Student reasoning about real-valued sequences

Insights from an example-generation study

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Table of contents

Introduction

"Active learning" is beneficial to students (Freeman et al., 2014)

- what exactly is active learning?
- which students?

Teaching staff are being asked to do more to encourage active learning with traditional cohorts (e.g. Loch and Borland, 2014)

- Blending and flipping
- which students?

What do we want students to do outside class?
What do we want students to do inside class?

Example generation is one option amongst many others

Teaching and learning with examples

Why bother?

Many expert mathematicians use examples when researching new topics (Zazkis and Chernoff, 2008)

“There is an underlying assumption that examples facilitate learning” *Chick* (2009, p.30)

Students who can access more (correct) examples of a concept will have better informed concept images (Meehan, 2002)

Student find arguments (and proofs) that are based on examples particularly convincing (Harel & Sowder, 1998),

Teaching and learning with examples

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However. . .

Student find arguments (and proofs) that are based on examples particularly convincing (Harel & Sowder, 1998), **even when such arguments are false** (Fischbein 1982, Chazan, 1993)

Students might extrapolate general features (sometimes in the form of definitions) from single prototype examples (Alcock & Simpson, 2002).

When individuals interact with examples, some may regard the examples as representatives for a wider class of objects, whilst others will see the examples as specific instances only (Mason & Pimm, 1984)

However, these are arguably linked to students not having rich enough example spaces (Michener, 1978)

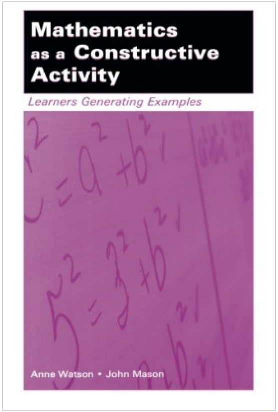
Example generation

One way to make working with examples more active is to have students generate their own (Watson & Mason, 2005)

- when they first meet a topic / definition
- when they are familiar with a topic

Students can attempt this with minimal guidance, so possibly something that is suitable for a blended / flipped approach. . .

What could possibly go wrong with that!?



Methodology

All students taking a first analysis course at a research intensive UK University (world top 50 in mathematics), mathematics majors and minors.

Example generation task (more about this later).

Mixed methods approach:

- Large scale (n=147) student task
  — Rasch Analysis
- Smaller scale (n=15) student task-based interviews
  — Phenomenographic coding

Here were are concentrating in the ways that students generated incorrect examples.

Reference sheet

Students had met the topic of real sequences before, and had access to a reference sheet of definitions.

A sequence is a list of real numbers

$$(a_1, a_2, a_3, a_4, \dots)$$

where $(a_n)_{n=1}^{\infty}$ denotes the whole sequence.

Definition: A sequence $(a_n)_{n=1}^{\infty}$ is *increasing* if and only if $\forall n \in \mathbb{N}$, $a_{n+1} \geq a_n$.

Definition: A sequence $(a_n)_{n=1}^{\infty}$ is *strictly increasing* if and only if $\forall n \in \mathbb{N}$, $a_{n+1} > a_n$.

Definition: A sequence $(a_n)_{n=1}^{\infty}$ is *monotonic* if and only if it is increasing or decreasing.

Definition: A sequence $(a_n)_{n=1}^{\infty}$ is *bounded above* if and only if $\exists U \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$, $a_n \leq U$.

Definition: U is an *upper bound* for the sequence $(a_n)_{n=1}^{\infty}$ if and only if $\forall n \in \mathbb{N}$, $a_n \leq U$.

Definition: A sequence $(a_n)_{n=1}^{\infty}$ is *bounded below* if and only if $\exists L \in \mathbb{R}$ s.t. $\forall n \in \mathbb{N}$, $a_n \geq L$.

Definition: L is an *lower bound* for the sequence $(a_n)_{n=1}^{\infty}$ if and only if $\forall n \in \mathbb{N}$,

The task

In both the large– and small-scale parts, the same (untimed) task was given.

Please give an example of each of the following, **or state that this is impossible**.
You can write your sequence in any way you choose:
As a list of numbers, as a formula, etc.
You do no need to prove your answers.

1. A strictly increasing sequence [-1.65]
2. An increasing sequence that is not strictly increasing [0.28]
3. A sequence that is both increasing and decreasing [0.89]
4. A sequence that is neither increasing nor decreasing [0.44]
5. A sequence that has no upper bound [-3.85]
6. A sequence that has neither an upper bound nor a lower bound [0.52]
7. A bounded, monotonic sequence [-0.10]
8. A sequence that tends to infinity [-2.90]
9. A sequence that tends to infinity that is not increasing [3.80]
10. A sequence that tends to infinity that is not bounded below [1.09]
11. A strictly increasing sequence that does not tend to infinity [1.47]

Results

For a more complete account see Edwards (2011).

There were plenty of good, intuitive, well thought out approaches and answers.

What I concentrate on here is on qualitatively different ways of answering incorrectly, in preparation for advice/suggestions for using example generation in active learning actitivites.

There isn't enough time to do justice to different reasons "why" students answered the way they did.

Issues with example generation

There were many situations where students did not use the definitions, or replaced them with everyday meanings (c.f. Cornu, 1991).

Q3. A sequence that is both increasing and decreasing

Edha: Erm, both increasing and decreasing, sine or cos curve?
Interviewer: What made you think of the sine or cos curve?
Edha: It just keeps going up and down
[Answer given: $a_n = (-1)n(n+1)$]

Q6. A sequence that has neither an upper bound nor a lower bound

Ben: Ok so here we want one that has a U at infinity and minus infinity to get both of those. So you could have -1, 2, -2, 3, -3.

Issues with example generation

Rarer, and less expected, were situations when the property overrides the constraint that the object be a sequence.

Q6. A sequence that has neither an upper bound nor a lower bound.

Student answer [pilot study]: $(-\infty, \infty)$

Q10. A sequence that tends to infinity that is not bounded below.

Mike: I suppose we could have something like, no - because we can never have a negative n can we?
I was thinking we could have something like $a_n = n^3$, something like that and that wouldn't be bounded below, and it would tend to infinity, but then we'd have to have a negative n, which we're not allowed.

Discussion

Students don't always have a good control over

- which objects are in an example space of an object
- similar definitions for different objects
- the representation of what they write

This is fine **and normal** — students were asked to think aloud!

But many students do not check answers and do not reflect on their thinking in mathematics.

Active classroom learning / flipped pedagogies

Example generation tasks are compatible and beneficial with a flipped classroom approach.

But, the flipped approach uses self-guided study for a topic initially, whereas this example generation task was after teaching. Still, several fundamental misconceptions were present.

More important is the feedback we give.

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