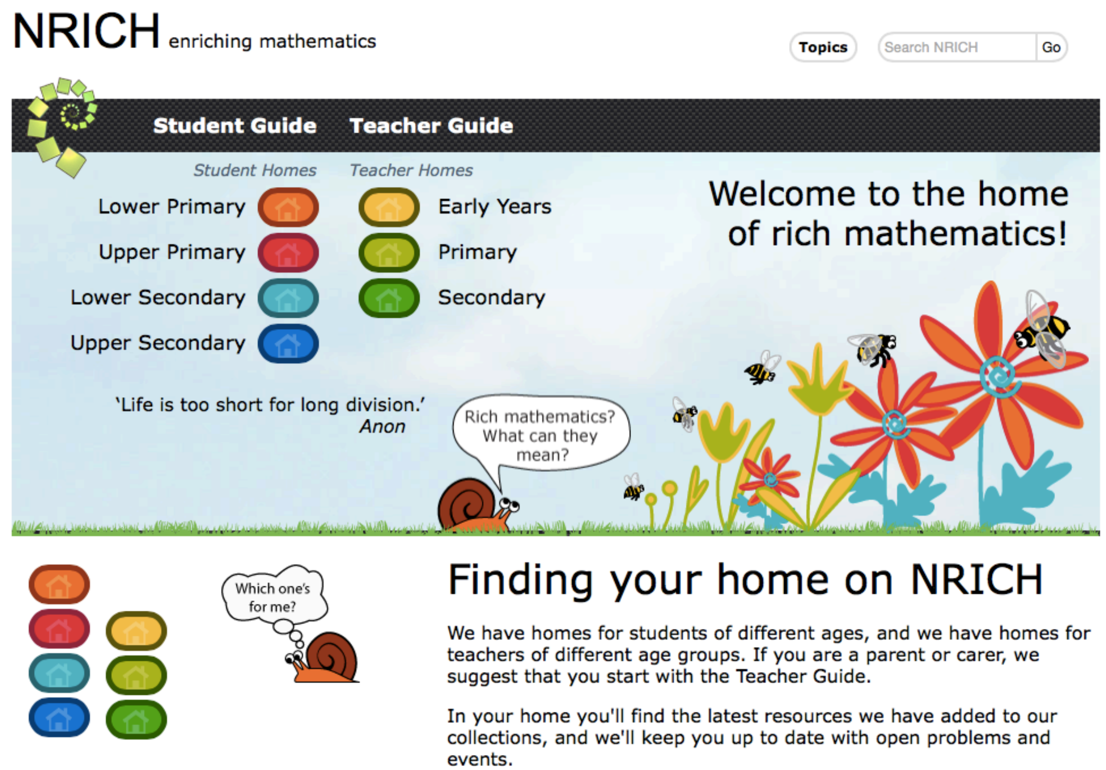
<click>

The puzzle-based learning subject at the University of Adelaide was the first structured subject using puzzles that I had seen, with a syllabus organised around various mathematical topics.



After mulling it over for a while, I realised that I was less concerned with the mathematical **topics** we covered, and more focussed on drawing out the different mathematical **processes** (which I'll say more about later). And so I discovered and fell completely in love with this book—another rich source of puzzles, and in the last chapter they are classified by content area.

nrich.maths.org



The Internet is a wide source of puzzles, including deep repositories like nrich maths.

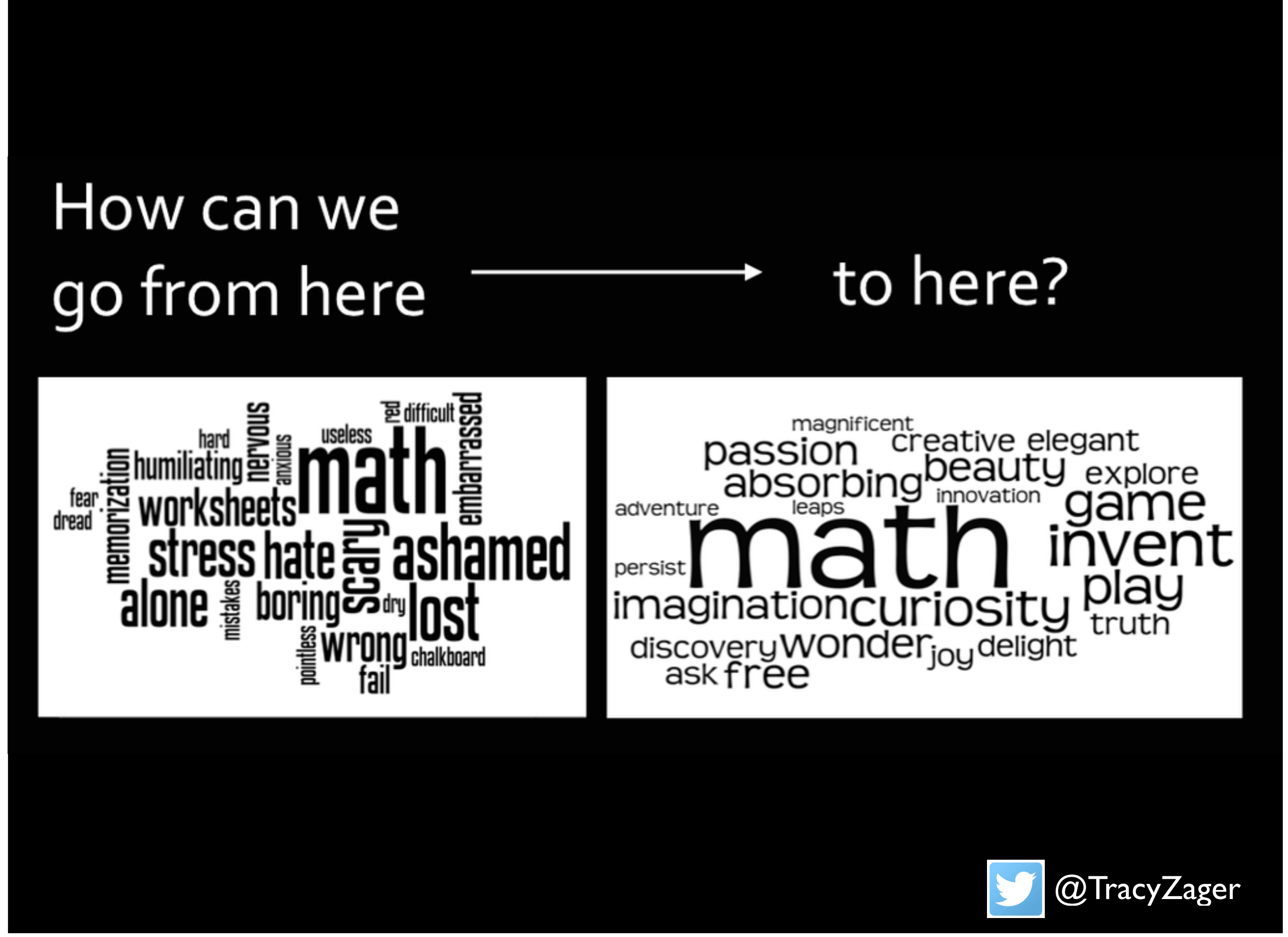
You'll notice that it is aimed from lower primary to upper secondary.

Here are some more fantastic resources aimed at primary and secondary educators, but with much for tertiary educators to learn from and adapt for their own classrooms.

Become connected -> Math Twitter Blogosphere

Tracy Zager, a maths coach in Maine in the US and Tracy works with pre-service and in-service elementary teachers

“Becoming the math teacher you wish you had: ideas and strategies from vibrant classrooms”



So in fact, Tracy has summed up in one slide (from a fabulous talk that I'd encourage you to watch) my overarching intention for this course.

The word cloud on the left is from students (actually it is from pre-service elementary teachers, which is distressing in itself).

The word cloud on the right is from mathematicians.

Why do some classrooms produce this response, when mathematics typically invokes this response from mathematicians?

My aim is to reproduce the collaborative, active and enjoyable environment in which mathematicians work.

Collaborative = Students work together. They communicate their thinking to one another.
Active = Students do, not watch. They explore. They discover. They create, and be creative.
Enjoyable = Students have fun. They play. They experience mathematical wonder.

I want my students to …

- appreciate **connections** between different areas of maths
- develop **multiple ways** to approach a problem

- experience the **messy process** of solving problems
- **revise** and **refine** their thinking
- get **stuck**

- work **collaboratively**
- improve maths **communication**

- acknowledge their **feelings** about maths
- be **alert** and **sensitised** to their students
- reflect on their own **thinking**
- build **confidence**

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Have fun!

- acknowledge their **feelings** about maths
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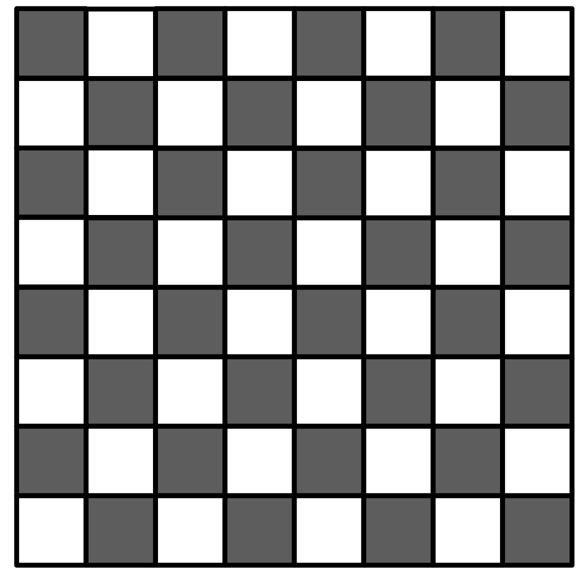
Chessboard Squares

It was once claimed that there are 204 squares on an ordinary chessboard. Can you justify this claim?



Thinking mathematically (2nd edn), Mason, Burton, Stacey

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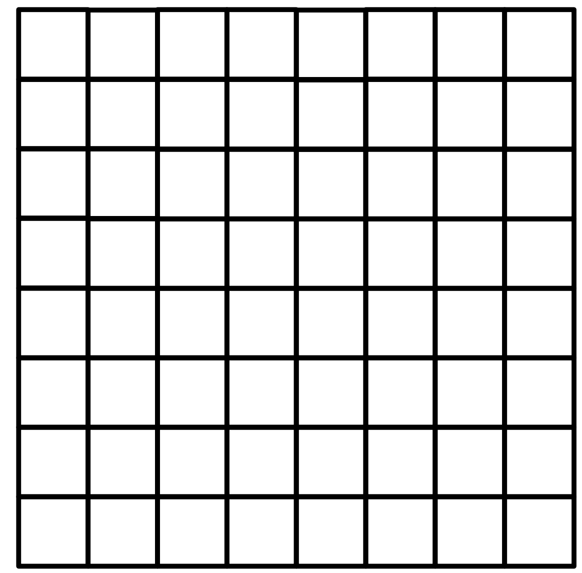
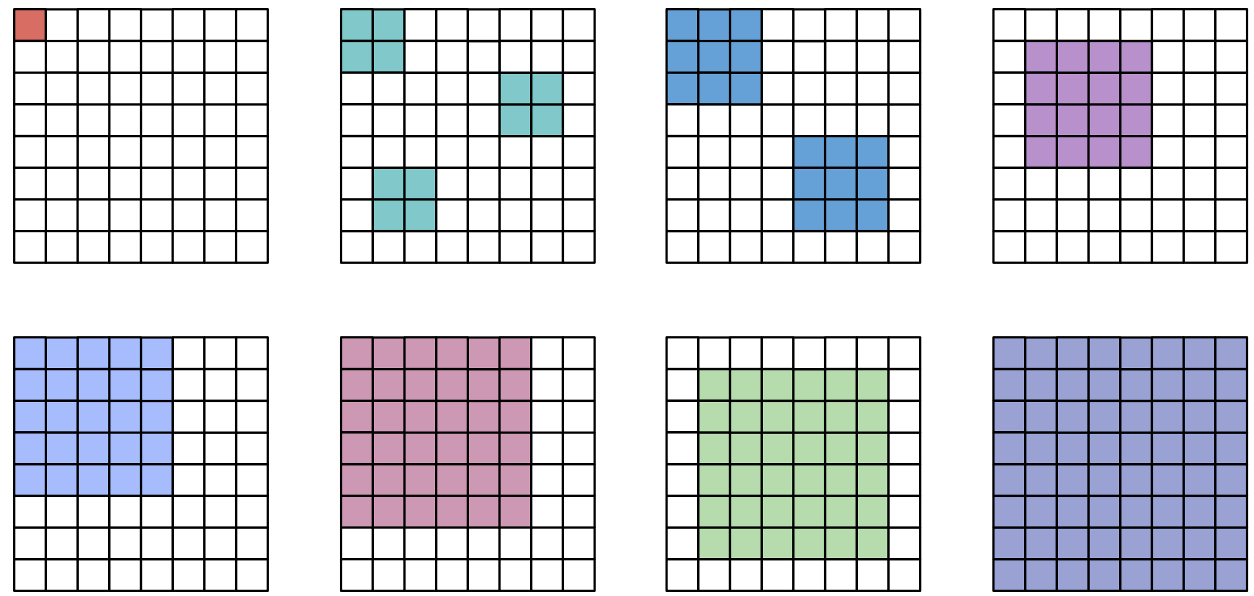


So a chessboard only has 64 squares. We must be counting some other squares. Talk to your neighbour about which kinds of squares we would be counting, and formulate a plan about how you would find them all.

Now, jump up a level and see if you can think about the mathematical processes that going on here.

Specialising

Chessboard Squares



Specialise— randomly, to get a feel for the problem.

Systematic

Chessboard Squares

Size	1×1	2×2	3×3	4×4	5×5	6×6	7×7	8×8
Number	64							1

Specialise—systematically.

Systematic

Chessboard Squares

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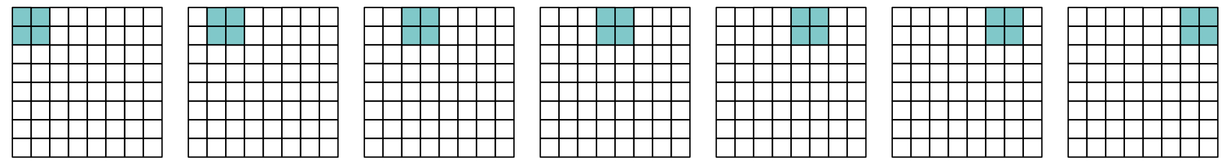

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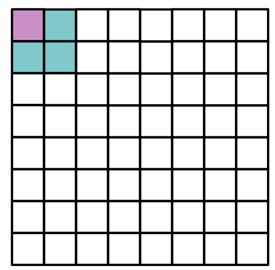

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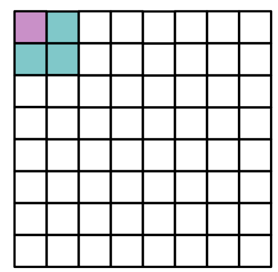

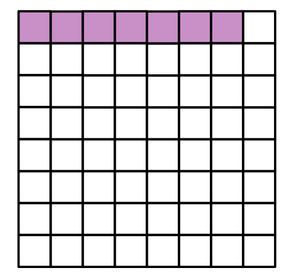

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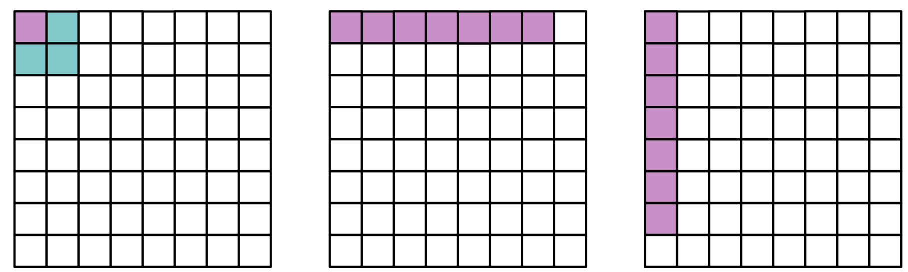

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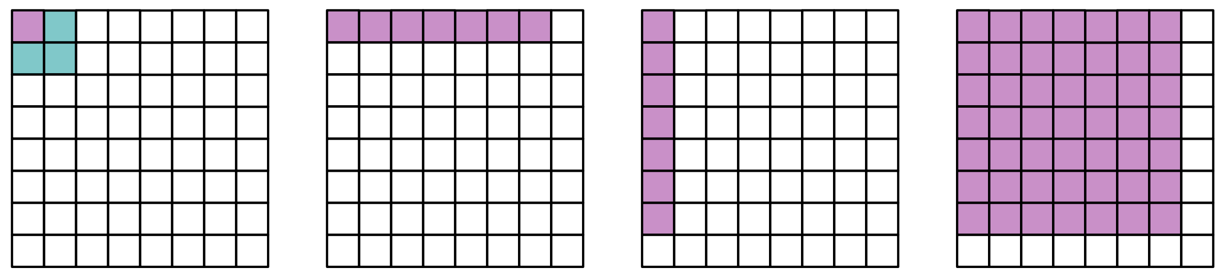
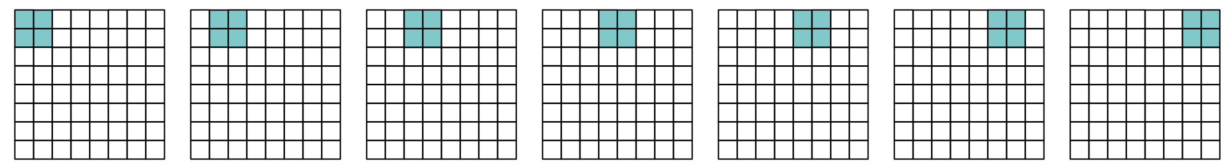
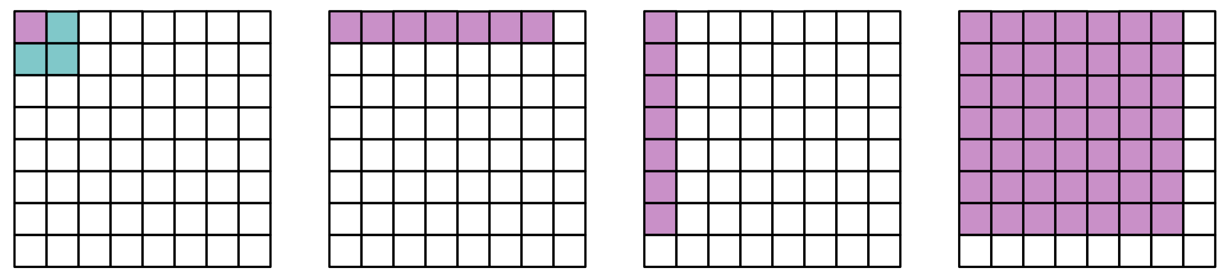


Systematic

Chessboard Squares

Size	1×1	2×2	3×3	4×4	5×5	6×6	7×7	8×8
Number	64	49						1





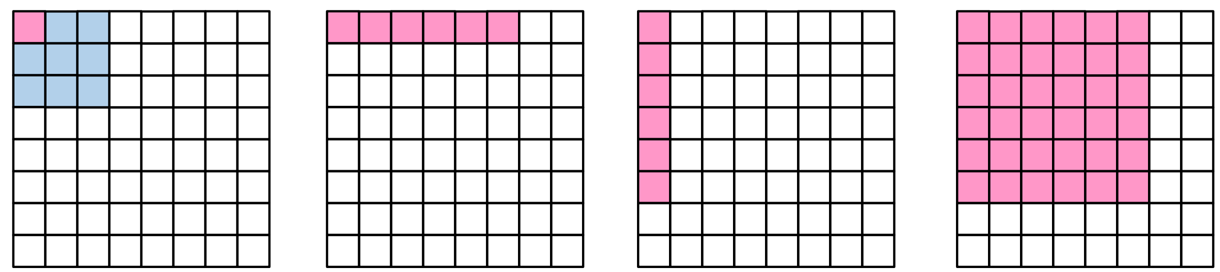
Noticing patterns

Chessboard Squares

Size	1×1	2×2	3×3	4×4	5×5	6×6	7×7	8×8
Number	64	49						1
Pattern	8^2	7^2						

Chessboard Squares

Size	1×1	2×2	3×3	4×4	5×5	6×6	7×7	8×8
Number	64	49	36					1
Pattern	8^2	7^2	6^2					1^2



Generalising by analogy

Chessboard Squares

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Number	64	49	36					1
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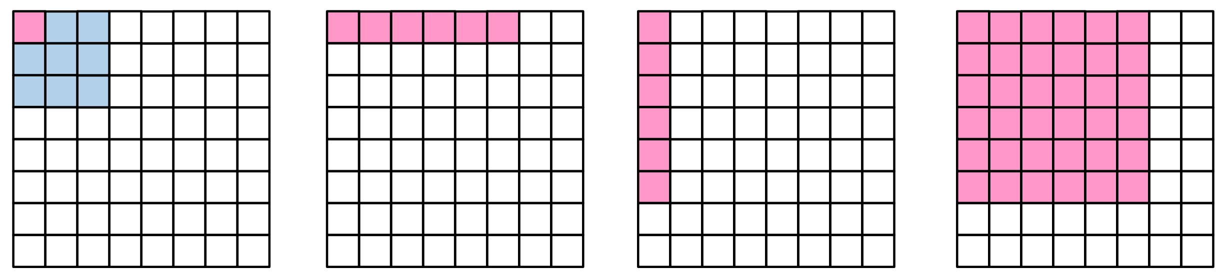


Chessboard Squares

Size	1×1	2×2	3×3	4×4	5×5	6×6	7×7	8×8
Number	64	49	36	25	16	9	4	1
Pattern	8^2	7^2	6^2	5^2	4^2	3^2	2^2	1^2

64 + 49 + 36 + 25 + 16 + 9 + 4 + 1 = 204

Here is our expression. The sum of the squares from 1 to 8. And it equals 204. Yay for us!

This problem took about 30 minutes of student work in Week 2 of the course. Each week has four hours of workshops (not lectures!)

Generalising by extension

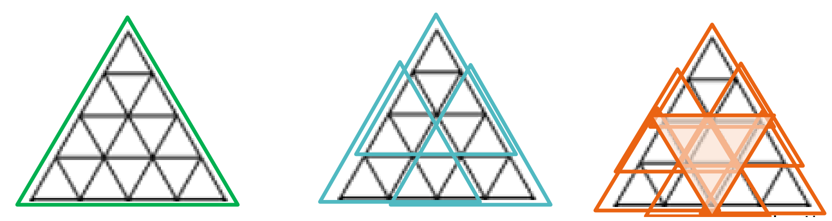
Chessboard Squares

So this is an example of a ‘low threshold, high ceiling’ task.
Low barrier for entry; everyone can begin.
High ceiling for more challenging mathematical exploration.

Generalising by extension

Chessboard Squares

- What about different sized chessboards?
- What about three dimensions?
- What about different types of grids?



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Low barrier for entry; everyone can begin.
High ceiling for more challenging mathematical exploration.

Specialising

- **randomly**
  (to get a feel for the problem)
- **systematically**
  (to prepare for generalising)
- **artfully**
  (to test any conjectures)

Generalising

Thinking mathematically (2nd edn), Mason, Burton, Stacey

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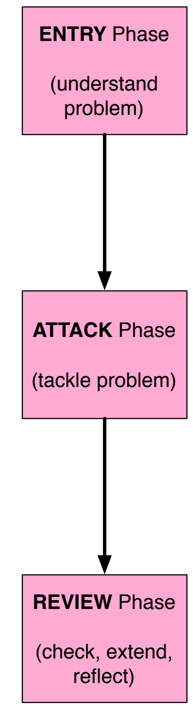

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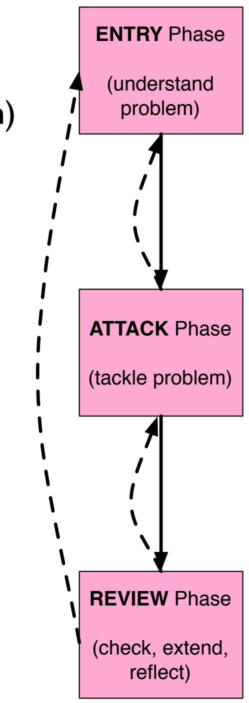

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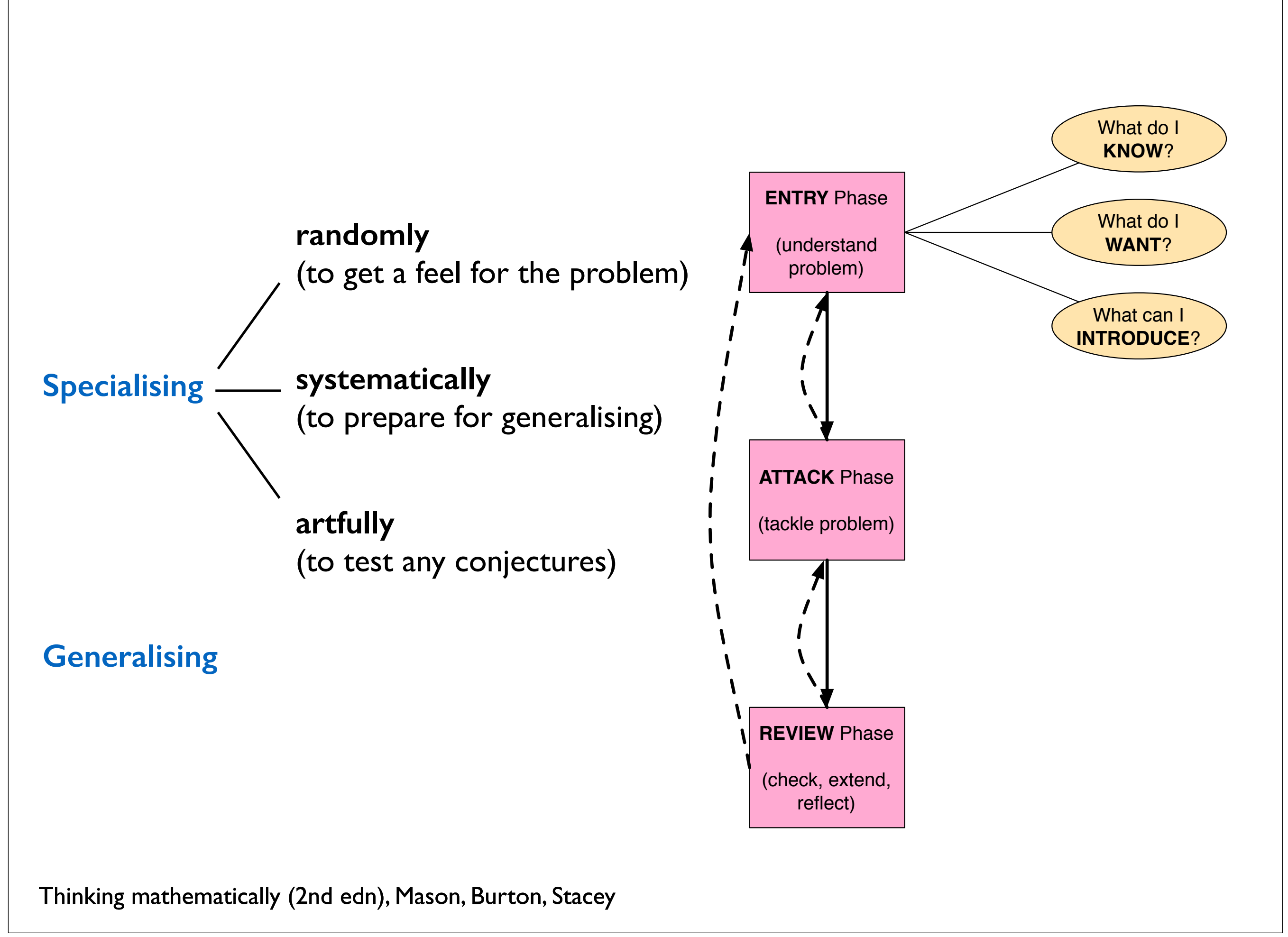
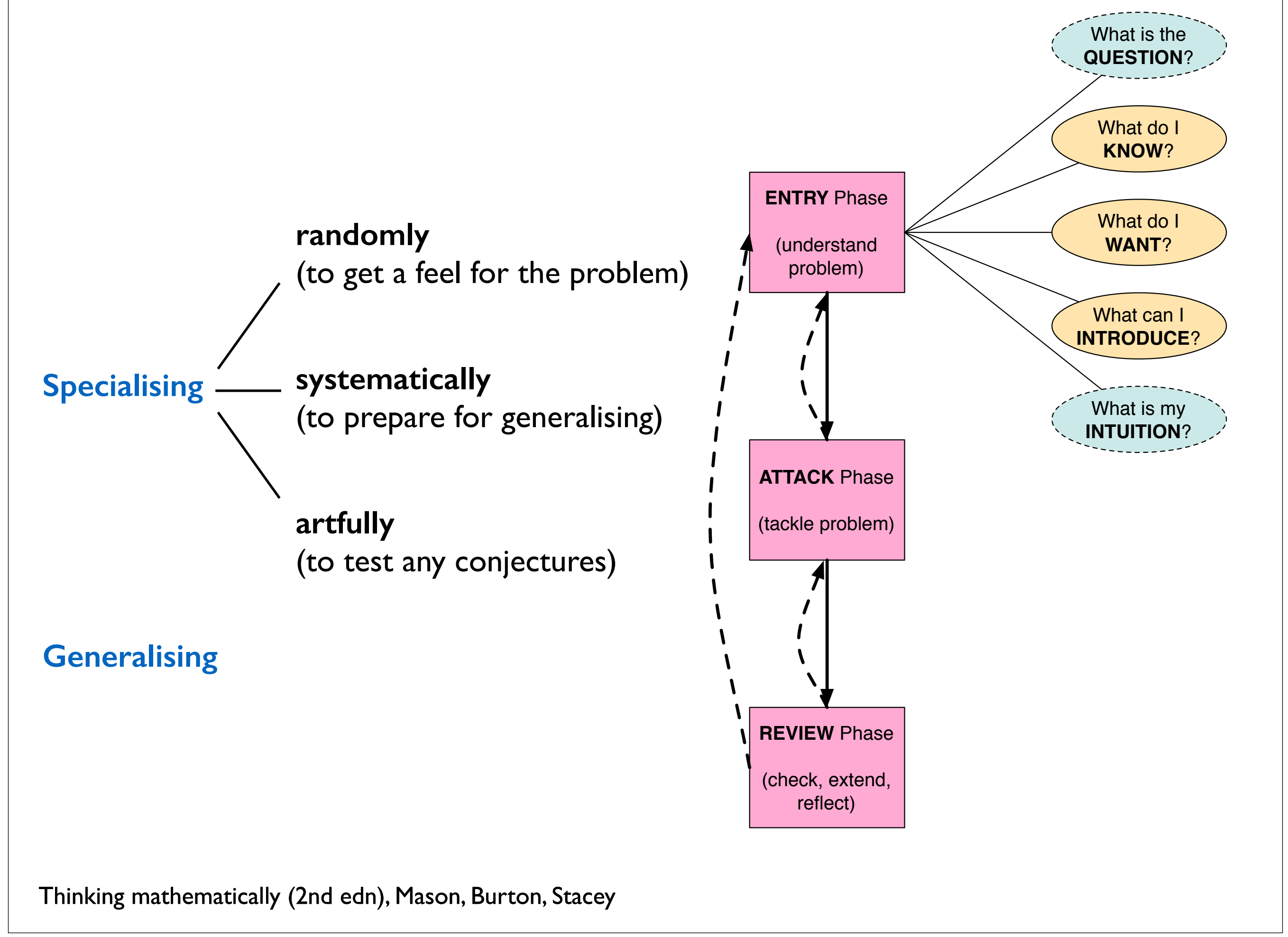


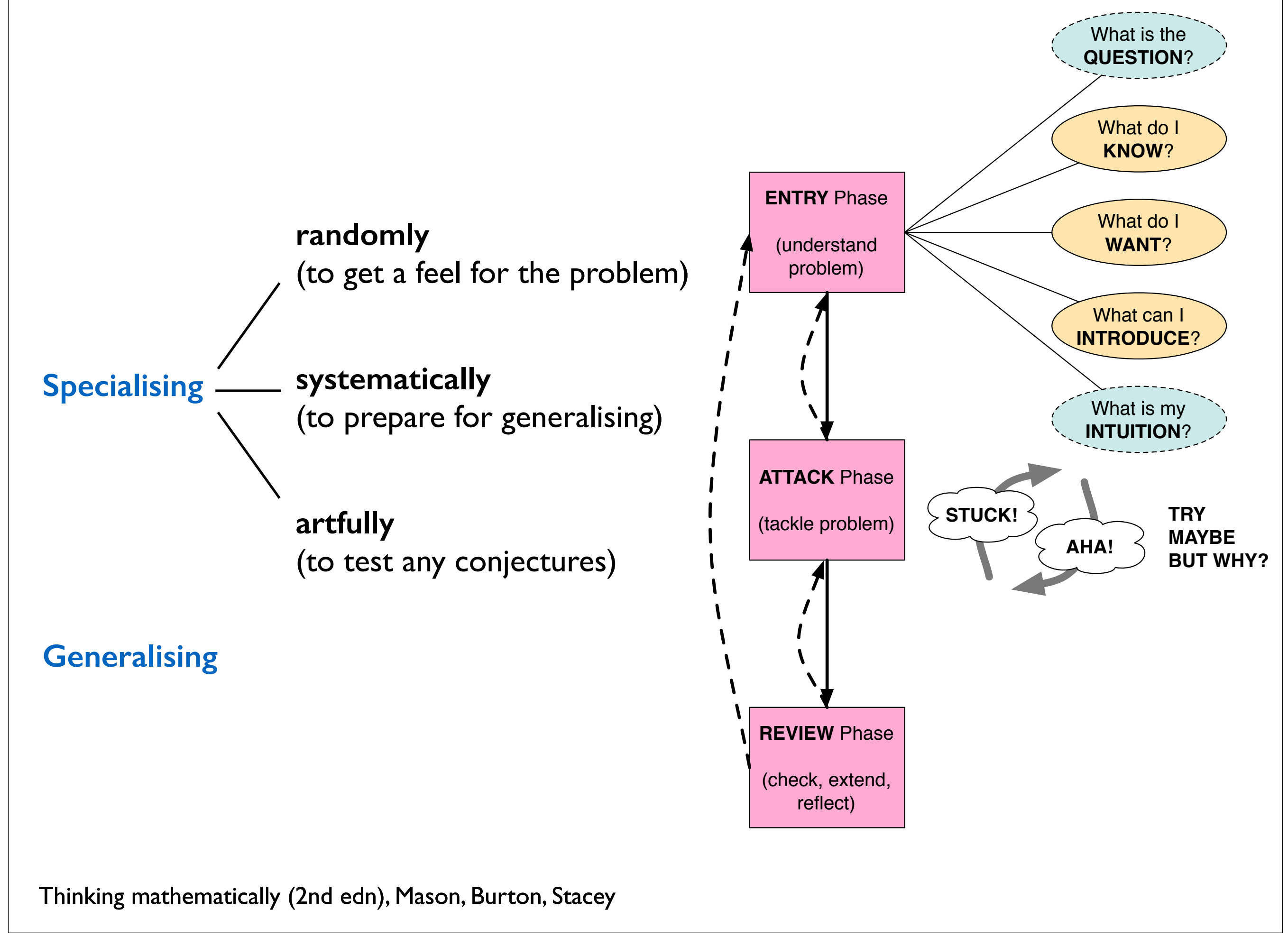
The Entry phase in which we work to understand the problem.

- What do I know (from the question or from experience)?
- What do I want? To find an answer? To prove something?
- What can I introduce? Definitions, notation, diagrams, tables, physical models, other ways to systematically record work.



Posing our own questions—and this often means students are so much more invested in finding a solution.

Looking at the role of intuition.



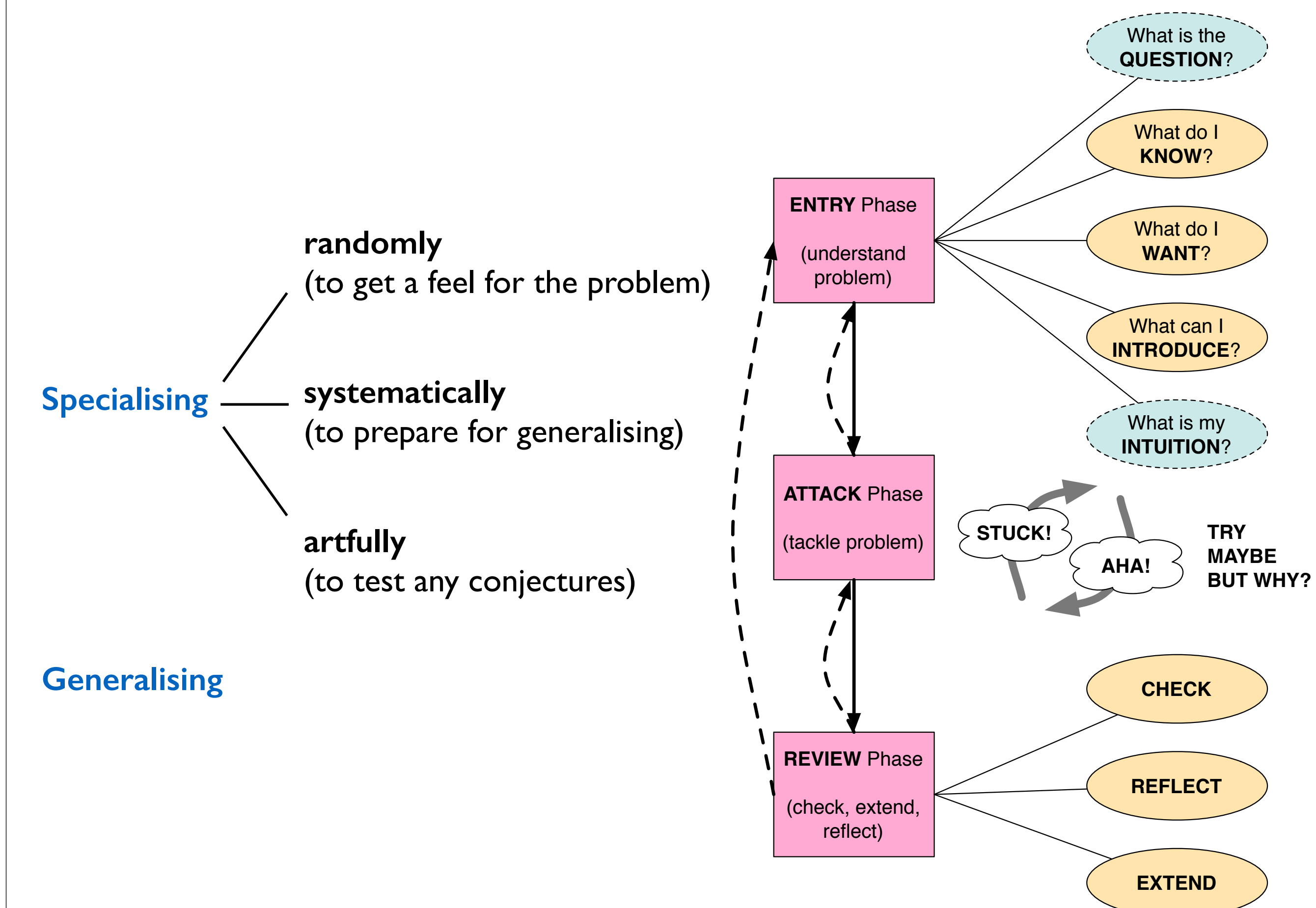
Finding strategies for 'getting unstuck'.

Working systematically to detect patterns or expose cases for which a theory might not hold.

Making conjectures.

Justifying and convincing—yourself, a friend, an enemy.

What it means to prove something and why we need proof. It's not sufficient to simply gather evidence to support your intuition. That is, detecting a pattern that holds for a few cases is not enough! Instead we need to uncover the truth.



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The Review phase which includes checking and extending work.

Week 1: Multiple representations, specialising and generalising, mathematical mindsets
Weeks 2-6: Entry phase
Weeks 7-12: Attack phase (noticing patterns, making and articulating conjectures, convincing, justifying, proving)
Review is emphasised throughout.

Students work mostly in groups and at their own pace on carefully-chosen problems that require a range of mathematical techniques but are purposefully selected to reinforce the weekly focus on a particular mathematical process.

Threaded throughout this is development of skills in mathematical communication, in a variety of forms and with a variety of dimensions.

Mathematical communication

- Written
- Verbal

- Informal
- Semi-formal
- Formal

- With self
- With peers
- With me

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- Written
- Verbal

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Group work

Mathematical communication

- Written
- Verbal

- Informal
- Semi-formal
- Formal

- With self
- With peers
- With me

Writing yourself notes

Colour Conundrum

Ms Green, Ms Brown and Ms Pink went to a hairdresser where they experimented with different colours and haircuts. When they left the hairdresser, the lady with the green hair said: 'Have you noticed that although our hair colours match our names, none of us has the same hair colour as our name?'. Ms Brown replied: 'Indeed, you are right! This is remarkable'. What hair colour did each lady have?

Rubric Writing

What I Know:

- I know that each three ladies cannot have the same hair colour as their name.
- I know that there is only three possible hair colours.
- I know that each lady can only have two of the possible hair colours.

What I need to find out:

- What hair colour each lady has.

AHA! If I know that each lady can only have two possible hair colours I then can arrange the information like so:

Mrs Brown can either have
- Pink hair
- Green hair
- Not brown hair

Mrs Green can either have
- Brown hair
- Pink hair
- Not green hair

Mrs Pink can either have
- Brown hair
- Green hair
- Not pink hair

By introducing a table for the information I know have sorted above, I then can use a process of elimination to find out the possible answers.

Person	Possibilities	
Mrs Brown	Pink	Green
Mrs Green	Pink	Brown
Mrs Pink	Green	Brown

STUCK! – Attempt one does not work. By giving Mrs Brown green hair and Mrs Green brown hair, Mrs Pink is left with no possible hair colours. So I now know by changing either Ms Brown or Mrs Green to pink hair this will free up an option for Mrs Pink.

CHECK

Person	Possibilities	
Mrs Brown	Pink	Green
Mrs Green	Pink	Brown
Mrs Pink	Green	Brown

AHA! This works, I now have each lady with a different hair colour to her name and where no hair colour is doubled up. Now I want to check to see if there is another option for the ladies.

CHECK

Person	Possibilities	
Mrs Brown	Pink	Green
Mrs Green	Pink	Brown
Mrs Pink	Green	Brown

AHA! This also works; by using that process of elimination I was again able to find another possible outcome for the ladies. Now I want to know if there are any more possible options. I will use this process of elimination until I have shown all possible options.

Person	Possibilities	
Mrs Brown	Pink	Green
Mrs Green	Pink	Brown
Mrs Pink	Green	Brown

Reflect:

After using the process of elimination I was able to conclude that there are only two possible outcomes that the three ladies can have. Either Mrs Brown has pink hair where Mrs Green has brown and Mrs Pink has green hair or Mrs Brown has green hair where Mrs Green has pink hair and Mrs Pink has brown.

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Assignments

Mathematical communication

- Written
- Verbal

- Informal
- Semi-formal
- Formal

- With self
- With peers
- With me

Mini-talks

Mathematical communication

- Written
- Verbal

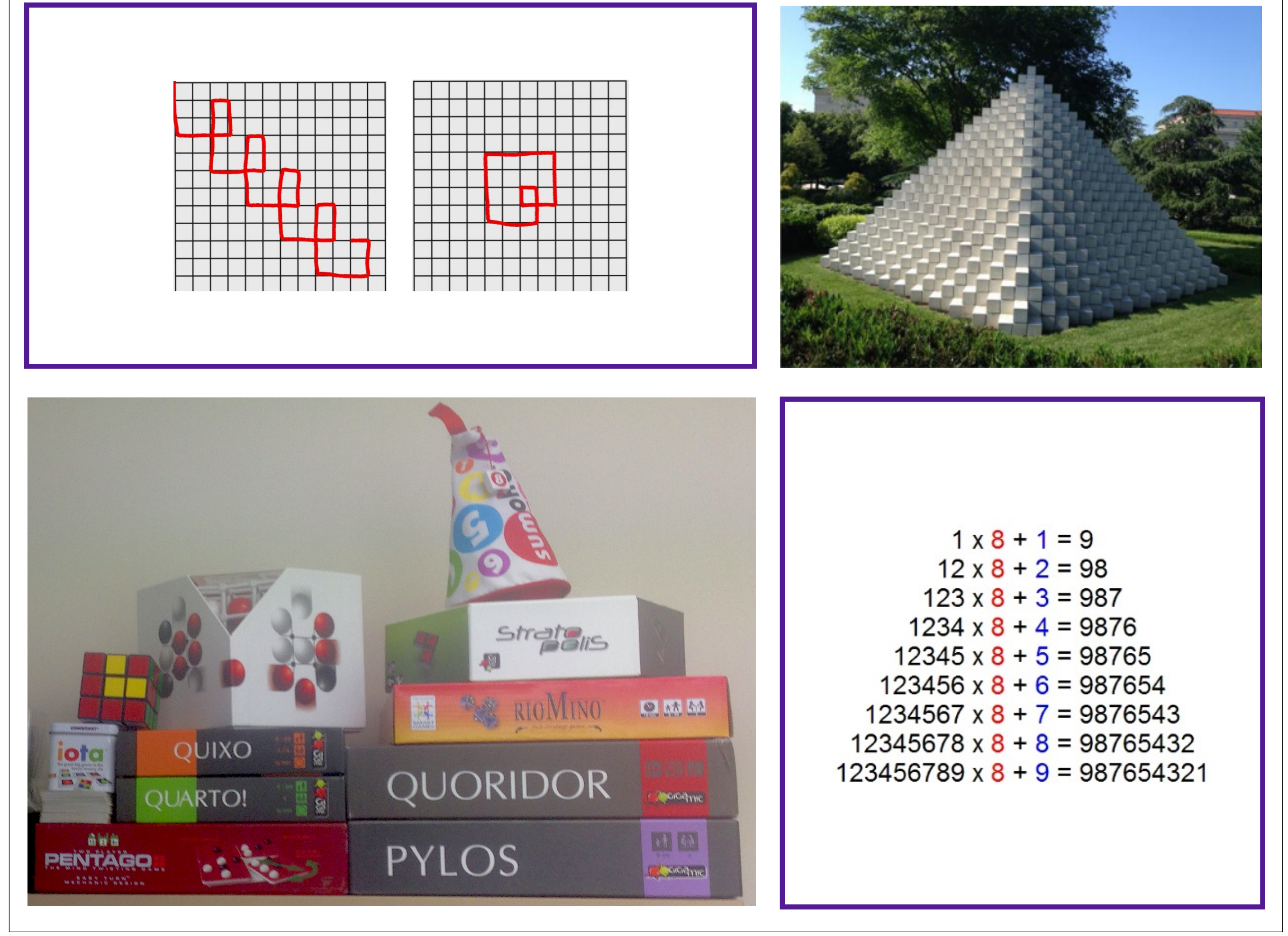
- Informal
- Semi-formal
- Formal

- With self
- With peers
- With me

Final project report

and

presentation



I want to conclude by talking about the major project for the course, which draws together all of these mathematical processes.

The project is an in-depth investigation by an individual student or pair of students on a topic of their choosing. By choosing their own question, they become deeply invested.

I have a few suggestions for topics, but many students propose their own.

< CLICK>

For example,

Project timeline

	Play around with ideas
Week 5	Discuss project topic in **individual meeting**
Week 6	
Week 7	
Week 8	Submit **draft** project report
Week 9	Receive written feedback, including from a **peer**
Week 10	Discuss progress in **individual meeting**
Week 11	
Week 12	
Week 13	Final 10-minute **presentations** to class
Week 14	Submit **final** project report

I ask students to play around with their ideas for a couple of weeks, then come and discuss their proposal with me in Week 5.

In Week 8, they hand up a draft report — and I deliberately ask for evidence of ideas that didn't work, rough notes and other evidence of mathematical exploration. So my students know that I value the process, not the end product, and that I'm encouraging them to be creative, to explore, and to follow their own path towards resolving the challenge.

I give them feedback, as does a fellow student (and I coach them on how to provide appropriate feedback). This mid-draft is to encourage the process of revising existing work and incorporating new work.

Students meet with me again in Week 10 to discuss their draft and the progress they are making.

In Week 14 they submit a final polished report, and an accompanying 10-minute oral presentation. And these skills are also progressively built throughout the course.

Evaluations

- I was not able to solve problems until I had this class. I dread solving problem-solving questions, but now I am **confident** I am able to think different ways in solving.
- I enjoyed the visual patterns at the beginning of each lesson. It helped me **think outside** the box.
- I was encouraged to use a variety of strategies to complete tasks and extend my thinking to a higher level. I was **constantly challenged** but never felt I was out of my depth.
- In some cases students were **developing quite high-level mathematical understandings** which **were well beyond their supposed status as mathematical novices**.

Despite the amount of marking, the last week is possibly the most enjoyable of the course because, having given their talks, students realise that they have become into enthusiastic, confident and capable problem solvers. In some cases, they are tackling problems that are known to be unsolved. (Although they didn't know this at the beginning!)

It's this that gives me optimism for the impact they will have on their own students in the future, encouraging them to become active, creative, problem solvers.

THANK YOU!

www.tinyurl.com/austms-15-aa

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